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A universality theorem for the sequential behaviour of minimal F-automata

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Abstract. In a sequence of previous articles, we have dealt with the final behaviour of certain types of automata, including F-automata. In the present paper, we tackle the study of the sequential behaviour of F-automata. We construct a pair of adjoint functors (E^*, N^*) between the categories $F\mathbf{A}$ of reachable F-automata and the category $F\mathbf{B}^*$ of sequential behaviours of F-automata. Thus, we provide a Goguen-like universality theorem for the sequential behaviour of F-automata.

The universality result presented in this paper is inspired by the fundamental adjunction between the category of deterministic automata and the category of their behaviours given by Goguen in [4] to prove that minimal realization is universal. Minimal realization is right adjoint to behaviours, and so behaviours are left adjoint to minimal realization, as both are functors between categories of machines and behaviours. The existence of such an adjunction yields several structural results on minimal realization. An adjunction between regular sets and finite state acceptors follows as a corollary of this universality result.

The algebraic theory of automata is a theory which models the work of the automatic devices which surround us in everyday life. It assumes time takes discrete values ($t = 0, 1, 2, \dots$); several kinds of automata have been devised within this medium. The most prominent automata seem to be *Mealy* and *Moore* automata. Both of them are tuples $(S, I, O, \delta, m, s_0)$, where (S, I, δ, s_0) is a semi-automaton with the initial state s_0 and the transition function $\delta : S \times I \longrightarrow S$, extended uniquely to a function $\delta : S \times I^* \longrightarrow S$ with the properties $\delta(s, \varepsilon) = \varepsilon$ for all $s \in S$, and $\delta(s, w_1 w_2) = \delta(\delta(s, w_1), w_2))$ for all $s \in S, w_1, w_2 \in I^*$. The

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output function is $m : S \times I \longrightarrow O$ for Mealy automata, and $m : S \longrightarrow O$ for Moore automata. The intended meaning is that δ constructs the next state of the automaton, i.e., $s_{n+1} = \delta(s_n, i_n)$, while m acts instantly. The formula $\beta(w) = m(\delta(s_0, w))$ defines the final behaviour $\beta : I^* \longrightarrow O$ for Moore automata, while $\beta(wi) = m(\delta(s_0, w), i)$ defines the final behaviour $\beta : I^+ \longrightarrow O$ for Mealy automata.

In a sequence of papers, we have dealt with semiautomata and other generalizations of Mealy and Moore automata: F-automata [5], *behavioural automata* [1], [2], and M-automata [3]. In the following, we resume the concept of F-automata, which is a common generalization of Mealy and Moore automata.

Recall that if $F : \mathbf{Set} \longrightarrow \mathbf{Set}$ is a functor which preserves surjections, an F-automaton is a tuple $\mathcal{A} = (S, I, O, \delta, \mu, s_0)$, where the reduct $\Sigma\mathcal{A} = (S, I, \delta, s_0)$ is a semiautomaton, and $\mu : S \times FI \longrightarrow O$. The automaton is said to be *reachable* if $\Sigma\mathcal{A}$ is reachable, i.e., if every $s \in S$ can be written in the form $s = \delta(s_0, w)$ for some $w \in I^*$. The *final behaviour* β of $\mathcal{A}$ is the function $\beta : I^* \times FI \longrightarrow O$ with $\beta(w, j) = \mu(\delta(s_0, w), j)$ for all $w \in I^*, j \in FI$. Mealy automata are obtained by taking $F := \mathrm{Id}_{\mathbf{Set}}$, while Moore automata are obtained by considering the constant functor $FI = \{\bullet\}$, $Ff = 1_{\{\bullet\}}$.

The category $F\mathbf{Aut}$ of F-automata is defined by a multi-sorted universal algebra, the morphisms being the triples (a, b, c) of mappings of sorts S, I, O, respectively, for which certain diagrams are commutative. Let $F\mathbf{A}$ be the subcategory of $F\mathbf{Aut}$, which consists of reachable F-automata and of those morphisms (a, b, c) for which b is surjective. The objects of the category $F\mathbf{Beh}$ of *behaviours* are the functions of the form $f : X^* \times FX \longrightarrow Y$, while the morphisms are the pairs (b, c) of mappings of sorts X and Y, respectively, for which certain diagrams are commutative. Let $F\mathbf{B}$ be the subcategory of $F\mathbf{Beh}$, which consists of the same objects and of those morphisms (b, c) for which b is surjective. The *external behaviour functor* $E : F\mathbf{A} \longrightarrow F\mathbf{B}$ is defined by $EA = \beta$ and $E(a, b, c) = (b, c)$. The *Nerode* F-automaton associated with a function $f : X^* \times FX \longrightarrow Y$ is an F-automaton Nf which realizes the function f, and for which the set S of states has the smallest cardinality among all the F-automata which realize the same function f. The correspondence $f \mapsto Nf$ turns out to be the object function of a functor $N : F\mathbf{B} \longrightarrow F\mathbf{A}$, which is called the *Nerode functor*. The Goguen-like theorem proved in [5] states that (E, N) is a pair of adjoint functors between $F\mathbf{A}$ and $F\mathbf{B}$.

So far, we have discussed the final behaviour; now, we pass to the sequential behaviour. Recall that unlike the final behaviour which exhibits just the final output, the sequential behaviour displays the entire sequence of outputs leading to the final one. The formal definition of the sequential behaviour for F-automata is the following.

Definiton 1. The *sequential behaviour* of an F-automaton $\mathcal{A}=(S, I, O, \delta, \mu, s_0)$, with final behaviour $\beta = \mu \circ \delta^0$, is the function $\beta^* : I^* \times FI \longrightarrow O^*$ defined by

$$\beta^*(\varepsilon, j) = \beta(\varepsilon, j)[= \mu(s_0, j)], \tag{1}$$

$$\beta^*(i_1 \dots i_n, j) = \beta(i_1, j) \dots \beta(i_1 \dots i_n, j). \tag{2}$$

Note that

$$\beta^*(wi, j) = \beta^*(w, j)\beta(wi, j) \quad \forall\, w \in I^+. \tag{3}$$

This is not true for all $w \in I^*$, just because for $w := \varepsilon$ we should have $\beta^*(i,j) = \beta^*(\varepsilon, j)\beta(i,j)$, meaning that $\beta^*(\varepsilon, j) = \varepsilon$, and this condition is not assumed in our approach.

While every function $f : X^* \times FX \longrightarrow Y$ can be realized as the final behaviour of a certain F-automaton (cf. Nf), only certain functions $g : X^* \times FX \longrightarrow Y^*$ occur as sequential behaviours of certain F-automata. Since the final behaviour β takes values in O, it follows from (1) and (3) that if a function $g : X^* \times FX \longrightarrow Y^*$ is the sequential behaviour of an F-automaton $\mathcal{A} = (S, X, Y, \delta, \mu, s_0)$, then

$$g(\varepsilon, j) \in Y \quad \text{and} \quad \forall w \in X^+ \forall x \in X \forall j \in FX \exists y \in Y g(wx, j) = g(w, j)y. \tag{4}$$

We are going to prove that the necessary condition (4) is also sufficient for g to be the sequential behaviour of an F-automaton.

Let $F\mathbf{B}^*$ be the category having as its objects the functions g which satisfy (4), while the morphisms in $F\mathbf{B}^*(g, g')$ are the pairs $(b, c) : X \times Y \longrightarrow X' \times Y'$ such that the identity $c^* \circ g = g' \circ (b^* \times Fb)$ holds

$$\begin{array}{ccc} X^* \times FX & \xrightarrow{\quad g \quad} & Y^* \\ {\scriptstyle b^* \times Fb}\downarrow & & \downarrow{\scriptstyle c^*} \\ X'^* \times FX' & \xrightarrow{\quad g' \quad} & Y'^* \end{array}$$

and b is a surjection.

Recall that if $\mathcal{C}$ is a category, then the class of its objects is denoted by $|\mathcal{C}|$.

Proposition 1. *A bijection $\Phi : |F\mathbf{B}| \longrightarrow |F\mathbf{B}^*|$ together with its inverse $\Psi : |F\mathbf{B}^*| \longrightarrow |F\mathbf{B}|$ are established iff the following identities hold:*

$$f(\varepsilon, j) = g(\varepsilon, j), \tag{5.1}$$
$$f(x, j) = g(x, j), \tag{5.2}$$
$$g(wx, j) = g(w, j)f(wx, j) \quad \forall w \in X^+. \tag{5.3}$$

Proof. If $f \in |F\mathbf{B}|$, then $g \in |F\mathbf{B}^*|$, because property (4) holds by (5.1) and (5.3). If $g \in |F\mathbf{B}^*|$, then (4) and (5.1) imply $f(\varepsilon, j) \in Y$, while (4) and (5.3) imply $g(w, j)y = g(w, j)f(wx, j)$, hence $f(wx, j) = y \in Y$, and so $f \in |F\mathbf{B}|$. Thus, Φ and Ψ are functions which act as it was claimed, and they are inverse to each other. □

The next step is to lift the object functions Φ and Ψ to functors.

Lemma 1. *For every $b : X \longrightarrow X'$, $c : Y \longrightarrow Y'$ and $\Phi f = g$, we have*

$$(b, c) \in F\mathbf{B}(f, f') \Longleftrightarrow (b, c) \in F\mathbf{B}^*(g, g').$$

Proof. According to the definition of morphisms in $F\mathbf{B}$ and $F\mathbf{B}^*$, we must prove that

$$c \circ f = f' \circ (b^* \times Fb) \Longleftrightarrow c^* \circ g = g' \circ (b^* \times Fb), \text{ i.e.,}$$

$$\begin{array}{ccc} X^* \times FX & \xrightarrow{f} & Y \\ {\scriptstyle b^* \times Fb}\downarrow & & \downarrow{\scriptstyle c} \\ X'^* \times FX' & \xrightarrow{f'} & Y' \end{array} \qquad\qquad \begin{array}{ccc} X^* \times FX & \xrightarrow{g} & Y^* \\ {\scriptstyle b^* \times Fb}\downarrow & & \downarrow{\scriptstyle c^*} \\ X'^* \times FX' & \xrightarrow{g'} & Y'^* \end{array}$$

For $w := \varepsilon$ and $w := x$, the two diagrams are identical.

If the left diagram is commutative, we prove the commutativity of the right diagram by induction on the length of the variable w.

If the right diagram is commutative for w, then it is also commutative for wx:

$$\begin{aligned} g'(b^*(wx), Fb(z)) &= g'(b^*(w)b(x), Fb(z)) = g'(b^*(w), Fb(z))f'(b^*(w)b(x), Fb(z)) \\ &= c^*(g(w), z)f'(b^*(wx), Fb(z)) = c^*(g(w), z)c(f(wx, z)) \\ &= c^*(g * (w), z)c(f(wx, z)) = c^*((g(w), z)f(wx, z)) = c^*(g(wx, z)). \end{aligned}$$

Moreover, the commutativity of the right diagram implies the commutativity of the left one, because

$$\begin{aligned} &g'(b^*(w), Fb(z))f'(b^*(wx), Fb(z)) \\ &\quad = g'(b^*(wx), Fb(z)) = c^*(g(wx, z)) = c^*(g(w), z)f(wx, z)) \\ &\quad = c^*(g(w), z)c^*(f(wx, z)) = g'(b^*(w), Fb(z))c(f(wx, z)). \end{aligned}$$

By comparing the first and the last term of this sequence of equalities, we get $f'(b^*(wx), Fb(z)) = c^*(f(wx, z))$. □

Lemma 1 can be paraphrased by the implications $(b, c) \in F\mathbf{B}(f, f') \Longrightarrow (b, c) \in F\mathbf{B}^*(\Phi f, \Phi f')$, and $(b, c) \in F\mathbf{B}^*(g, g') \Longrightarrow (b, c) \in F\mathbf{B}(\Psi g, \Psi g')$. Therefore, the mappings Φ and Ψ can be extended to functors $\Phi : F\mathbf{B} \longrightarrow F\mathbf{B}^*$ and $\Psi : F\mathbf{B}^* \longrightarrow F\mathbf{B}$, by setting $\Phi(b, c) = (b, c)$ and $\Psi(b, c) = (b, c)$, respectively. Proposition 1 is thus strengthened as follows:

Proposition 2. *The functors Φ and its inverse Ψ establish an isomorphism between $F\mathbf{B}$ and $F\mathbf{B}^*$.*

Moreover, Φ and Ψ enable us to establish a Goguen-like theorem for the sequential behaviour.

Theorem 1. *The functors $E^* \stackrel{def}{=} \Phi \circ E : F\mathbf{A} \longrightarrow F\mathbf{B}^*$ and $N^* \stackrel{def}{=} N \circ \Psi : F\mathbf{B}^* \longrightarrow F\mathbf{A}$ are a pair of adjoint functors.*

Proof. By Proposition 2, since adjointness is clearly preserved by isomorphisms. □

Proposition 3. *$E^*\mathcal{A}$ is the sequential behaviour of $\mathcal{A}$, for every $\mathcal{A} \in |F\mathbf{A}|$.*

Proof. $E^*\mathcal{A} = \Phi E\mathcal{A} = \Phi\beta = \beta^*$, because the properties (1), (2) and (3) show that conditions (5) are satisfied for $f := \beta$ and $g := \beta^*$. □

Now, we are able to prove that every function $g \in |F\mathbf{B}^*|$ can be realized by an F-automaton, as we have claimed.

Proposition 4. *The F-automaton $N^*g = N(\Psi g)$ realizes the sequential behaviour g with the least number of states among the F-automata which realize g.*

Proof. The sequential behaviour of N^*g is $E^*N^*g = \Phi EN\Psi g = \Phi\Psi g = g$ by Proposition 3 and Theorem 1. If $\mathcal{A} \in |F\mathbf{A}|$ satisfies $E^*\mathcal{A} = g$, then from $\Phi E\mathcal{A} = g$ it follows that $E\mathcal{A} = \Psi g$. Therefore, $\mathcal{A}$ has at least as many states as $N(\Psi g)$. □

Conclusion

All these results show that Theorem 1 represents a Goguen-like universality theorem for the sequential behaviour of F-automata. It was proved that minimal realization is a right adjoint to behaviours, and so behaviours are left adjoint to minimal realization, while both are functors between categories of F-automata and sequential behaviours.

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