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Completely continuous commutator of Marcinkiewicz integral

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Abstract. Let $\mathcal{M}_\Omega$ be the higher-dimensional Marcinkiewicz integral introduced by Stein. In this paper, by Fourier transform estimates, approximation and a sufficient condition for strongly pre-compact set in $L^p(L^2[1, 2], l^2; \mathbb{R}^n)$, the authors proved that if $b \in \mathrm{CMO}(\mathbb{R}^n)$ and $\Omega \in L(\ln L)^{\frac{3}{2}}(S^{n-1})$, then for $p \in (1, \infty)$, the commutator generated by b and $\mathcal{M}_\Omega$ is a completely continuous operator on $L^p(\mathbb{R}^n)$.

1. Introduction

As an analogy to the classical Littlewood–Paley g-function, Marcinkiewicz [20] introduced the operator defined by

$$\mathcal{M}(f)(x) = \left(\int_0^\pi \frac{|F(x+t) - F(x-t) - 2F(x)|^2}{t^3} \,\mathrm{d}t \right)^{\frac{1}{2}},$$

where $F(x) = \int_0^x f(t)\mathrm{d}t$. This operator is now called the Marcinkiewicz integral. Zygmund [26] proved that $\mathcal{M}$ is bounded on $L^p([0, 2\pi])$ for $p \in (1, \infty)$. Stein [21] generalized the Marcinkiewicz operator to the case of higher dimension. Let Ω be homogeneous of degree zero, integrable and have mean value zero on the unit sphere S^{n-1}. Define the Marcinkiewicz integral operator $\mathcal{M}_\Omega$ by

$$\mathcal{M}_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t} f(x)|^2 \frac{\mathrm{d}t}{t^3} \right)^{\frac{1}{2}}, \tag{1.1}$$

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where

$$F_{\Omega,t}f(x)=\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}f(y)\mathrm{d}y$$

for $f\in\mathcal{S}(\mathbb{R}^n)$. This operator has been studied by many authors (see [1], [6], [12], [13], and the related references therein). STEIN [21] proved that if $\Omega\in \mathrm{Lip}_\alpha(S^{n-1})$ with $\alpha\in(0,1]$, then $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbb{R}^n)$ for $p\in(1,2]$. Benedek, Calderón and Panzon showed that the $L^p(\mathbb{R}^n)$ ($p\in(1,\infty)$) boundedness of $\mathcal{M}_\Omega$ holds true under the condition that $\Omega\in C^1(S^{n-1})$. Using the one-dimensional result and Riesz transforms similarly as in the case of singular integrals (see [4]) and interpolation, WALSH [24] proved that for $p\in(1,\infty)$, $\Omega\in L(\ln L)^{1/r}(\ln\ln L)^{2(1-2/r')}(S^{n-1})$ is a sufficient condition such that $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbb{R}^n)$, where $r=\min\{p,p'\}$ and $p'=p/(p-1)$. DING, FAN and PAN [12] proved that if $\Omega\in H^1(S^{n-1})$ (the Hardy space on S^{n-1}), then $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbb{R}^n)$ for all $p\in(1,\infty)$; AL-SALMAM *et al.* [1] proved that $\Omega\in L(\ln L)^{\frac12}(S^{n-1})$ is a sufficient condition such that $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbb{R}^n)$ for all $p\in(1,\infty)$.

The commutator of $\mathcal{M}_\Omega$ is also of interest and has been considered by many authors. Let $b\in\mathrm{BMO}(\mathbb{R}^n)$, the commutator generated by $\mathcal{M}_\Omega$ and b is defined by

$$\mathcal{M}_{\Omega,b}f(x)=\left(\int_0^\infty\Big|\int_{|x-y|\le t}\big(b(x)-b(y)\big)\frac{\Omega(x-y)}{|x-y|^{n-1}}f(y)dy\Big|^2\frac{dt}{t^3}\right)^{\frac12}. \tag{1.2}$$

TORCHINSKY and WANG [22] showed that if $\Omega\in\mathrm{Lip}_\alpha(S^{n-1})$ ($\alpha\in(0,1]$), then $\mathcal{M}_{\Omega,b}$ is bounded on $L^p(\mathbb{R}^n)$ with bound $C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$ for all $p\in(1,\infty)$. HU and YAN [19] proved that $\Omega\in L(\ln L)^{\frac32}(S^{n-1})$ is a sufficient condition such that $\mathcal{M}_{\Omega,b}$ is bounded on $L^2(\mathbb{R}^n)$. CHEN and LU [5] improved the result in [19] and showed that if $\Omega\in L(\ln L)^{\frac32}(S^{n-1})$, then $\mathcal{M}_{\Omega,b}$ is bounded on $L^p(\mathbb{R}^n)$ with bound $C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$ for all $p\in(1,\infty)$.

Let $\mathrm{CMO}(\mathbb{R}^n)$ be the closure of $C_0^\infty(\mathbb{R}^n)$ in the $\mathrm{BMO}(\mathbb{R}^n)$ topology, which coincides with $\mathrm{VMO}(\mathbb{R}^n)$, the space of functions of vanishing mean oscillation introduced by COIFMAN and WEISS [11], see also [3]. UCHIYAMA [23] proved that if S is a Calderón–Zygmund operator, and $b\in\mathrm{BMO}(\mathbb{R}^n)$, then the commutator of S defined by

$$[b,S]f(x)=b(x)Sf(x)-S(bf)(x)$$

is a compact operator on $L^p(\mathbb{R}^n)$ ($p\in(1,\infty)$) if and only if $b\in\mathrm{CMO}(\mathbb{R}^n)$. CHEN, DING and WANG [8] considered the compactness of $\mathcal{M}_{\Omega,b}$ on $L^p(\mathbb{R}^n)$, and proved that if Ω satisfies certain regularity condition of Dini type, then for $p\in(1,\infty)$,

$\mathcal{M}_{\Omega,b}$ is compact on $L^p(\mathbb{R}^n)$ if and only if $b \in \mathrm{CMO}(\mathbb{R}^n)$. The purpose of this paper is to prove that, in order to guarantee the compactness of $\mathcal{M}_{\Omega,b}$ on $L^p(\mathbb{R}^n)$, the regularity condition of Ω is superfluous. To formulate our main result, we first recall some definitions.

Definition 1.1. Let $\mathcal{X}$ be a normed linear space and $\mathcal{X}^*$ be its dual space, $\{x_k\} \subset \mathcal{X}$ and $x \in \mathcal{X}$. If for all $f \in \mathcal{X}^*$,

$$\lim_{k\to\infty} |f(x_k) - f(x)| = 0,$$

then $\{x_k\}$ is said to converge to x weakly, or $x_k \rightharpoonup x$.

Definition 1.2. Let $\mathcal{X}$, $\mathcal{Y}$ be two Banach spaces and S be a bounded operator from $\mathcal{X}$ to $\mathcal{Y}$.

(i) If for each bounded set $\mathcal{G} \subset \mathcal{X}$, $S\mathcal{G} = \{Sx : x \in \mathcal{G}\}$ is a strongly pre-compact set in $\mathcal{Y}$, then S is called a compact operator from $\mathcal{X}$ to $\mathcal{Y}$;

(ii) if for $\{x_k\} \subset \mathcal{X}$ and $x \in \mathcal{X}$,

$$x_k \rightharpoonup x \text{ in } \mathcal{X} \Rightarrow \|Sx_k - Sx\|_{\mathcal{Y}} \to 0,$$

then S is said to be a completely continuous operator.

It is well known that if $\mathcal{X}$ is a reflexive space and S is completely continuous from $\mathcal{X}$ to $\mathcal{Y}$, then S is also compact from $\mathcal{X}$ to $\mathcal{Y}$. On the other hand, if S is a linear compact operator from $\mathcal{X}$ to $\mathcal{Y}$, then S is also a completely continuous operator. However, if S is not linear, then the compactness of S does not imply that S is completely continuous. For example, the operator

$$Sx = \|x\|_{l^2}$$

is compact from l^2 to $\mathbb{R}$, but not completely continuous.

The main result in this paper can be stated as follows.

Theorem 1.1. *Let Ω be homogeneous of degree zero and have mean value zero on S^{n-1}. Suppose that $\Omega \in L(\ln L)^{3/2}(S^{n-1})$. Then for $b \in \mathrm{CMO}(\mathbb{R}^n)$ and $p \in (1, \infty)$, $\mathcal{M}_{\Omega,b}$ is completely continuous on $L^p(\mathbb{R}^n)$.*

Remark 1.1. Recently, Chen and Hu [7] considered the compactness of the commutator of homogeneous singular integral operators defined by

$$T_\Omega f(x) = \mathrm{p.\,v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\mathrm{d}y,$$

here Ω is homogeneous of degree zero, integrable on S^{n-1} and has mean value zero. Using the idea of approximating T_Ω by a sequence of operators with smooth kernels, Chen and Hu considered the compactness of the commutator of T_Ω when Ω satisfies

$$\sup_{\zeta\in S^{n-1}}\int_{S^{n-1}}|\Omega(\eta)|\left(\ln\frac{1}{|\eta\cdot\zeta|}\right)^{\theta}\mathrm{d}\eta<\infty \tag{1.3}$$

for some $\theta>2$. It should be pointed out that this idea comes from WATSON's paper [25]. In this paper, we will also employ the idea of Watson. However, the operators $\mathcal{M}$ and $\mathcal{M}_{\Omega,\,b}$ are not linear, the proof of Theorem 1.1 involves much more technical problems, such as an appropriate sufficient condition of strongly pre-compact sets in space $L^p(L^2[1,\,2],\,l^2;\,\mathbb{R}^n)$ (see Lemma 3.4 below), and the argument in this paper is more complicated.

We make some conventions. In what follows, C always denotes a positive constant that is independent of the main parameters involved, but whose value may differ from line to line. We use the symbol $A\lesssim B$ to denote that there exists a positive constant C such that $A\le CB$. For a set $E\subset\mathbb{R}^n$, χ_E denotes its characteristic function. Let M be the Hardy–Littlewood maximal operator. For $r\in(0,\,\infty)$, we use M_r to denote the operator $M_rf(x)=\big(M(|f|^r)(x)\big)^{1/r}$.

2. Approximation

Let Ω be homogeneous of degree zero, integrable on S^{n-1}. For $t\in[1,\,2]$ and $j\in\mathbb{Z}$, set

$$K_t^j(x)=\frac{1}{2^j}\frac{\Omega(x)}{|x|^{n-1}}\chi_{\{2^{j-1}t<|x|\le 2^jt\}}(x). \tag{2.1}$$

As it was proved in [15], there exists a constant $\alpha\in(0,\,1)$ such that for $t\in[1,\,2]$ and $\xi\in\mathbb{R}^n\backslash\{0\}$,

$$|\widehat{K_t^j}(\xi)|\lesssim\|\Omega\|_{L^\infty(S^{n-1})}\,\min\{1,\,|2^j\xi|^{-\alpha}\}. \tag{2.2}$$

Moreover, if $\int_{S^{n-1}}\Omega(x')\mathrm{d}x'=0$, then

$$|\widehat{K_t^j}(\xi)|\lesssim\|\Omega\|_{L^1(S^{n-1})}\,\min\{1,\,|2^j\xi|\}. \tag{2.3}$$

Let

$$\widetilde{\mathcal{M}}_\Omega f(x)=\left(\int_1^2\sum_{j\in\mathbb{Z}}\big|F_jf(x,\,t)\big|^2\mathrm{d}t\right)^{\frac12},$$

with

$$F_jf(x,\,t)=\int_{\mathbb{R}^n}K_t^j(x-y)f(y)\mathrm{d}y.$$

For $b\in\mathrm{BMO}(\mathbb{R}^n)$, let $\widetilde{\mathcal{M}}_{\Omega,\,b}$ be the commutator of $\widetilde{\mathcal{M}}_{\Omega}$ defined by

$$\widetilde{\mathcal{M}}_{\Omega,\,b}f(x)=\Big(\int_1^2\sum_{j\in\mathbb{Z}}\big|F_{j,\,b}f(x,\,t)\big|^2\mathrm{d}t\Big)^{\frac{1}{2}},$$

with

$$F_{j,\,b}f(x,\,t)=\int_{\mathbb{R}^n}\big(b(x)-b(y)\big)K_t^j(x-y)f(y)\mathrm{d}y.$$

A trivial computation leads to that

$$\mathcal{M}_{\Omega}f(x)\approx\widetilde{\mathcal{M}}_{\Omega}f(x),\quad \mathcal{M}_{\Omega,\,b}f(x)\approx\widetilde{\mathcal{M}}_{\Omega,\,b}f(x).\tag{2.4}$$

Let $\phi\in C_0^{\infty}(\mathbb{R}^n)$ be a nonnegative function such that $\int_{\mathbb{R}^n}\phi(x)\mathrm{d}x=1$, $\mathrm{supp}\,\phi\subset\{x:\,|x|\le 1/4\}$. For $l\in\mathbb{Z}$, let $\phi_l(y)=2^{-nl}\phi(2^{-l}y)$. It is easy to verify that for any $\beta\in(0,\,1)$,

$$|\widehat{\phi_l}(\xi)-1|\lesssim\min\{1,\,|2^l\xi|^{\beta}\}.\tag{2.5}$$

Let

$$F_j^lf(x,\,t)=\int_{\mathbb{R}^n}K_t^j*\phi_{j-l}(x-y)f(y)\,\mathrm{d}y.$$

Define the operator $\widetilde{\mathcal{M}}^l_{\Omega}$ by

$$\widetilde{\mathcal{M}}^l_{\Omega}f(x)=\Big(\int_1^2\sum_{j\in\mathbb{Z}}\big|F_j^lf(x,\,t)\big|^2\mathrm{d}t\Big)^{\frac{1}{2}}\tag{2.6}$$

For $b\in\mathrm{BMO}(\mathbb{R}^n)$, let $\widetilde{\mathcal{M}}^l_{\Omega,\,b}$ be the commutator of $\widetilde{\mathcal{M}}^l_{\Omega}$, that is,

$$\widetilde{\mathcal{M}}^l_{\Omega,\,b}f(x)=\Big(\int_1^2\sum_{j\in\mathbb{Z}}\big|F_{j,\,b}^lf(x,\,t)\big|^2\mathrm{d}t\Big)^{\frac{1}{2}},\tag{2.7}$$

with

$$F_{j,\,b}^lf(x,\,t)=\int_{\mathbb{R}^n}\big(b(x)-b(y)\big)K_t^j*\phi_{j-l}(x-y)f(y)\,\mathrm{d}y.$$

For $j\in\mathbb{Z}$ and $l\in\mathbb{N}$, let

$$U_{l,\,j;\,t}(y)=K_t^j*\phi_{l-j}(y)-K_t^j(y).$$

Let $E_0 = \{x' \in S^{n-1} : |\Omega(x')| \le 2\}$ and $E_d = \{x' \in S^{n-1} : 2^d < |\Omega(x')| \le 2^{d+1}\}$ for $d \in \mathbb{N}$. Denote by Ω_d the restriction of Ω to E_d, namely, $\Omega_d(x') = \Omega(x')\chi_{E_d}(x')$. Set

$$U_{l,\,j;\,d,\,t}(y) = K^j_{d,\,t} * \phi_{l-j}(y) - K^j_{d,\,t}(y),$$

with

$$K^j_{d,\,t}(y) = \frac{1}{2^j}\frac{\Omega_d(x)}{|x|^{n-1}}\chi_{\{2^{j-1}t<|x|\le 2^j t\}}(x).$$

Lemma 2.1. *Let Ω be homogeneous of degree zero and $\Omega \in L^1(S^{n-1})$. Then for $p \in (1,\,\infty)$,*

$$\Big\|\Big(\sum_{l\in\mathbb{Z}}|U_{l,\,j;\,t} * f_l(x)|^2\Big)^{\frac12}\Big\|_{L^p(\mathbb{R}^n)} \lesssim \|\Omega\|_{L^1(S^{n-1})}\Big\|\Big(\sum_{l\in\mathbb{Z}}|f_l|^2\Big)^{\frac12}\Big\|_{L^p(\mathbb{R}^n)}.$$

For the proof of Lemma 2.1, see [19].

Lemma 2.2. *Let Ω be homogeneous of degree zero and have mean value zero, $\widetilde{\mathcal{M}}^l_\Omega$ be the operator defined by (2.6). Suppose that $\Omega \in L\ln L(S^{n-1})$, then for $l \in \mathbb{N}$ and $p \in (1,\,\infty)$,*

$$\|\widetilde{\mathcal{M}}_\Omega f - \widetilde{\mathcal{M}}^l_\Omega f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}. \tag{2.8}$$

Proof. It is obvious that

$$\big|\widetilde{\mathcal{M}}_\Omega f(x) - \widetilde{\mathcal{M}}^l_\Omega f(x)\big| \le \Big(\int_1^2 \sum_j \big|U_{l,\,j;\,t} * f(x)\big|^2 \mathrm{d}t\Big)^{\frac12}.$$

Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be a radial function such that $\operatorname{supp}\psi \subset \{1/4 \le |\xi| \le 4\}$ and

$$\sum_{i\in\mathbb{Z}}\psi(2^{-i}\xi) = 1, \quad |\xi| \ne 0.$$

Define the multiplier operator S_i by

$$\widehat{S_i f}(\xi) = \psi(2^{-i}\xi)\widehat{f}(\xi).$$

Let

$$\mathrm{D}_1 f(x) = \sum_{m=-\infty}^{0}\Big(\int_1^2\sum_j\Big|U_{l,\,j;\,t} * (S_{m-j}f)(x)\Big|^2\mathrm{d}t\Big)^{\frac12},$$

$$\mathrm{D}_2 f(x)=\sum_{d=1}^{\infty}\sum_{m=Nd}^{\infty}\left(\int_1^2\sum_j\left|U_{l,j;d,t}*(S_{m-j}f)(x)\right|^2\mathrm{d}t\right)^{\frac12}$$

and

$$\mathrm{D}_3 f(x)=\sum_{d=1}^{\infty}\sum_{m=1}^{Nd}\left(\int_1^2\sum_{j\in\mathbb{Z}}\left|U_{l,\,j;\,d,\,t}*(S_{m-j}f)(x)\right|^2\mathrm{d}t\right)^{\frac12}.$$

It then follows that for $f\in\mathcal{S}(\mathbb{R}^n)$,

$$\left\|\left(\int_1^2\sum_j\left|U_{l,\,j;\,t}*f(x)\right|^2\mathrm{d}t\right)^{\frac12}\right\|_{L^p(\mathbb{R}^n)}\le\sum_{i=1}^{3}\|\mathrm{D}_i f\|_{L^p(\mathbb{R}^n)}.$$

We now estimate the term D_1. By Fourier transform estimate, we know that

$$\begin{aligned}&\left\|\left(\int_1^2\sum_j\left|U_{l,\,j;\,t}*(S_{m-j}f)(x)\right|^2\mathrm{d}t\right)^{\frac12}\right\|_{L^2(\mathbb{R}^n)}^2\\&=\int_1^2\int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}}\left|U_{l,\,j;\,t}*(S_{m-j}f)(x)\right|^2\mathrm{d}x\mathrm{d}t\\&=\int_1^2\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n}|\widehat{K_t^j}(\xi)|^2|\widehat{\phi_{j-l}}(\xi)-1|^2|\psi(2^{-m+j}\xi)|^2|\widehat{f}(\xi)|^2\mathrm{d}\xi\mathrm{d}t\\&\lesssim\|\Omega\|_{L^1(S^{n-1})}\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n}|2^{j-l}\xi|^2|\psi(2^{-m+j}\xi)|^2|\widehat{f}(\xi)|^2\mathrm{d}\xi\\&\le 2^{2m}2^{-2l}\|\Omega\|_{L^1(S^{n-1})}^2\|f\|_{L^p(\mathbb{R}^n)}^2.\end{aligned}\tag{2.9}$$

On the other hand, for $p\in(2,\infty)$, applying the Minkowski inequality and Lemma 2.1, we have that

$$\begin{aligned}&\left\|\left(\int_1^2\sum_j\left|U_{l,\,j;\,t}*(S_{m-j}f)(x)\right|^2\mathrm{d}t\right)^{\frac12}\right\|_{L^p(\mathbb{R}^n)}^2\\&\le\int_1^2\left(\int_{\mathbb{R}^n}\left(\sum_{j\in\mathbb{Z}}\left|U_{l,\,j;\,t}*(S_{m-j}f)(x)\right|^2\right)^{\frac p2}\mathrm{d}x\right)^{\frac2p}\mathrm{d}t\\&\le\|\Omega\|_{L^1(S^{n-1})}^2\|f\|_{L^p(\mathbb{R}^n)}^2.\end{aligned}\tag{2.10}$$

To estimate

$$\left\|\left(\int_1^2\sum_j\left|U_{l,\,j;\,t}*(S_{m-j}f)(x)\right|^2\mathrm{d}t\right)^{\frac12}\right\|_{L^p(\mathbb{R}^n)}$$

for $p \in (1,\, 2)$, we consider the mapping $\mathcal{F}$ defined by

$$\mathcal{F}:\ \{h_j(x)\}_{j\in\mathbb{Z}} \longrightarrow \{U_{l,\,j;\,t} * h_j(x)\}.$$

Note that for any $t \in (1,\, 2)$,

$$\big|U_{l,\,j;\,t} * h_j(x)\big| \lesssim MM_\Omega h_j(x) + M_\Omega h_j(x),$$

with M_Ω the maximal operator defined by

$$M_\Omega h(x) = \sup_{r>0} \frac{1}{|B(x,\, r)|} \int_{B(x,\, r)} |\Omega(x-y)||f(y)|\mathrm{d}y.$$

It is well known that M_Ω is bounded on $L^p(\mathbb{R}^n)$ with bound $C\|\Omega\|_{L^1(S^{n-1})}$ for all $p \in (1,\, \infty)$. A straightforward computation then tells us that for $p_0 \in (1,\, \infty)$

$$\int_{\mathbb{R}^n} \int_1^2 \sum_{j\in\mathbb{Z}} \big|U_{l,\,j;\,t} * h_j(x)\big|^{p_0} \mathrm{d}t\mathrm{d}x \lesssim \|\Omega\|^{p_0}_{L^1(S^{n-1})} \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} |h_j(x)|^{p_0} \mathrm{d}x. \tag{2.11}$$

Also, we have that

$$\sup_{j\in\mathbb{Z}} \sup_{t\in[1,\,2]} \big|U_{l,\,j;\,t} * h_j(x)\big| \lesssim \|\Omega\|_{L^1(S^{n-1})} \sup_{j\in\mathbb{Z}} |h_j(x)|,$$

which implies that for $p_1 \in (1,\, \infty)$,

$$\Big\|\sup_{j\in\mathbb{Z}} \sup_{t\in[1,\,2]} \big|U_{l,\,j;\,t} * h_j\big|\Big\|_{L^{p_1}(\mathbb{R}^n)} \lesssim \|\Omega\|_{L^1(S^{n-1})} \Big\|\sup_{j\in\mathbb{Z}} |h_j|\Big\|_{L^{p_1}(\mathbb{R}^n)}. \tag{2.12}$$

For $p \in (1,\, 2)$, interpolating the inequalities (2.11) and (2.12) (with $p_0 \in (1,\, 2)$, $p_1 \in (2,\, \infty)$ and $1/p = 1/2 + (2-p_0)/(2p_1)$) leads to that

$$\Big\|\Big(\int_1^2 \sum_{j\in\mathbb{Z}} \big|U_{l,\,j;\,t} * h_j\big|^2 \mathrm{d}t\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^n)} \lesssim \|\Omega\|_{L^1(S^{n-1})} \Big\|\Big(\sum_{j\in\mathbb{Z}} |h_j|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^n)},$$

and so

$$\begin{aligned}
&\Big\|\Big(\int_1^2 \sum_{j} \Big|U_{l,\,j;\,t} * (S_{m-j}f)(x)\Big|^2 \mathrm{d}t\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^n)} \\
&\qquad \lesssim \|\Omega\|_{L^1(S^{n-1})} \Big\|\Big(\sum_{j\in\mathbb{Z}} |S_{m-j}f|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^n)} \lesssim \|\Omega\|_{L^1(S^{n-1})} \|f\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

This, along with (2.10), states that for $p \in (1,\, \infty)$,

$$\Big\|\Big(\int_1^2 \sum_j \Big|U_{l,\,j;\,t} * (S_{m-j}f)(x)\Big|^2 \mathrm{d}t\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^n)} \lesssim \|\Omega\|_{L^1(S^{n-1})}\|f\|_{L^p(\mathbb{R}^n)}. \quad (2.13)$$

Interpolating the inequalities (2.9) and (2.13) gives us that for $p \in (1,\, \infty)$,

$$\Big\|\Big(\int_1^2 \sum_j \Big|U_{l,\,j;\,t} * (S_{m-j}f)(x)\Big|^2 \mathrm{d}t\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^n)} \lesssim 2^{t_p m}\|\Omega\|_{L^1(S^{n-1})}\|f\|_{L^p(\mathbb{R}^n)},$$

with $t_p \in (0,\, 1)$ a constant depending only on p. Therefore,

$$\|\mathrm{D}_1 f\|_{L^p(\mathbb{R}^n)} \lesssim \|\Omega\|_{L^1(S^{n-1})}\|f\|_{L^p(\mathbb{R}^n)}.$$

We turn our attention to the term D_2. Again by the Plancherel theorem and the Fourier transform estimates (2.2) and (2.5), we have that

$$\begin{aligned}
&\Big\|\Big(\int_1^2 \sum_{j\in\mathbb{Z}} \Big|U_{l,\,j;\,d,\,t} * (S_{m-j}f)(x)\Big|^2 \mathrm{d}t\Big)^{\frac{1}{2}}\Big\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad= \int_1^2 \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} |\widehat{K_t^j}(\xi)|^2 |\widehat{\phi_{j-l}}(\xi) - 1|^2 |\psi(2^{-m+j}\xi)|^2 |\widehat{f}(\xi)|^2 \mathrm{d}\xi \mathrm{d}t \\
&\quad\lesssim \|\Omega_d\|_{L^\infty(S^{n-1})}^2 \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} |2^j\xi|^{-2\alpha} |2^{j-l}\xi|^{\alpha} \psi(2^{-m+j}\xi)|^2 |\widehat{f}(\xi)|^2 \mathrm{d}\xi \\
&\quad\lesssim \|\Omega_d\|_{L^\infty(S^{n-1})}^2 2^{-l\alpha} 2^{-m\alpha} \|f\|_{L^2(\mathbb{R}^n)}^2. \qquad (2.14)
\end{aligned}$$

As in the inequality (2.13), we have that for $p \in (1,\, \infty)$,

$$\Big\|\Big(\int_1^2 \sum_{j\in\mathbb{Z}} \Big|U_{l,\,j;\,d,\,t} * (S_{m-j}f)(x)\Big|^2 \mathrm{d}t\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^n)}^2 \lesssim \|\Omega_d\|_{L^1(S^{n-1})}^2 \|f\|_{L^p(\mathbb{R}^n)}^2. \quad (2.15)$$

Interpolating the inequalities (2.14) and (2.15) then gives that

$$\Big\|\Big(\int_1^2 \sum_{j\in\mathbb{Z}} \Big|U_{l,\,j;\,d,\,t} * (S_{m-j}f)(x)\Big|^2 \mathrm{d}t\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^n)} \lesssim \|\Omega_d\|_{L^\infty(S^{n-1})} 2^{-m\delta_p} \|f\|_{L^p(\mathbb{R}^n)}.$$

This in turn implies that

$$\|\mathrm{D}_2 f\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{d=1}^{\infty} 2^d \sum_{m=Nd}^{\infty} 2^{-m\delta_p} \|f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)},$$

if we choose $N \in \mathbb{N}$ such that $N > 2\delta_p$.

It remains to consider the term D_3. Again as (2.13), we have that for $p \in (1,\infty)$,

$$\Big\|\Big(\int_1^2 \sum_j \Big|U_{l,\,j;\,d,\,t} * (S_{m-j}f)(x)\Big|^2 \mathrm{d}t\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^n)} \lesssim \|\Omega_d\|_{L^1(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}.$$

This, in turn implies that

$$\|\mathrm{D}_3 f\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{d=1}^{\infty} 2^d \sum_{m=1}^{Nd} \|\Omega_d\|_{L^1(S^{n-1})}\|f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

Combining the estimates for D_1, D_2 and D_3 leads to (2.8). □

The following result shows that $\{\widetilde{\mathcal{M}}^l_\Omega\}_{l\in\mathbb{N}}$ approximate to $\widetilde{\mathcal{M}}_\Omega$ properly, and will be useful in the proof of Theorem 1.1.

Theorem 2.1. *Let Ω be homogeneous of degree zero and have mean value zero. Suppose that $\Omega \in L(\ln L)^\gamma(S^{n-1})$ for some $\gamma \in (1,\infty)$, then for $l \in \mathbb{N}$ and $p \in (1,\infty)$,*

$$\|\widetilde{\mathcal{M}}_\Omega f - \widetilde{\mathcal{M}}^l_\Omega f\|_{L^p(\mathbb{R}^n)} \lesssim l^{-\delta_p}\|f\|_{L^p(\mathbb{R}^n)},$$

with δ_p a constant depending only on p and n.

Proof. By the estimates in [17], we know that if Ω satisfies (1.3) for some $\theta \in (0,\infty)$, then, for $\xi \in \mathbb{R}^n\backslash\{0\}$, $j \in \mathbb{Z}$ and $t \in [1,2]$,

$$|\widehat{K^j_t}(\xi)| \lesssim \ln^{-\theta}(|2^j\xi|).$$

For each $\xi \in \mathbb{R}^n\backslash\{0\}$ and $l \in \mathbb{N}$, let j_0 be the integer such that $2^{l/2-1} < |2^{j_0}\xi| \le 2^{l/2}$. A trivial computation involving the Fourier transform estimates (2.1)–(2.3) leads to that

$$\sum_{j\in\mathbb{Z}} \big|\widehat{K^j_t}(\xi)\widehat{\phi}(2^{j-l}\xi) - \widehat{K^j_t}(\xi)\big|^2 \lesssim \sum_{j\in\mathbb{Z}:\,j\le j_0} |2^{j-l}\xi|^2 + \sum_{j\in\mathbb{Z}:\,j>j_0} \ln^{-2\gamma}(|2^j\xi|) \lesssim l^{-2\theta+1}.$$

This, via the Plancherel theorem, leads to

$$\big\|\widetilde{\mathcal{M}}_\Omega f - \widetilde{\mathcal{M}}^l_\Omega f\big\|_{L^2(\mathbb{R}^n)} \lesssim l^{-\theta+\frac{1}{2}}\|f\|_{L^2(\mathbb{R}^n)}. \tag{2.16}$$

On the other hand, it was pointed out in [18] that, if $\Omega \in L(\ln L)^\gamma(S^{n-1})$ for $\gamma \in (1,\infty)$, then Ω satisfies (1.3) for $\theta \in (1,\gamma)$. Therefore, by interpolating the inequalities (2.8) and (2.16), we know that under the hypothesis of Theorem 2.1,

$$\big\|\widetilde{\mathcal{M}}_\Omega f - \widetilde{\mathcal{M}}^l_\Omega f\big\|_{L^p(\mathbb{R}^n)} \lesssim l^{-\gamma+1/2+\epsilon}\|f\|_{L^p(\mathbb{R}^n)},$$

with $\epsilon \in (0,\gamma-1/2)$. This completes the proof of Theorem 2.1. □

3. Proof of Theorem 1.1

To prove Theorem 1.1, we will use some lemmas.

Lemma 3.1. *Let Ω be homogeneous of degree zero and belong to $L^1(S^{n-1})$, K_t^j be defined as in* (2.1)*. Then for $l \in \mathbb{N}$, $s \in (1, \infty]$, $j_0 \in \mathbb{Z}_-$ and $y \in \mathbb{R}^n$ with $|y| < 2^{j_0-4}$,*

$$\sum_{j>j_0}\sum_{k\in\mathbb{Z}} 2^{kn/s}\bigg(\int_{2^k<|x|\le 2^{k+1}} \big|K_t^j * \phi_{j-l}(x+y) - K_t^j * \phi_{j-l}(x)\big|^{s'}\mathrm{d}x\bigg)^{\frac{1}{s'}} \lesssim 2^{l(n+1)}2^{-j_0}|y|.$$

Proof. We follow the argument used in [25] (see also [7]), with suitable modification. Observe that $\operatorname{supp} K_t^j * \phi_{j-l} \subset \{x : 2^{j-2} \le |x| \le 2^{j+2}\}$, and

$$\|\phi_{j-l}(\cdot+y) - \phi_{j-l}(\cdot)\|_{L^{s'}(\mathbb{R}^n)} \lesssim 2^{(j-l)n/s}2^{j-l}|y|.$$

Thus, for all $k \in \mathbb{N}$,

$$\sum_{j\in\mathbb{Z}} 2^{k\frac{n}{s}}\bigg(\int_{2^k<|x|\le 2^{k+1}} |K_t^j * \phi_{j-l}(x+y) - K_t^l * \phi_{l-j}(x)|^{s'}\mathrm{d}x\bigg)^{\frac{1}{s'}}$$
$$\lesssim \sum_{j\in\mathbb{Z}:|j-k|\le 3} 2^{kn/s}\|K_t^j\|_{L^1(\mathbb{R}^n)}\|\phi_{j-l}(\cdot+y) - \phi_{j-l}(\cdot)\|_{L^{s'}(\mathbb{R}^n)} \lesssim 2^{l(n+1)s}\frac{|y|}{2^k}.$$

This, in turn, leads to that

$$\sum_{j>j_0}\sum_{k\in\mathbb{Z}} 2^{kn/s}\bigg(\int_{2^k<|x|\le 2^{k+1}} \big|K_t^j * \phi_{j-l}(x+y) - K_t^j * \phi_{j-l}(x)\big|^{s'}\mathrm{d}x\bigg)^{\frac{1}{s'}}$$
$$= \sum_{k>j_0-3}\sum_{j\in\mathbb{Z}} 2^{\frac{kn}{s}}\bigg(\int_{2^k<|x|\le 2^{k+1}} \big|K_\Omega^j * \phi_{j-l}(x+y) - K_\Omega^j * \phi_{j-l}(x)\big|^{s'}\mathrm{d}x\bigg)^{\frac{1}{s'}}$$
$$\lesssim 2^{l(n+1)}2^{-j_0}|y|,$$

and completes the proof of Lemma 3.1. □

For $t \in [1, 2]$ and $j \in \mathbb{Z}$, let K_t^j be defined as in (2.1), ϕ and ϕ_l (with $l \in \mathbb{Z}$) be as in Section 2. By Lemma 2.2 and the $L^p(\mathbb{R}^n)$ boundedness of $\mathcal{M}_\Omega$, we see that if Ω is homogeneous of degree zero, has mean value zero and $\Omega \in L\ln L(S^{n-1})$, then for $p \in (1, \infty)$, $\widetilde{\mathcal{M}}_{\Omega}^{l}$ is bounded on $L^p(\mathbb{R}^n)$ with bound independent of l. For $j_0 \in \mathbb{Z}$, define the operator $\widetilde{\mathcal{M}}_{\Omega}^{l,\,j_0}$ by

$$\widetilde{\mathcal{M}}_{\Omega}^{l,\,j_0} f(x) = \Big(\int_1^2 \sum_{j\in\mathbb{Z}: j>j_0} \Big|\int_{\mathbb{R}^n} K_t^j * \phi_{j-l}(x-y) f(y)\mathrm{d}y\Big|^2 \mathrm{d}t\Big)^{\frac{1}{2}},$$

and the commutator $\widetilde{\mathcal{M}}_{\Omega,\,b}^{l,\,j_0}$ by

$$\widetilde{\mathcal{M}}_{\Omega,\,b}^{l,\,j_0} f(x) = \Big(\int_1^2 \sum_{j\in\mathbb{Z}: j>j_0} \Big|\int_{\mathbb{R}^n} \big(b(x)-b(y)\big) K_t^j * \phi_{j-l}(x-y) f(y)\mathrm{d}y\Big|^2 \mathrm{d}t\Big)^{\frac{1}{2}},$$

with $b \in \mathrm{BMO}(\mathbb{R}^n)$.

Lemma 3.2. *Let Ω be homogeneous of degree zero and have mean value zero. Suppose that $\Omega \in L(\ln L)^{\gamma}(S^{n-1})$ for some $\gamma \in (1, \infty)$, then for $p \in (1, \infty)$, $l \in \mathbb{N}$ and $j_0 \in \mathbb{Z}$, $\widetilde{\mathcal{M}}_{\Omega}^{l,\,j_0}$ is bounded on $L^p(\mathbb{R}^n)$ with bound independent of l and j_0.*

PROOF. Let $p \in (1, \infty)$ and $l \in \mathbb{Z}$. By Theorem 2.1, it follows that $\widetilde{\mathcal{M}}_{\Omega}^{l}$ is bounded on $L^p(\mathbb{R}^n)$ with bounded independent of l. Observe that

$$\widetilde{\mathcal{M}}_{\Omega}^{l,\,j_0} f(x) \lesssim \widetilde{\mathcal{M}}_{\Omega}^{l} f(x) + \mathcal{N}_{\Omega}^{l,\,j_0} f(x),$$

with

$$\mathcal{N}_{\Omega}^{l,\,j_0} f(x) = \Big(\int_1^2 \sum_{j\le j_0} |F_j^l f(x,\,t)|^2\Big)^{\frac{1}{2}}.$$

Thus, it suffices to prove that $\mathcal{N}_{\Omega}^{l,\,j_0}$ is bounded on L^p with bound independent of j_0 and l. To this aim, we first note that if $\mathrm{supp}\, f \subset Q$ for a cube Q having side length 2^{j_0}, then $\mathrm{supp}\mathcal{N}_{\Omega}^{l,\,j_0} f \subset 20\sqrt{n}Q$. On the other hand, if $\{Q_k\}_k$ is a sequence of cubes with disjoint interiors and having side length 2^{j_0}, then the cubes $\{20\sqrt{n}Q_k\}$ have bounded overlaps. Thus, we may assume that $\mathrm{supp}\, f \subset Q$, with Q a cube centered at $h \in \mathbb{R}^n$ and having side length 2^{j_0}. For such a $f \in L^p(\mathbb{R}^n)$, we see that if $x \in 20\sqrt{n}Q$, then

$$\int_1^2 \sum_{j>j_0+20n} |F_j^l f(x,\,t)|^2 \mathrm{d}t = 0.$$

Therefore, for $x \in 20\sqrt{n}Q$,

$$\mathcal{N}_{\Omega}^{l,\,j_0} f(x) \le \widetilde{\mathcal{M}}_{\Omega}^{l} f(x) + \Big(\int_1^2 \sum_{j_0<j\le j_0+20n} |F_j^l f(x,\,t)|^2 \mathrm{d}t\Big)^{\frac{1}{2}} \lesssim \widetilde{\mathcal{M}}_{\Omega}^{l} f(x) + M_{\Omega} M f(x).$$

The desired $L^p(\mathbb{R}^n)$ boundedness of $\widetilde{\mathcal{M}}_{\Omega}^{l,\,j_0}$ then follows directly. □

Lemma 3.3. *Let Ω be homogeneous of degree zero and integrable on S^{n-1}. Then for $b \in C_0^\infty(\mathbb{R}^n)$, $l \in \mathbb{N}$, $j_0 \in \mathbb{Z}_-$ and $p \in (1, \infty)$,*

$$\big\|\widetilde{\mathcal{M}}^{l,\,j_0}_{\Omega,\,b} f - \widetilde{\mathcal{M}}^{l}_{\Omega,\,b} f\big\|_{L^p(\mathbb{R}^n)} \lesssim 2^{j_0}\|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. Let $b \in C_0^\infty(\mathbb{R}^n)$ with $\|\nabla b\|_{L^\infty(\mathbb{R}^n)} = 1$. By the fact that $\operatorname{supp} K^j_\Omega * \phi_{j-l} \subset \{x : 2^{j-2} \le |x| \le 2^{j+2}\}$, it is easy to verify that

$$\begin{aligned}
&\sum_{j\le j_0}\int_{\mathbb{R}^n}\big|K^j_t*\phi_{j-l}(x-y)\big||x-y||f(y)|\mathrm{d}y\\
&\quad\lesssim \sum_{j\le j_0}\sum_{k\in\mathbb{Z}}2^k\int_{2^k<|x-y|\le 2^{k+1}}\big|K^j_t*\phi_{j-l}(x-y)\big||f(y)|\mathrm{d}y\\
&\quad\lesssim \sum_{j\le j_0}\sum_{|k-j|\le 3}2^k\int_{2^k<|x-y|\le 2^{k+1}}\big|K^j_t*\phi_{j-l}(x-y)\big||f(y)|\mathrm{d}y \lesssim 2^{j_0}M_\Omega Mf(x).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\Big|\widetilde{\mathcal{M}}^{l,\,j_0}_{\Omega,\,b} f(x) - \widetilde{\mathcal{M}}^{l}_{\Omega,\,b} f(x)\Big|^2\\
&\quad\le \sum_{j<j_0}\int_1^2\bigg|\int_{\mathbb{R}^n}\big(b(x)-b(y)\big)K^j_t*\phi_{j-l}(x-y)f(y)\bigg|^2\mathrm{d}t\\
&\quad\lesssim \int_1^2\bigg(\sum_{j\le j_0}\int_{\mathbb{R}^n}|x-y|\big|K^j_t*\phi_{j-l}(x-y)f(y)|\mathrm{d}y\bigg)^2\mathrm{d}t \lesssim \{2^{j_0}M_\Omega Mf(x)\}^2.
\end{aligned}$$

The desired conclusion now follows immediately. □

Let $p,\, r \in [1, \infty)$, $q \in [1, \infty]$, $L^p(L^q([1, 2]), l^r; \mathbb{R}^n)$ be the space of sequences of functions defined by

$$L^p(L^q([1, 2]), l^r; \mathbb{R}^n) = \big\{\vec{f} = \{f_k\}_{k\in\mathbb{Z}} : \|\vec{f}\|_{L^p(L^q([1,2]),\,l^r;\,\mathbb{R}^n)} < \infty\big\},$$

with

$$\|\vec{f}\|_{L^p(L^q([1,2]),\,l^r;\,\mathbb{R}^n)} = \bigg\|\bigg(\int_1^2\Big(\sum_{k\in\mathbb{Z}}|f_k(x,\,t)|^r\Big)^{\frac{q}{r}}\mathrm{d}t\bigg)^{\frac{1}{q}}\bigg\|_{L^p(\mathbb{R}^n)}.$$

With usual addition and scalar multiplication, $L^p(L^q([1, 2]), l^r; \mathbb{R}^n)$ is a Banach space.

Lemma 3.4. *Let $p \in (1, \infty)$, $\mathcal{G} \subset L^p(L^2([1, 2]), l^2; \mathbb{R}^n)$. Suppose that $\mathcal{G}$ satisfies the following five conditions:*

(a) *$\mathcal{G}$ is bounded, that is, there exists a constant C such that for all $\vec{f} \in \mathcal{G}$, $\|\vec{f}\|_{L^p(L^2([1,2]),\, l^2;\, \mathbb{R}^n)} \le C$;*

(b) *for each fixed $\epsilon > 0$, there exists a constant $A > 0$, such that for all $\vec{f} = \{f_k\}_{k\in\mathbb{Z}} \in \mathcal{G}$,*

$$\Big\|\Big(\int_1^2 \sum_{k\in\mathbb{Z}} |f_k(\cdot,\, t)|^2 \mathrm{d}t\Big)^{\frac{1}{2}} \chi_{\{|\cdot|>A\}}(\cdot)\Big\|_{L^p(\mathbb{R}^n)} < \epsilon;$$

(c) *for each fixed $\epsilon > 0$ and $N \in \mathbb{N}$, there exists a constant $\varrho > 0$, such that for all $\vec{f} = \{f_k\}_{k\in\mathbb{Z}} \in \mathcal{G}$,*

$$\Big\|\sup_{|h|\le\varrho}\Big(\int_1^2 \sum_{|k|\le N} |f_k(x,\, t) - f_k(x+h,\, t)|^2 \mathrm{d}t\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^n)} < \epsilon;$$

(d) *for each fixed $\epsilon > 0$ and $N \in \mathbb{N}$, there exists a constant $\sigma \in (0,\, 1/2)$ such that for all $\vec{f} = \{f_k\}_{k\in\mathbb{Z}} \in \mathcal{G}$,*

$$\Big\|\sup_{|s|\le\sigma}\Big(\int_1^2 \sum_{|k|\le N} |f_k(\cdot,\, t+s) - f_k(\cdot,\, t)|^2 \mathrm{d}t\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^n)} < \epsilon,$$

(e) *for each fixed $D > 0$ and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $\{f_k\}_{k\in\mathbb{Z}} \in \mathcal{G}$,*

$$\Big\|\Big(\int_1^2 \sum_{|k|>N} |f_k(\cdot,\, t)|^2 \mathrm{d}t\Big)^{\frac{1}{2}} \chi_{B(0,\, D)}\Big\|_{L^p(\mathbb{R}^n)} < \epsilon.$$

Then $\mathcal{G}$ is a strongly pre-compact set in $L^p(L^2([1,\, 2]),\, l^2;\, \mathbb{R}^n)$.

Proof. We employ the argument used in the proof of [9, Theorem 5], with some refined modifications. We claim that for each fixed $\epsilon > 0$, there exists a $\delta = \delta_\epsilon > 0$, and a mapping Φ_ϵ on $L^p(L^2([1,\, 2]),\, l^2;\, \mathbb{R}^n)$, such that $\Phi_\epsilon(\mathcal{G}) = \{\Phi_\epsilon(\vec{f}) :\ \vec{f} \in \mathcal{G}\}$ is a strong pre-compact set in $L^p(L^2([1,\, 2]),\, l^2;\, \mathbb{R}^n)$, and for any $\vec{f},\, \vec{g} \in \mathcal{G}$,

$$\|\Phi_\epsilon(\vec{f}) - \Phi_\epsilon(\vec{g})\|_{L^p(L^2([1,2]),\, l^2;\, \mathbb{R}^n)} < \delta \Rightarrow \|\vec{f} - \vec{g}\|_{L^p(L^2([1,2]),\, l^2;\, \mathbb{R}^n)} < 9\epsilon.$$

If we can prove this, then by Lemma 6 in [9], we see that $\mathcal{G}$ is a strongly pre-compact set in $L^p(L^2[1,\, 2],\, l^2;\, \mathbb{R}^n)$.

Now let $\epsilon > 0$. We choose $A > 1$ large enough as in assumption (b), $N \in \mathbb{N}$ such that for all $\{f_k\}_{k\in\mathbb{Z}} \in \mathcal{G}$,

$$\left\|\Big(\int_1^2 \sum_{|k|>N} |f_k(\cdot,\, t)|^2 \mathrm{d}t\Big)^{\frac{1}{2}} \chi_{B(0,\,2A)}\right\|_{L^p(\mathbb{R}^n)} < \epsilon.$$

Let $\varrho \in (0,\, 1/2)$ be small enough as in assumption (c), and $\sigma \in (0,\, 1/2)$ be small enough such that (d) holds true. Let Q be the largest cube centered at the origin such that $2Q \subset B(0,\, \varrho)$, $Q_1,\, \dots,\, Q_J$ be J copies of Q such that they are non-overlapping, and $\overline{B(0,\, A)} \subset \overline{\cup_{j=1}^J Q_j} \subset B(0,\, 2A)$, $I_1,\, \dots,\, I_L \subset [1,\, 2]$ be non-overlapping intervals with the same length $|I|$, such that $|s-t| \le \sigma$ for all $s,\, t \in I_j$ $(j = 1,\, \dots,\, L)$ and $\cup_{j=1}^N I_j = [1,\, 2]$. Define the mapping Φ_ϵ on $L^p(L^2([1,\, 2]),\, l^2;\, \mathbb{R}^n)$ by

$$\Phi_\epsilon(\vec{f})(x,\, t) = \Big\{\dots, 0,\, \dots, 0,\, \sum_{i=1}^J\sum_{j=1}^L m_{Q_i\times I_j}(f_{-N})\chi_{Q_i\times I_j}(x,t),$$
$$\sum_{i=1}^J\sum_{j=1}^L m_{Q_i\times I_j}(f_{-N+1})\chi_{Q_i\times I_j}(x,t), \dots, \sum_{i=1}^J\sum_{j=1}^L m_{Q_i\times I_j}(f_N)\chi_{Q_i\times I_j}(x,t), 0, \dots, \Big\},$$

where, and in the following,

$$m_{Q_i\times I_j}(f_k) = \frac{1}{|Q_i|}\frac{1}{|I_j|}\int_{Q_i\times I_j} f_k(x,\, t)\mathrm{d}x\mathrm{d}t.$$

Note that

$$|m_{Q_i\times I_j}(f_k)| \le \Big(\frac{1}{|Q_i||I_j|}\int_{I_j}\int_{Q_i} |f_k(y,\, t)|^2 \,\mathrm{d}y\mathrm{d}t\Big)^{\frac{1}{2}}.$$

For $\vec{f} = \{f_k\}_{k\in\mathbb{Z}}$ and $p \in [2,\, \infty)$, we have that by the Hölder inequality,

$$\|\Phi_\epsilon(\vec{f})\|^p_{L^p(L^2([1,\,2]),l^2;\mathbb{R}^n)} = |Q|^{1-\frac{p}{2}}|I|^{1-\frac{p}{2}}\sum_{i=1}^J\sum_{j=1}^L\Big(\int_{I_j}\int_{Q_i}\sum_{k\in\mathbb{Z}}|f_k(y,t)|^2\mathrm{d}y\mathrm{d}t\Big)^{\frac{p}{2}}$$
$$\le \sum_{i=1}^J\sum_{j=1}^L\int_{I_j}\int_{Q_i}\Big(\sum_{k\in\mathbb{Z}}|f_k(y,\, t)|^2\Big)^{\frac{p}{2}}\mathrm{d}y\mathrm{d}t \le \|\vec{f}\|^p_{L^p(L^2([1,\,2]),\,l^2,\,\mathbb{R}^n)}.$$

On the other hand, we have that

$$\sup_{-N\le k\le N}\sup_{t\in[1,\,2]}\Big|\sum_{i=1}^J\sum_{j=1}^L m_{Q_i\times I_j}(f_k)\chi_{Q_i\times I_j}(x,t)\Big| \lesssim \sup_{k\in\mathbb{Z}}\sup_{t\in[1,\,2]}|f_k(x,\, t)|,$$

which implies that for $p_1 \in (1, \infty)$,

$$\|\Phi_\epsilon(\vec{f})\|_{L^{p_1}(L^\infty([1,2]),\, l^\infty;\, \mathbb{R}^n)} \lesssim \|\vec{f}\|_{L^{p_1}(L^\infty([1,2]),\, l^\infty;\, \mathbb{R}^n)}. \tag{3.1}$$

We also have that for $p_0 \in (1, \infty)$,

$$|m_{Q_i\times I_j}(f_k)| \le \Big(\frac{1}{|Q_i||I_j|}\int_{I_j}\int_{Q_i}|f_k(y,\, t)|^{p_0}\,\mathrm{d}y\mathrm{d}t\Big)^{\frac{1}{p_0}},$$

and so

$$\|\Phi_\epsilon(\vec{f})\|_{L^{p_0}(L^{p_0}([1,2]),\, l^{p_0};\, \mathbb{R}^n)} \lesssim \|\vec{f}\|_{L^{p_0}(L^{p_0}([1,2]),\, l^{p_0};\, \mathbb{R}^n)}. \tag{3.2}$$

By interpolation, we deduce from (3.1) and (3.2) that for any $p \in (1, 2)$,

$$\|\Phi_\epsilon(\vec{f})\|_{L^p(L^2([1,2]),\, l^2;\, \mathbb{R}^n)} \lesssim \|\vec{f}\|^p_{L^p(L^2([1,2]),\, l^2,\, \mathbb{R}^n)}.$$

Thus, $\Phi_\epsilon(\mathcal{G}) = \{\Phi_\epsilon(\vec{f}) : \vec{f} \in \mathcal{G}\}$ is a strongly pre-compact set in $L^p(L^2([1, 2]), l^2; \mathbb{R}^n)$. Denote $\mathcal{D} = \cup_{i=1}^J Q_i$. Write

$$\begin{aligned}
&\big\|\vec{f}\chi_{\mathcal{D}} - \Phi_\epsilon(\vec{f})\big\|_{L^p(L^2([1,2]),\, l^2;\, \mathbb{R}^n)}\\
&\quad\le \Big\|\Big(\int_1^2\sum_{|k|\le N}\big|f_k(\cdot,t)\chi_{\mathcal{D}} - \sum_{i=1}^J\sum_{j=1}^L m_{Q_i\times I_j}(f_k)\chi_{Q_i\times I_j}(\cdot,t)\big|^2\mathrm{d}t\Big)^{\frac12}\Big\|_{L^p(\mathbb{R}^n)}\\
&\qquad+\Big\|\Big(\int_1^2\sum_{|k|>N}\big|f_k(\cdot,\, t)\big|^2\Big)^{\frac12}\chi_{B(0,\,2A)}\Big\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

Let

$$E = \Big\|\sup_{|h|\le\varrho}\Big(\int_1^2\sum_{|k|\le N}|f_k(\cdot,\, t) - f_k(\cdot + h,\, t)|^2\mathrm{d}t\Big)^{\frac12}\Big\|^p_{L^p(\mathbb{R}^n)},$$

$$F = \Big\|\sup_{|s|\le\sigma}\Big(\int_1^2\sum_{|k|\le N}|f_k(\cdot,\, t) - f_k(\cdot,\, t+s)|^2\mathrm{d}t\Big)^{\frac12}\Big\|^p_{L^p(\mathbb{R}^n)}.$$

Noting that for $x \in Q_i$ with $1 \le i \le J$,

$$\begin{aligned}
&\Big\{\int_1^2\sum_{|k|\le N}\big|f_k(x,\, t)\chi_{\mathcal{D}} - \sum_{i=1}^J\sum_{j=1}^L m_{Q_i\times I_j}(f_k)\chi_{Q_i\times I_j}(x,t)\big|^2\mathrm{d}t\Big\}^{\frac12}\\
&\quad\lesssim |Q|^{-\frac12}|I|^{-\frac12}\Big\{\sum_{j=1}^L\int_{I_j}\int_{Q_i}\int_{I_j}\sum_{|k|\le N}\big|f_k(x,\, t) - f_k(y,\, s)\big|^2\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}t\Big\}^{\frac12}
\end{aligned}$$

$$\lesssim |Q|^{-\frac{1}{2}}\bigg\{\int_{2Q}\int_1^2\sum_{|k|\le N}|f_k(x,\,s)-f_k(x+h,\,s)|^2\mathrm{d}s\,\mathrm{d}h\bigg\}^{\frac{1}{2}}$$
$$+|I|^{-\frac{1}{2}}\bigg\{\sum_{j=1}^{L}\int_{I_j}\int_{I_j}\sum_{|k|\le N}|f_k(x,\,t)-f_k(x,\,s)|^2\mathrm{d}t\,\mathrm{d}s\bigg\}^{\frac{1}{2}},$$

we then get that

$$\sum_{i=1}^{J}\int_{Q_i}\bigg\{\int_1^2\Big(\sum_{|k|\le N}\Big|f_k(x,\,t)-\sum_{l=1}^{J}m_{Q_l}(f_k)\chi_{Q_l}(x)\Big|^2\Big)\mathrm{d}t\bigg\}^{\frac{p}{2}}\mathrm{d}x\lesssim E+F.$$

It then follows from the assumption (b) that for all $\vec{f}\in\mathcal{G}$,

$$\|\vec{f}-\Phi_\epsilon(\vec{f})\|_{L^p(L^2([1,\,2]),\,l^2;\,\mathbb{R}^n)}$$
$$\le\|\vec{f}\chi_{\mathcal{D}}-\Phi_\epsilon(\vec{f})\|_{L^p(L^2([1,\,2]),\,l^2;\,\mathbb{R}^n)}+\bigg\|\Big(\int_1^2\sum_{k\in\mathbb{Z}}|f_k(\cdot,\,t)|^2\mathrm{d}t\Big)^{\frac{1}{2}}\chi_{\{|\cdot|>A\}}(\cdot)\bigg\|_{L^p(\mathbb{R}^n)}<3\epsilon.$$

Note that

$$\begin{aligned}&\|\vec{f}-\vec{g}\|_{L^p(L^2([1,\,2]),\,l^2;\,\mathbb{R}^n)}\\&\quad\le\|\vec{f}-\Phi_\epsilon(\vec{f})\|_{L^p(L^2([1,\,2]),\,l^2;\,\mathbb{R}^n)}+\|\Phi_\epsilon(\vec{f})-\Phi_\epsilon(\vec{g})\|_{L^p(L^2([1,\,2]),\,l^2;\,\mathbb{R}^n)}\\&\qquad+\|\vec{g}-\Phi_\epsilon(\vec{g})\|_{L^p(L^2([1,\,2]),\,l^2;\,\mathbb{R}^n)}.\end{aligned}$$

Our claim then follows directly. This completes the proof of Lemma 3.4. □

For $b\in\mathrm{BMO}(\mathbb{R}^n)$, set

$$F_{j,\,b}^{l}f(x,\,t)=\int_{\mathbb{R}^n}\big(b(x)-b(y)\big)K_t^j*\phi_{j-l}(x-y)f(y)\,\mathrm{d}y.$$

Proof of Theorem 1.1. Let $j_0\in\mathbb{Z}_-$, $b\in C_0^\infty(\mathbb{R}^n)$ with $\mathrm{supp}\,b\subset B(0,\,R)$, $p\in(1,\,\infty)$ and $\delta\in(0,\,1)$. Without loss of generality, we may assume that $\|b\|_{L^\infty(\mathbb{R}^n)}+\|\nabla b\|_{L^\infty(\mathbb{R}^n)}=1$. We claim that

(i) for each fixed $\epsilon>0$, there exists a constant $A>0$ such that

$$\bigg\|\Big(\int_1^2\sum_{j\in\mathbb{Z}}|F_{j,\,b}^{l}f(x,\,t)|^2\mathrm{d}t\Big)^{\frac{1}{2}}\chi_{\{|\cdot|>A\}}(\cdot)\bigg\|_{L^p(\mathbb{R}^n)}<\epsilon\|f\|_{L^p(\mathbb{R}^n)};$$

(ii) for $s\in(1,\,\infty)$,

$$\Big(\int_1^2\sum_{j>j_0}|F^l_{j,\,b}f(x,\,t)-F^l_{j,\,b}f(x+h,\,t)|^2\mathrm{d}t\Big)^{\frac{1}{2}} \lesssim 2^{-j_0}|h|\Big(\widetilde{\mathcal{M}}^{l,\,j_0}_{\Omega}f(x)+2^{l(n+1)}M_sf(x)\Big);$$

(iii) for each $\epsilon>0$ and $N\in\mathbb{N}$, there exists a constant $\sigma\in(0,\,1/2)$ such that

$$\Big\|\sup_{|s|\le\sigma}\Big(\int_1^2\sum_{|j|\le N}|F^l_{j,\,b}f(x,\,s+t)-F^l_{j,\,b}f(x,\,t)|^2\mathrm{d}t\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^n)}<\epsilon\|f\|_{L^p(\mathbb{R}^n)};$$

(iv) for each fixed $D>0$ and $\epsilon>0$, there exists $N\in\mathbb{N}$ such that

$$\Big\|\Big(\int_1^2\sum_{j>N}|F^l_{j,\,b}f(\cdot,\,t)|^2\mathrm{d}t\Big)^{\frac{1}{2}}\chi_{B(0,\,D)}\Big\|_{L^p(\mathbb{R}^n)}<\epsilon\|f\|_{L^p(\mathbb{R}^n)}.$$

We now prove claim (i). Let $t\in[1,\,2]$. For each fixed $x\in\mathbb{R}^n$ with $|x|>4R$, observe that $\operatorname{supp}K^j_t*\phi_{j-l}\subset\{2^{j-2}\le|x|\le 2^{j+2}\}$, and $\int_{|z|<R}\big|K^j_t*\phi_{j-l}(x-z)\big|\mathrm{d}z\neq 0$ only if $2^j\approx|x|$. A trivial computation leads to that

$$\begin{aligned}&\int_{|z|<R}\big|K^j_t*\phi_{j-l}(x-z)\big|\mathrm{d}z\\ &\lesssim\Big(\int_{|z|<R}\big|K^j_t*\phi_{j-l}(x-z)\big|^2\mathrm{d}z\Big)^{\frac{1}{2}}R^{\frac{n}{2}}\lesssim\Big(\int_{\frac{|x|}{2}\le|z|<2|x|}\big|K^j_t*\phi_{j-l}(z)\big|^2\mathrm{d}z\Big)^{\frac{1}{2}}R^{\frac{n}{2}}\\ &\lesssim\|K^j_t\|_{L^1(S^{n-1})}\|\phi_{j-l}\|_{L^2(\mathbb{R}^n)}R^{\frac{n}{2}}\lesssim 2^{nl/2}|x|^{-\frac{n}{2}}R^{\frac{n}{2}}.\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}\Big(\int_{|y|<R}|K^j_t*\phi_{j-l}(x-y)||f(y)|^s\mathrm{d}y\Big)^{\frac{1}{s}}\\ &=\sum_{j\in\mathbb{Z}:\,2^j\approx|x|}\Big(\int_{|x|/2\le|y-x|\le 2|x|}|K^j_t*\phi_{j-l}(x-y)||f(y)|^s\mathrm{d}y\Big)^{\frac{1}{s}}\lesssim\Big(M_\Omega M(|f|^s)(x)\Big)^{\frac{1}{s}}.\end{aligned}$$

Another application of the Hölder inequality then yields

$$\sum_{j\in\mathbb{Z}}\big|F^l_{j,\,b}f(x,\,t)\big|^2\lesssim$$

$$\lesssim \sum_{j\in\mathbb{Z}}\Big(\int_{|y|<R}|K_t^j*\phi_{j-l}(x-y)||f(y)|^s\mathrm{d}y\Big)^{\frac{2}{s}}\times\Big(\int_{|y|<R}|K_t^j*\phi_{j-l}(x-y)|\mathrm{d}y\Big)^{\frac{2}{s'}}$$
$$\lesssim 2^{\frac{nl}{s'}}|x|^{-\frac{n}{s'}}R^{\frac{n}{s'}}\Big(M_\Omega M(|f|^s)(x)\Big)^{\frac{2}{s}}.$$

This, in turn implies our claim (i).

We turn our attention to claim (ii). Write

$$|F_{j,\,b}^l f(x,\,t)-F_{j,\,b}^l f(x+h,\,t)|\le|b(x)-b(x+h)||F_j^l f(x,\,t)|+\mathrm{J}_j^l f(x,\,t),$$

with

$$\mathrm{J}_j^l f(x,t)=\Big|\int_{\mathbb{R}^n}\big(K_t^j*\phi_{j-l}(x-y)-K_t^j*\phi_{j-l}(x+h-y)\big)\big(b(x+h)-b(y)\big)f(y)\mathrm{d}y\Big|.$$

It follows from Lemma 3.1 that

$$\Big(\sum_{j>j_0}|\mathrm{J}_j^l f(x,t)|^2\Big)^{\frac{1}{2}}\lesssim\sum_{j>j_0}\int_{\mathbb{R}^n}\big|K_t^j*\phi_{j-l}(x-y)-K_t^j*\phi_{j-l}(x+h-y)\big||f(y)|\mathrm{d}y$$
$$\lesssim 2^{l(n+1)}|h|2^{-j_0}M_s f(x).$$

Therefore,

$$\Big(\int_1^2\sum_{j>j_0}|F_{j,\,b}^l f(x,\,t)-F_{j,\,b}^l f(x+h,\,t)|^2\mathrm{d}t\Big)^{\frac{1}{2}}$$
$$\lesssim|h|\widetilde{\mathcal{M}}_\Omega^{l,\,j_0}f(x)+2^{l(n+1)}2^{-j_0}|h|M_s f(x).\quad(3.3)$$

The claim (ii) now follows from the (3.3) and Lemma 3.2.

We now verify claim (iii). For each fixed $\sigma\in(0,\,1/2)$ and $t\in[1,\,2]$, let

$$U_{t,\,\sigma}^j(z)=\frac{1}{2^j}\frac{|\Omega(z)|}{|z|^{n-1}}\chi_{\{2^j(t-\sigma)\le|z|\le 2^j t\}}+\frac{1}{2^j}\frac{|\Omega(z)|}{|z|^{n-1}}\chi_{\{2^{j+1}t\le|z|\le 2^{j+1}(t+\sigma)\}},$$

and

$$G_{l,\,t,\,\sigma}^j f(x)=\int_{\mathbb{R}^n}\big(U_{t,\,\sigma}^j*|\phi_{l-j}|\big)(x-y)|f(y)|\mathrm{d}y.$$

Note that

$$\|U_{t,\,\sigma}^j*|\phi_{l-j}|\|_{L^1(\mathbb{R}^n)}\lesssim\sigma.$$

By the Young inequality, it is obvious that for $p_1\in(1,\,\infty)$,

$$\Big\|\sup_{|j|\le N}\sup_{t\in[1,\,2]}|G_{l,\,t,\,\sigma}^j f|\Big\|_{L^{p_1}(\mathbb{R}^n)}\lesssim\sigma\|f\|_{L^{p_1}(\mathbb{R}^n)},\quad(3.4)$$

and for $p_0 \in (1, \infty)$,

$$\int_{\mathbb{R}^n} \int_1^2 \sum_{|j| \le N} \big|G^j_{l,\,t,\,\sigma} f(x)\big|^{p_0} \mathrm{d}t\mathrm{d}x \lesssim N\sigma^{p_0} \|f\|^{p_0}_{L^{p_0}(\mathbb{R}^n)}. \tag{3.5}$$

We get from (3.4) and (3.5) that for $p \in (1, 2)$,

$$\Big\| \Big(\int_1^2 \sum_{|j| \le N} \big|G^j_{l,\,t,\,\sigma} f(x)\big|^2 \mathrm{d}t \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{R}^n)} \lesssim N\sigma \|f\|_{L^p(\mathbb{R}^n)}. \tag{3.6}$$

On the other hand, for $p \in [2, \infty)$, we obtain from the Minkowski inequality and the Young inequality that

$$\begin{aligned}
&\Big\| \Big(\int_1^2 \sum_{|j| \le N} |G^j_{l,\,t,\,\sigma} f(x)|^2 \mathrm{d}t \Big)^{\frac{1}{2}} \Big\|^2_{L^p(\mathbb{R}^n)} \\
&\lesssim \Big\{ \int_{\mathbb{R}^n} \Big(\int_1^2 \Big(\sum_{|j| \le N} \int_{\mathbb{R}^n} \big(U^j_{l,\,t,\,\sigma} * |\phi_{l-j}|\big)(x-y)|f(y)|\mathrm{d}y \Big)^2 \mathrm{d}t \Big)^{\frac{p}{2}} \mathrm{d}x \Big\}^{\frac{2}{p}} \\
&\lesssim \int_1^2 \Big\{ \sum_{|j| \le N} \Big(\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \big(U^j_{l,\,t,\,\sigma} * |\phi_{l-j}|\big)(x-y)|f(y)|\mathrm{d}y \Big)^p \mathrm{d}x \Big)^{\frac{1}{p}} \Big\}^2 \mathrm{d}t \\
&\lesssim (2N\sigma)^2 \|f\|^2_{L^p(\mathbb{R}^n)}.
\end{aligned} \tag{3.7}$$

Since

$$\sup_{|s| \le \sigma} |F^l_{j,\,b} f(x,\,t) - F^l_{j,\,b} f(x,\,t+s)| \le G^j_{l,\,t,\,\sigma} f(x),$$

our claim (iii) now follows from (3.6) and (3.7) immediately if we choose $\sigma = \epsilon/(2N)$.

It remains to prove (iv). Let $D > 0$ and $N \in \mathbb{N}$ such that $2^{N-2} > D$. Then for $j > N$ and $x \in \mathbb{R}^n$ with $|x| \le D$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \big|K^j_t * \phi_{j-l}(x-y) f(y)\big| \mathrm{d}y = \int_{\mathbb{R}^n} \big|K^j_t * \phi_{j-l}(x-y) f(y)\big| \chi_{\{|y| \le 2^{j+3}\}}(y)\mathrm{d}y \\
&\lesssim \int_{|y| \le 2^{j+3}} |f(y)|\mathrm{d}y \|K^j_t\|_{L^1(\mathbb{R}^n)} \|\phi_{j-l}\|_{L^\infty(\mathbb{R}^n)} \lesssim 2^{nl} 2^{-\frac{nj}{p}} \|f\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

Therefore,

$$\Big\| \Big(\int_1^2 \sum_{j > N} |F^l_{j,\,b} f(\cdot,\,t)|^2 \mathrm{d}t \Big)^{\frac{1}{2}} \chi_{B(0,\,D)} \Big\|_{L^p(\mathbb{R}^n)} \lesssim 2^{nl} \Big(\frac{D}{2^N} \Big)^{\frac{n}{p}} \|f\|_{L^p(\mathbb{R}^n)}.$$

We can now conclude the proof of Theorem 1.1. Let $p \in (1, \infty)$. Our claims (i)–(iv), via Lemma 3.2 and Lemma 3.4, prove that for $b \in C_0^\infty(\mathbb{R}^n)$, $l \in \mathbb{N}$ and $j_0 \in \mathbb{Z}_-$, the operator $\mathcal{F}_{j_0}^l$ defined by

$$\mathcal{F}_{j_0}^l : f(x) \to \{\dots, 0, \dots, F_{j_0, b}^l f(x, t), F_{j_0+1, b}^l f(x, t), \dots\}$$

is compact, and completely continuous from $L^p(\mathbb{R}^n)$ to $L^p(L^2([1, 2]), l^2; \mathbb{R}^n)$. Thus, $\widetilde{\mathcal{M}}_{\Omega, b}^{l, j_0}$ is completely continuous on $L^p(\mathbb{R}^n)$. This, via Lemma 3.3 and Theorem 2.1, shows that for $b \in C_0^\infty(\mathbb{R}^n)$, $\widetilde{\mathcal{M}}_{\Omega, b}$ is completely continuous on $L^p(\mathbb{R}^n)$. Note that

$$\left|\mathcal{M}_{\Omega, b} f_k(x) - \mathcal{M}_{\Omega, b} f(x)\right| \lesssim \mathcal{M}_{\Omega, b}(f_k - f)(x) \lesssim \widetilde{\mathcal{M}}_{\Omega, b}(f_k - f)(x).$$

Thus, for $b \in C_0^\infty(\mathbb{R}^n)$, $\mathcal{M}_{\Omega, b}$ is completely continuous on $L^p(\mathbb{R}^n)$. Recalling that when $\Omega \in L(\ln L)^{\frac{3}{2}}(S^{n-1})$, $\mathcal{M}_{\Omega, b}$ is bounded on $L^p(\mathbb{R}^n)$ with bound $C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$ (see [5]), we finally obtain that for $b \in \mathrm{CMO}(\mathbb{R}^n)$, $\mathcal{M}_{\Omega, b}$ is completely continuous on $L^p(\mathbb{R}^n)$. □

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