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Estimates of fractional integral operator with variable kernel

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Abstract. In this paper, we use interpolation and iterative methods to study the fractional integral operator $\mathcal{F}_{\Omega,\alpha}$ with variable kernel. We obtain the sharp size condition on Ω to ensure the (L^q, L^p) boundedness of $\mathcal{F}_{\Omega,\alpha}$ for $0 < \alpha < n$, $1 < p < \infty$. We also obtain some corresponding estimates of the rough bilinear fractional integral.

1. Introduction and main results

Let S^{n-1} be the unit sphere in Euclidean $\mathbb{R}^n$ $(n \geq 2)$, and $d\sigma$ be the area element on S^{n-1} induced by the Lebesgue measure on $\mathbb{R}^n$. A function $\Omega(x, z)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ is said to belong to $L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$, $r \geq 1$, if it satisfies the following conditions: for any x, $z \in \mathbb{R}^n$ and $\lambda \geq 0$,

$$\Omega(x, \lambda z) = \Omega(x, z), \tag{1.1}$$

$$\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})} = \sup_{x\in\mathbb{R}^n} \left(\int_{S^{n-1}} |\Omega(x, z')|^r d\sigma(z') \right)^{\frac{1}{r}} < \infty, \tag{1.2}$$

where $z' = \frac{z}{|z|}$, for any $z \in \mathbb{R}^n \setminus \{0\}$.

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If $\Omega(x,z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$ satisfies the mean zero property

$$\int_{S^{n-1}} \Omega(x,z')d\sigma(z') = 0 \quad \text{for all } x \in \mathbb{R}^n, \tag{1.3}$$

then the famous Calderón–Zygmund singular integral operator with variable kernel is defined on the space $\mathcal{S}(\mathbb{R}^n)$ of all Schwartz functions f by

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x,y)}{|y|^n} f(x-y)dy, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

For the operator T, the following result is well known.

Theorem A ([2], [3], [9])**.** *Let $n \geq 2$. If the function Ω satisfies conditions* (1.1), (1.3) *and $\Omega(x,z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$, then the following inequality holds:*

$$\|Tf\|_{L^p(\mathbb{R}^n)} \leq C_{r,p} \|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})} \|f\|_{L^p(\mathbb{R}^n)},$$

provided that

(1) $\frac{1}{r} < \frac{1}{p'}\frac{n}{n-1}$ if $1 < p \leq 2$ $\quad \left(p' = \frac{p}{p-1}\right)$;

(2) $\frac{1}{r} < \frac{1}{p}\frac{1}{n-1} + \frac{1}{p'}$ if $2 \leq p < \infty$.

In this paper, we will study the fractional integral operator

$$\mathcal{F}_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x,y)}{|y|^{n-\alpha}} f(x-y)dy,$$

where $0 < \alpha < n$, $f \in \mathcal{S}(\mathbb{R}^n)$ and Ω satisfies (1.1) and (1.2). In this case, the kernel of $\mathcal{F}_{\Omega,\alpha}$ has less singularity in a neighborhood of the origin than the kernel of singular integral operator T, and one does not need to assume the cancellation condition (1.3) on Ω in the definition of $\mathcal{F}_{\Omega,\alpha}$. On the other hand, when $\Omega = 1$, $\mathcal{F}_{\Omega,\alpha}$ is the Riesz potential

$$\Re_\alpha(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy,$$

which plays significant roles in analysis, partial differential equations, probability theory and in many other fields of mathematics, via the Hardy–Littlewood–Sobolev embedding theory.

In 1971, Muckenhoupt and Wheeden [13] studied the power-weighted (L^q, L^p) boundedness of $\mathcal{F}_{\Omega,\alpha} f$ for all $0 < \alpha < n$. In the unweighted case, their theorem can be stated as follows.

Theorem B ([13]). *Let $n \geq 2$. Suppose that $0 < \alpha < n$, $1 < q < n/\alpha$ and $1/p = 1/q - \alpha/n$. If $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$ for $r > q'$, then there exists a constant C independent of f and Ω, such that*

$$\|\mathcal{F}_{\Omega,\alpha} f\|_{L^p(\mathbb{R}^n)} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^q(\mathbb{R}^n)},$$

and there is no such C if $r < q'$.

As we see, Muckenhoupt and Wheeden pointed out that the inequality in Theorem B cannot be improved if $r < q'$ for all $0 < \alpha < n$ and all indices p, q satisfying $1 < q < n/\alpha$ and $1/p = 1/q - \alpha/n$. However, when $0 < \alpha < 1/2$, the condition in Theorem B in fact can be improved. To address this interesting phenomenon, Chen, Ding and Fan [4], [5], [10] published a series of papers to study the fractional integral operator $\mathcal{F}_{\Omega,\alpha}$, among other things (see also [1], [6], [8] for some related results). We list one result related to this paper in the following theorem.

Theorem C ([4]). *Let $n \geq 2$, $0 < \alpha < 1/2$ and $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$. If $r > 2\rho(n-1)/(n-2\alpha)$, where $\rho = (1/2 - \alpha/n)(1/p' - \alpha/n)$, then*

$$\|\mathcal{F}_{\Omega,\alpha} f\|_{L^p(\mathbb{R}^n)} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^q(\mathbb{R}^n)}$$

with $1/p = 1/q - \alpha/n$.

Note that $\rho = (1/2 - \alpha/n)(1/p' - \alpha/n)$, $1/p = 1/q - \alpha/n$, the size condition $r > 2\rho(n-1)/(n-2\alpha)$ in Theorem C is equivalent to

$$r > \frac{n-1}{n} q',$$

which is obviously better than the size condition $r > q'$ in Theorem B since S^{n-1} is compact.

We also can show that the size condition Ω in Theorem C is the sharp one.

Theorem 1. *If in the inequality (1.2) we take $r = \frac{n-1}{n} q'$, the transform of $\mathcal{F}_{\Omega,\alpha} f$ of an $f \in L^q$ $(\frac{n}{n-\alpha} < q < \frac{n}{\alpha})$ needs not be in L^p $(1 < p < 2)$.*

Proof. We will modify the proof for a similar problem on the singular integral (see page 223 in [2] by Calderón and Zygmund). Denote $K_{\Omega,\alpha}(x, y) = \frac{\Omega(x,y')}{|y|^{n-\alpha}}$. We have

$$\mathcal{F}_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} K_{\Omega,\alpha}(x, x-y) f(y) dy.$$

Take for $f(y)$ the function equal to 1 for $|y| \leq 1$ and equal to zero elsewhere. Then $f \in L^q(\mathbb{R}^n)$ for any $q \geq 1$. Let r be any positive number. We define a function $\Omega(x, y')$ on $\mathbb{R}^n \times S^{n-1}$ by assuming $\Omega(x, y') = 0$ for $|x| \leq 20$. When $|x| > 20$, denote the subset S_x of S^{n-1} by

$$S_x = \left\{ y' \in S^{n-1} : \left| y' - \frac{x}{|x|} \right| < \frac{10}{|x|} \right\},$$

and define $\Omega(x, y')$ as

(a) equal to $|x|^{(n-1)/r}$ if $y' \in S_x$;

(b) equal to zero if $y' \notin S_x$.

Let $A(S_x)$ denote the surface area of S_x. Since $A(S_x) \approx |x|^{-(n-1)}$ uniformly for all $|x| \geq 20$, by the definition, we have that

$$\sup_{x \in \mathbb{R}^n} \int_{S^{n-1}} |\Omega(x, y')|^r d\sigma(y) = \sup_{x \in \mathbb{R}^n} \int_{S_x} |x|^{(n-1)} d\sigma(y) < C.$$

This shows $\Omega(x, y') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$.

On the other hand, we notice that for sufficiently large $|x|$,

$$\left| (x-y)' - \frac{x}{|x|} \right| = \left| \frac{(x-y)}{|x-y|} - \frac{x}{|x|} \right| < \frac{6}{|x|}$$

uniformly for all $|y| \leq 1$. Hence, by the choice of f and Ω, we have that

$$\begin{aligned} |\mathcal{F}_{\Omega,\alpha} f(x)| &= \left| \int_{\mathbb{R}^n} K_{\Omega,\alpha}(x, x-y) f(y) dy \right| = \left| \int_{|y| \leq 1} \frac{\Omega(x, (x-y)')}{|x-y|^{n-\alpha}} dy \right| \\ &\approx |x|^{-n+\alpha} \left| \int_{|y| \leq 1} \Omega(x, (x-y)') dy \right| \approx \frac{C}{|x|^\eta} \quad \text{as } |x| \to \infty, \end{aligned}$$

where $\eta = n - \alpha - (n-1)/r$. Now, in order to ensure $\mathcal{F}_{\Omega,\alpha} f(x)$ to be in L^p, we must assume that $\eta > n/p$, which is equivalent to $r > \frac{n-1}{n} q'$. This completes the proof. □

Inspired by Theorem A, Theorem B and Theorem C, it naturally raises the following two questions.

Question 1: How to extend Theorem C to the case of $2 < p < \infty$?

Question 2: Does Theorem B hold at the endpoint $r = q'$?

Question 2 was solved in the case $1 < p \leq 2$ (see Theorem D). Thus, similar to Question 1, we need to address this question in the case $2 < p < \infty$.

Now, we state our first main result about the fractional integral operator. The following theorem solves Question 1.

Theorem 2. *For $0 < \alpha < 1/2$, $n \geq 2$, let $1 < q < n/\alpha$, $1/p = 1/q - \alpha/n$ and $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$. If $2 \leq p < \infty$ and $\frac{1}{r} < \frac{1}{q'} + \frac{n-2\alpha}{pn(n-1)}$, then*

$$\|\mathcal{F}_{\Omega,\alpha} f\|_{L^p(\mathbb{R}^n)} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^q(\mathbb{R}^n)}.$$

It is easy to check that the theorem improves the result in Theorem B in the case $0 < \alpha < 1/2$. We will prove Theorem 2 by using an interpolation between the $L^{\frac{2n}{n+2\alpha}} \to L^2$ estimate and the inequality in Theorem B.

The reader might notice that both Theorem 1 and Theorem C have a restriction $0 < \alpha < \frac{1}{2}$. One naturally expects to remove this restriction. We note that Chen, Ding, Fan in [4] improved Theorem C in the case $1 < p \leq 2$.

Theorem D ([4])**.** *Let $n \geq 2$. Suppose that $0 < \alpha < n$, $1 < q < n/\alpha$, $1/p = 1/q - \alpha/n$, $1 < p \leq 2$ and $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$. If $r \geq q'$, then*

$$\|\mathcal{F}_{\Omega,\alpha} f\|_{L^p(\mathbb{R}^n)} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^q(\mathbb{R}^n)}.$$

As an application of Theorem 2, in the following theorem we extend Theorem D to the full range $1 < p < \infty$, which solves Question 2.

Theorem 3. *Let $n \geq 2$, $1 < q < n/\alpha$, $1/p = 1/q - \alpha/n$, $0 < \alpha < n$ and $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$. If $r \geq q'$, then*

$$\|\mathcal{F}_{\Omega,\alpha} f\|_{L^p(\mathbb{R}^n)} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^q(\mathbb{R}^n)}.$$

We summarize the above results in Theorems 1–3 and Theorems B–D in the following

Theorem 4. *Let $n \geq 2$. Suppose that $1 < q < n/\alpha$ and $1/p = 1/q - \alpha/n$. We have the following conclusions.*

(1) *For $0 < \alpha < 1/2$ and $1 < p \leq 2$, there exists a constant $C > 0$ such that for all $f \in L^q(\mathbb{R}^n)$ and all $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$*

$$\|\mathcal{F}_{\Omega,\alpha} f\|_{L^p(\mathbb{R}^n)} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^q(\mathbb{R}^n)}$$

if and only if

$$r > \frac{n-1}{n} q'.$$

(2) *For $0 < \alpha < 1/2$ and $2 \leq p < \infty$, there exists a constant $C > 0$ such that for all $f \in L^q(\mathbb{R}^n)$ and all $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$,*

$$\|\mathcal{F}_{\Omega,\alpha} f\|_{L^p(\mathbb{R}^n)} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^q(\mathbb{R}^n)}$$

if

$$\frac{1}{r} < \frac{1}{q'} + \frac{n-2\alpha}{pn(n-1)}.$$

(3) *For $1/2 < \alpha \leq n$ and $1 < p < \infty$, there exists a constant $C > 0$ such that for all $f \in L^q(\mathbb{R}^n)$ and all $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$,*

$$\|\mathcal{F}_{\Omega,\alpha} f\|_{L^p(\mathbb{R}^n)} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^q(\mathbb{R}^n)}$$

if and only if $r \geq q'$.

Remark 1. If we consider another fractional integral

$$\mathcal{L}_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x, y)}{|y|^{n-\alpha}} f(x + y)dy,$$

all conclusions in Theorem 4 still hold if we replace $\mathcal{F}_{\Omega,\alpha}$ by $\mathcal{L}_{\Omega,\alpha}$.

We notice that some authors considered fractional Marcinkiewicz integrals with variable kernels. For instance, in [12] (see also [11]), the authors study the fractional Marcinkiewicz integrals with variable kernels in the form of

$$\mu_{\Omega,\alpha}(f)(x) = \left(\int_0^\infty \left|\int_{|x-y|<t} \frac{\Omega(x, x-y)}{|x-y|^{n-1}} f(y)dy\right|^2 \frac{dt}{t^{3-2\alpha}}\right)^{\frac{1}{2}}, \quad 0 < \alpha \leq 1.$$

Also, some people study the fractional integral of Marcinkiewicz type

$$M_{\Omega,\alpha}(f)(x) = \left(\int_0^\infty \left|\int_{|x-y|<t} \frac{\Omega(x, x-y)}{|x-y|^{n-1-\alpha}} f(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}}, \quad 0 < \alpha \leq 1.$$

For $1 < p \leq 2$, Lin *et al.* [12] gave some estimates on $\mu_{\Omega,\alpha}$.

Theorem E ([12])**.** *Let $n \geq 2$ and $0 < \alpha < \frac{1}{2}$. If Ω satisfies* (1.1), (1.2) *and* (1.3) *for $r = 2$, then there exists a constant C independent of f such that*

$$\|\mu_{\Omega,\alpha}(f)\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^{\frac{2n}{n+2\alpha}}(\mathbb{R}^n)}.$$

In addition, for $\frac{n}{n-\alpha} < p < 2$ and $1/p = 1/q - \alpha/n$, if $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^2(S^{n-1})$ and satisfies $L^{2,\alpha}$ Dini conditions, then there exists a constant C independent of f such that

$$\|\mu_{\Omega,\alpha}(f)\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^q(\mathbb{R}^n)}.$$

It is easy to see that both integrals $\mu_{\Omega,\alpha}(f)(x)$ and $M_{\Omega,\alpha}(f)(x)$ are pointwise dominated by the fractional integral $\mathcal{F}_{\Omega,\alpha}f$. In the fractional case, we may assume that both Ω and f are non-negative. Thus, by the Minkowski integral inequality, we obtain

$$\begin{aligned}|\mu_{\Omega,\alpha}(f)(x)| &\leq \int_{\mathbb{R}^n} \frac{\Omega(x,x-y)}{|x-y|^{n-1}} f(y) \left(\int_0^\infty \chi_{|x-y|<t}(t) \frac{dt}{t^{3-2\alpha}}\right)^{\frac{1}{2}} dy \\ &= \int_{\mathbb{R}^n} \frac{\Omega(x,x-y)}{|x-y|^{n-1}} f(y) \left(\int_{|x-y|}^\infty \frac{dt}{t^{3-2\alpha}}\right)^{\frac{1}{2}} dy \\ &\simeq \int_{\mathbb{R}^n} \frac{\Omega(x,x-y)}{|x-y|^{n-\alpha}} f(y) dy = \mathcal{F}_{\Omega,\alpha}f(x).\end{aligned}$$

Similarly,

$$\begin{aligned}|M_{\Omega,\alpha}(f)(x)| &\leq \int_{\mathbb{R}^n} \frac{\Omega(x,x-y)}{|x-y|^{n-1-\alpha}} f(y) \left(\int_0^\infty \chi_{\{|x-y|<t\}}(t) \frac{dt}{t^3}\right)^{\frac{1}{2}} dy \\ &= \int_{\mathbb{R}^n} \frac{\Omega(x,x-y)}{|x-y|^{n-1-\alpha}} f(y) \left(\int_{|x-y|}^\infty \frac{dt}{t^3}\right)^{\frac{1}{2}} dy \\ &\simeq \int_{\mathbb{R}^n} \frac{\Omega(x,x-y)}{|x-y|^{n-\alpha}} f(y) dy = \mathcal{F}_{\Omega,\alpha}f(x).\end{aligned}$$

As an application of our results, we will study the rough bilinear fractional integral

$$\widetilde{B}_{\Omega,\alpha}(f,g)(x) = \int_{\mathbb{R}^n} f(x+y)g(x-y) \frac{\Omega(x,y')}{|y|^{n-\alpha}} dy,$$

where $0 < \alpha < n$. The following result is an easy consequence of our results together with an application of Hölder's inequality, which is an improvement of Proposition 1 in [7].

Theorem 5. *Let $n \geq 2$. Suppose that $p > \frac{n-\alpha}{n}, 1 < p_1, p_2 < \infty$, $1/p = 1/p_1 + 1/p_2 - \alpha/n$ and $1/\sigma = 1/p_1 + 1/p_2$. We have the following conclusions.*

(1) *For $0 < \alpha < 1/2$ and $1 < p \leq 2$, there exists a constant $C > 0$ such that for all $f \in L^{p_1}(\mathbb{R}^n)$, $g \in L^{p_2}(\mathbb{R}^n)$ and all $\Omega(x,z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$,*

$$\|\widetilde{B}_{\Omega,\alpha}(f,g)\|_{L^p(\mathbb{R}^n)} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}$$

if

$$r > \frac{n-1}{n}\sigma'.$$

(2) *For* $0 < \alpha < 1/2$ *and* $2 \le p < \infty$, *there exists a constant* $C > 0$ *such that for all* $f \in L^{p_1}(\mathbb{R}^n)$, $g \in L^{p_2}(\mathbb{R}^n)$ *and all* $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$,

$$\|\widetilde{B}_{\Omega,\alpha}(f,g)\|_{L^p(\mathbb{R}^n)} \le C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}$$

if

$$\frac{1}{r} < \frac{1}{\sigma'} + \frac{n-2\alpha}{pn(n-1)}.$$

(3) *For* $1/2 < \alpha \le n$ *and* $1 < p < \infty$, *there exists a constant* $C > 0$ *such that for all* $f \in L^{p_1}(\mathbb{R}^n)$, $g \in L^{p_2}(\mathbb{R}^n)$ *and all* $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$,

$$\|\widetilde{B}_{\Omega,\alpha}(f,g)\|_{L^p(\mathbb{R}^n)} \le C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}$$

if $r \ge \sigma'$.

This paper is organized as follows. In Section 2, we prove Theorem 2. Theorem 3 and Theorem 5 will be proved in Section 3.

Throughout the paper, the letter C always denotes a positive constant that may vary at each occurrence, but is independent of all essential variables.

2. Proof of Theorem 2

We invoke the interpolation methods used in [3]. Suppose now $2 \le p < \infty$, consider the kernel

$$\Omega(x, z', \xi) = |\Omega(x, z')|^{\ell_1(\xi)} \operatorname{sgn} \Omega(x, z'),$$

where ξ is a complex parameter. For $f \in C_0^\infty$, consider also the function $f(x,\xi)$ and $g(x,\xi)$,

$$f(x,\xi) = |f(x)|^{\ell_2(\xi)} \operatorname{sgn} f(x), \quad g(x,\xi) = |g(x)|^{\ell_3(\xi)} \operatorname{sgn} g(x),$$

where $g(x)$ is a simple function, and $\ell_i(\xi) = a_i\xi + b_i$ $(i = 1, 2, 3)$ are linear functions whose coefficients will be determined later.

We set

$$G(\xi) = \int_{x\in\mathbb{R}^n} \int_{y\in\mathbb{R}^n} \frac{\Omega(x, x-y, \xi)}{|x-y|^{n-\alpha}} f(y,\xi)dy\ g(x,\xi)dx.$$

Consider the point $\left(\frac{1}{p}, \frac{1}{r}\right)$ in the square

$$(Q) \quad 0 \le s \le 1, \quad 0 \le t \le 1, \quad \text{where } \frac{1}{r} < \frac{1}{q'} + \frac{n-2\alpha}{pn(n-1)}.$$

The point $\left(\frac{1}{p}, \frac{1}{r}\right)$ lies below the segment joining $\left(\frac{1}{2}, \frac{n-2\alpha}{2(n-1)}\right)$ and $\left(0, \frac{n-\alpha}{n}\right)$. We may assume $\frac{1}{r} > \frac{1}{q'} = 1 - \frac{\alpha}{n} - \frac{1}{p}$, therefore, $\left(\frac{1}{p}, \frac{1}{r}\right)$ lies on the segment from point $\left(\frac{1}{p_1}, \frac{1}{r_1}\right)$ to point $\left(\frac{1}{2}, \frac{1}{r_0}\right)$, where $\frac{1}{r_1} < 1 - \frac{\alpha}{n} - \frac{1}{p_1}$ and $\frac{1}{r_0} < \frac{n-2\alpha}{2(n-1)}$.

There is an s, $0 < s < 1$, such that

$$\frac{1}{r} = \frac{1-s}{r_0} + \frac{s}{r_1}, \quad \frac{1}{p} = \frac{1-s}{2} + \frac{s}{p_1}, \quad \frac{1}{q} = \frac{1-s}{q_0} + \frac{s}{q_1}, \tag{2.1}$$

where $q_0 = \frac{2n}{n+2\alpha}$, $\frac{1}{q_1} = \frac{1}{p_1} + \frac{\alpha}{n}$.

Let

$$\lambda_1(\xi) = \frac{1-\xi}{r_0} + \frac{\xi}{r_1}, \quad \lambda_2(\xi) = \frac{1-\xi}{2} + \frac{\xi}{p_1}, \quad \lambda_3(\xi) = \frac{1-\xi}{q_0} + \frac{\xi}{q_1},$$

and define functions ℓ_1, ℓ_2, ℓ_3 by

$$\ell_1(\xi) = r\lambda_1(\xi), \quad \ell_2(\xi) = q\lambda_3(\xi), \quad \ell_3(\xi) = \frac{p}{p-1}[1 - \lambda_2(\xi)].$$

Then, for $\xi = s$ we have $\ell_1(s) = \ell_2(s) = \ell_3(s) = 1$.

For $\operatorname{Re} \xi = 0$,

$$\operatorname{Re} \ell_1(\xi) = \frac{r}{r_0}, \quad \operatorname{Re} \ell_2(\xi) = \frac{q}{q_0}, \quad \operatorname{Re} \ell_3(\xi) = \frac{p'}{2}.$$

Using Hölder's inequality and the L^2 boundedness of $\mathcal{F}_{\Omega,\alpha}$ ($p = 2$ in Theorem C), we obtain

$$\begin{aligned}
|G(\xi)| &\le \|g(\cdot,\xi)\|_{L^2(\mathbb{R}^n)} \left(\int_{x\in\mathbb{R}^n} \left| \int_{y\in\mathbb{R}^n} \frac{\Omega(x, x-y, \xi)}{|x-y|^{n-\alpha}} f(y,\xi) dy \right|^2 dx \right)^{\frac{1}{2}} \\
&\le C \left\| |g|^{\frac{p'}{2}} \right\|_{L^2(\mathbb{R}^n)} \left\| |f|^{\frac{q}{q_0}} \right\|_{L^{q_0}(\mathbb{R}^n)} \|\Omega(\cdot,\cdot,\xi)\|_{L^\infty(\mathbb{R}^n)\times L^{r_0}(S^{n-1})} \\
&\le C \|g\|_{L^{p'}(\mathbb{R}^n)}^{\frac{p'}{2}} \|f\|_{L^q(\mathbb{R}^n)}^{\frac{q}{q_0}} \sup_x \left(\int_{S^{n-1}} |\Omega(x,z')|^r d\sigma(z') \right)^{\frac{1}{r_0}}.
\end{aligned}$$

For $\operatorname{Re} \xi = 1$,

$$\operatorname{Re} \ell_1(\xi) = \frac{r}{r_1}, \quad \operatorname{Re} \ell_2(\xi) = \frac{q}{q_1}, \quad \operatorname{Re} \ell_3(\xi) = \frac{p'}{p_1'}.$$

For $0<\alpha<\frac{1}{2}$, by Hölder's inequality and Theorem B, we have

$$\begin{aligned}|G(\xi)| &\le \|g(\cdot,\xi)\|_{L^{p_1'}(\mathbb{R}^n)}\left(\int_{x\in\mathbb{R}^n}\left|\int_{y\in\mathbb{R}^n}\frac{\Omega(x,x-y,\xi)}{|x-y|^{n-\alpha}}f(y,\xi)dy\right|^{p_1}dx\right)^{\frac{1}{p_1}}\\ &\le C\left\||g|^{\frac{p'}{p_1'}}\right\|_{L^{p_1'}(\mathbb{R}^n)}\left\||f|^{\frac{q}{q_1}}\right\|_{L^{q_1}(\mathbb{R}^n)}\|\Omega(\cdot,\cdot,\xi)\|_{L^\infty(\mathbb{R}^n)\times L^{r_1}(S^{n-1})}\\ &\le C\|g\|^{\frac{p'}{p_1'}}_{L^{p'}(\mathbb{R}^n)}\|f\|^{\frac{q}{q_1}}_{L^q(\mathbb{R}^n)}\sup_x\left(\int_{S^{n-1}}|\Omega(x,z')|^r d\sigma(z')\right)^{\frac{1}{r_1}}.\end{aligned}$$

By the three-line theorem, we get

$$|G(s)|\le C\|g\|^{p'\left(\frac{1-s}{2}+\frac{s}{p_1'}\right)}_{L^{p'}(\mathbb{R}^n)}\|f\|^{q\left(\frac{1-s}{q_0}+\frac{s}{q_1}\right)}_{L^q(\mathbb{R}^n)}\sup_x\left\{\left(\int_{S^{n-1}}|\Omega(x,z')|^r d\sigma(z')\right)\right\}^{\frac{1-s}{r_0}+\frac{s}{r_1}}.$$

According to (2.1), the exponent of both $\|g\|_{L^{p'}(\mathbb{R}^n)}$ and $\|f\|_{L^q(\mathbb{R}^n)}$ is 1, and the exponent of $\sup_x\{\cdots\}$ is $\frac{1}{r}$.

Now, recalling that ℓ_1, ℓ_2, ℓ_3 are all equal to 1 at $\xi=s$, we have

$$\|\mathcal{F}_{\Omega,\alpha}f\|_{L^p(\mathbb{R}^n)}\le C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^q(\mathbb{R}^n)}\text{ with }\frac{1}{r}<\frac{1}{q'}+\frac{n-2\alpha}{pn(n-1)}.$$

Theorem 2 is proved. □

3. Proof of Theorem 3 and Theorem 5

3.1. Proof of Theorem 3. By Theorem D, it suffices to show the case $2<p<\infty$. Without loss of generality, we may assume that both Ω and f are non-negative. Since we have obtained a better result when $0<\alpha<1/2$ in Theorem 2, our strategy is to use a two-step iteration to extend α to the full range $(0,n)$. The first step is to use the result of Theorem 2 to obtain the boundedness of $\mathcal{F}_{\Omega,\alpha}$ under the condition $r=q'$ when $1/2\le\alpha\le n/2$. Then continue this process to obtain the theorem for $\alpha\in(n/2,n)$. We now begin our proof of step 1 by fixing an $\varepsilon_0\in[\sqrt{2}-1,\frac{1}{2})$ and letting $p_0=(2+\varepsilon_0)n$. For this choice of p_0, it is easy to see that, for any $\frac{1}{2}\le\alpha\le\frac{n}{2}$, we have $\alpha-\varepsilon_0\le n\left(\frac{1}{2}-\frac{1}{p}\right)$ if $p\ge p_0$. Let δ be a number to be chosen later. We write

$$\mathcal{F}_{\Omega,\alpha}f(x)\approx\int_{\mathbb{R}^n}\frac{\Omega(x,y)}{|y|^{n-\alpha}}f(x-y)dy$$

$$= \int_{|y|\le\delta} \frac{\Omega(x,y)}{|y|^{n-\alpha}} f(x-y)dy + \int_{|y|>\delta} \frac{\Omega(x,y)}{|y|^{n-\alpha}} f(x-y)dy := H_1 + H_2.$$

To estimate H_1, it is easy to see that

$$\begin{aligned} H_1 &= \int_{|y|\le\delta} \frac{|y|^{\alpha-\varepsilon_0}\,\Omega(x,y)}{|y|^{n-\varepsilon_0}} f(x-y)dy \\ &\le \delta^{\alpha-\varepsilon_0} \int_{\mathbb{R}^n} \frac{\Omega(x,y)}{|y|^{n-\varepsilon_0}} f(x-y)dy \le \delta^{\alpha-\varepsilon_0} \mathcal{F}_{\Omega,\varepsilon_0} f(x). \end{aligned}$$

For H_2, using Hölder's inequality and noticing that the condition $1/p = 1/q - \alpha/n$ implies $(\alpha-n)q' + n < 0$, we obtain

$$\begin{aligned} H_2 &\le \left(\int_{|y|>\delta} \left(\frac{\Omega(x,y)}{|y|^{n-\alpha}} \right)^{q'} dy \right)^{\frac{1}{q'}} \|f\|_{L^q(\mathbb{R}^n)} \\ &= \left(\int_\delta^\infty \int_{S^{n-1}} \Omega(x,y')^{q'} r^{(\alpha-n)q'} r^{n-1} dy' dr \right)^{\frac{1}{q'}} \|f\|_{L^q(\mathbb{R}^n)} \\ &\le \|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^{q'}(S^{n-1})} \left(\int_\delta^\infty r^{(\alpha-n)q'+n-1} dr \right)^{\frac{1}{q'}} \|f\|_{L^q(\mathbb{R}^n)} \\ &\le \|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})} \delta^{\alpha-\frac{n}{q}} \|f\|_{L^q(\mathbb{R}^n)}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{F}_{\Omega,\alpha} f(x) &\le \delta^{\alpha-\varepsilon_0} \mathcal{F}_{\Omega,\varepsilon_0} f(x) + \delta^{\alpha-\frac{n}{q}} \|f\|_{L^q(\mathbb{R}^n)} \|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})} \\ &= \delta^{\alpha-\varepsilon_0} \big(\mathcal{F}_{\Omega,\varepsilon_0} f(x) + \delta^{\varepsilon_0-\frac{n}{q}} \|f\|_{L^q(\mathbb{R}^n)} \|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})} \big). \end{aligned}$$

Now we take

$$\delta = \left(\frac{\mathcal{F}_{\Omega,\varepsilon_0} f(x)}{\|f\|_{L^q(\mathbb{R}^n)} \|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}} \right)^{\frac{1}{\varepsilon_0-\frac{n}{q}}},$$

and denote $\kappa_0 = \frac{\alpha-\varepsilon_0}{\varepsilon_0-\frac{n}{q}}$. It follows that

$$\mathcal{F}_{\Omega,\alpha} f(x) \le (\mathcal{F}_{\Omega,\varepsilon_0} f(x))^{1+\kappa_0} \left(\|f\|_{L^q(\mathbb{R}^n)} \|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})} \right)^{-\kappa_0}. \tag{3.1}$$

In order to obtain the L^p boundedness of $\mathcal{F}_{\Omega,\alpha}$, we should make some estimates on $\mathcal{F}_{\Omega,\varepsilon_0}$. For $p \ge p_0 = (2+\varepsilon_0)n$, it is not difficult to see that

$$\left\| (\mathcal{F}_{\Omega,\varepsilon_0} f)^{1+\kappa_0} \right\|_{L^p(\mathbb{R}^n)} = \|\mathcal{F}_{\Omega,\varepsilon_0} f\|^{1+\kappa_0}_{L^{p(1+\kappa_0)}(\mathbb{R}^n)}.$$

Note that $p(1+\kappa_0) \geq 2$, and

$$\frac{1}{\tilde{q}'} < \frac{1}{\tilde{q}'} + \frac{n-2\varepsilon_0}{p\,(1+\kappa_0)n(n-1)}$$

for any $\tilde{q} > 1$. Using Theorem 2, we obtain

$$\begin{aligned}\left\|(\mathcal{F}_{\Omega,\varepsilon_0} f)^{1+\kappa_0}\right\|_{L^p(\mathbb{R}^n)} &= \|\mathcal{F}_{\Omega,\varepsilon_0} f\|^{1+\kappa_0}_{L^{p(1+\kappa_0)}(\mathbb{R}^n)} \\ &\leq C\|\Omega\|^{1+\kappa_0}_{L^\infty(\mathbb{R}^n)\times L^{\tilde{q}'}(S^{n-1})}\|f\|^{1+\kappa_0}_{L^{\tilde{q}}(\mathbb{R}^n)},\end{aligned} \tag{3.2}$$

where

$$\frac{1}{\tilde{q}} = \frac{1}{p(1+\kappa_0)} + \frac{\varepsilon_0}{n}.$$

The condition $\frac{1}{p} = \frac{1}{q} - \frac{\alpha}{n}$ and a trivial calculation yield

$$\frac{1}{p(1+\kappa_0)} + \frac{\varepsilon_0}{n} = \frac{1}{p\left(1+\frac{\alpha-\varepsilon_0}{\varepsilon_0-\frac{n}{q}}\right)} + \frac{\varepsilon_0}{n} = \frac{n-q\varepsilon_0}{qn} + \frac{\varepsilon_0}{n} = \frac{1}{q},$$

which implies $\tilde{q} = q$.

Combining the above conclusion with the estimates of (3.1) and (3.2), for $\frac{1}{2} \leq \alpha \leq \frac{n}{2}$ and $p \geq p_0 = (2+\varepsilon_0)n$, we obtain that

$$\begin{aligned}\|\mathcal{F}_{\Omega,\alpha} f\|_{L^p(\mathbb{R}^n)} &\leq C\left\|(\mathcal{F}_{\Omega,\varepsilon_0} f)^{1+\kappa_0}\right\|_{L^p(\mathbb{R}^n)}\left(\|f\|_{L^q(\mathbb{R}^n)}\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\right)^{-\kappa_0} \\ &\leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^q(\mathbb{R}^n)},\end{aligned} \tag{3.3}$$

where $1/q = 1/p + \alpha/n$ and $r = q'$.

In the following, we need to discuss the boundedness of $\mathcal{F}_{\Omega,\alpha}$ for $2 < p < p_0$. We also note that the following $L^{q_1} \to L^2$ boundedness of $\mathcal{F}_{\Omega,\alpha}$ for $\alpha \leq \frac{n}{2}$ was established in [4]:

$$\|\mathcal{F}_{\Omega,\alpha} f\|_{L^2(\mathbb{R}^n)} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^{r_1}(S^{n-1})}\|f\|_{L^{q_1}(\mathbb{R}^n)}, \tag{3.4}$$

where $q_1 = \frac{2n}{n+2\alpha}$ and $r_1 = q_1'$.

Interpolating between (3.4) and the following inequality

$$\|\mathcal{F}_{\Omega,\alpha} f\|_{L^{p_0}(\mathbb{R}^n)} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^{r_0}(S^{n-1})}\|f\|_{L^{q_0}(\mathbb{R}^n)},$$

we get that, for all $2 < p < p_0$,

$$\|\mathcal{F}_{\Omega,\alpha} f\|_{L^p(\mathbb{R}^n)} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\|f\|_{L^q(\mathbb{R}^n)}, \tag{3.5}$$

where
$$\frac{1}{p}=\frac{\theta}{p_0}+\frac{1-\theta}{2},\quad \frac{1}{r}=\frac{\theta}{r_0}+\frac{1-\theta}{r_1},\quad \frac{1}{q}=\frac{\theta}{q_0}+\frac{1-\theta}{q_1}.$$
Thus, a trivial calculation yields $\theta=1-2/p$ and $r=q'$. Therefore, we complete the proof of the theorem for $1/2\le\alpha\le n/2$.

Our second step is to extend α to the range $\frac{n}{2}<\alpha<n$, by invoking the result obtained in the previous step. For any $\alpha\in(\frac{n}{2},n)$, we can find a small positive number ϵ such that $\frac{n}{2}<\alpha\le n-\epsilon$. Let $\varepsilon_1=\frac{n}{2}$ and $p_1=\frac{n}{\epsilon}$. When $p\ge p_1$, we have $\alpha-\varepsilon_1<n\left(\frac{1}{2}-\frac{1}{p}\right)$. Let δ_1 be a number to be chosen later and write
$$\mathcal{F}_{\Omega,\alpha}f(x)=\int_{|y|\le\delta_1}\frac{\Omega(x,y)}{|y|^{n-\alpha}}f(x-y)dy+\int_{|y|>\delta_1}\frac{\Omega(x,y)}{|y|^{n-\alpha}}f(x-y)dy:=H_3+H_4.$$
It yields that, by a similar argument as the estimate of H_1 and H_2,
$$H_3\le\int_{|y|\le\delta_1}\frac{|y|^{\alpha-\varepsilon_1}\Omega(x,y)}{|y|^{n-\varepsilon_1}}f(x-y)dy\le{\delta_1}^{\alpha-\varepsilon_1}\mathcal{F}_{\Omega,\varepsilon_1}f(x),$$
$$H_4\le\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}{\delta_1}^{\alpha-\frac{n}{q}}\|f\|_{L^q(\mathbb{R}^n)}.$$
Thus,
$$\mathcal{F}_{\Omega,\alpha}f(x)\le{\delta_1}^{\alpha-\varepsilon_1}(\mathcal{F}_{\Omega,\varepsilon_1}f(x)+{\delta_1}^{\varepsilon_1-\frac{n}{q}}\|f\|_{L^q(\mathbb{R}^n)}\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}).$$
We now take
$$\delta_1=\left(\frac{\mathcal{F}_{\Omega,\varepsilon_1}f(x)}{\|f\|_{L^q(\mathbb{R}^n)}\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}}\right)^{\frac{1}{\varepsilon_1-\frac{n}{q}}},$$
and denote $\kappa_1=\frac{\alpha-\varepsilon_1}{\varepsilon_1-\frac{n}{q}}$. It is not difficult to see that
$$\mathcal{F}_{\Omega,\alpha}f(x)\le(\mathcal{F}_{\Omega,\varepsilon_1}f(x))^{1+\kappa_1}\left(\|f\|_{L^q(\mathbb{R}^n)}\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\right)^{-\kappa_1}\tag{3.6}$$
and
$$\left\|(\mathcal{F}_{\Omega,\varepsilon_1}f)^{1+\kappa_1}\right\|_{L^p(\mathbb{R}^n)}=\|\mathcal{F}_{\Omega,\varepsilon_1}f\|^{1+\kappa_1}_{L^{p(1+\kappa_1)}(\mathbb{R}^n)}.$$
In order to use the result of step 1, it is necessary to ensure that $p(1+\kappa_1)\ge 2$. Note that in this case the inequality $\alpha-\varepsilon_1\le n\left(\frac{1}{2}-\frac{1}{p}\right)$ is equivalent to $p(1+\kappa_1)\ge 2$ if α lies in the interval $\left(\frac{n}{2},n\right)$. Hence, for $p\ge p_1=\frac{n}{\epsilon}$, by virtue of the result of step 1, we get
$$\|\mathcal{F}_{\Omega,\varepsilon_1}f\|^{1+\kappa_1}_{L^{p(1+\kappa_1)}(\mathbb{R}^n)}\le C\|\Omega\|^{1+\kappa_1}_{L^\infty(\mathbb{R}^n)\times L^{\tilde{q}'}(S^{n-1})}\|f\|^{1+\kappa_1}_{L^{\tilde{q}}(\mathbb{R}^n)},\tag{3.7}$$

where $\frac{1}{\tilde{\tilde{q}}} = \frac{1}{p(1+\kappa_1)} + \frac{\varepsilon_1}{n}$. Also, a trivial calculation yields $\frac{1}{p(1+\kappa_1)} + \frac{\varepsilon_1}{n} = \frac{1}{q}$, which implies $\tilde{\tilde{q}} = q$.

The estimates of (3.6) and (3.7) give us that, for all $p \geq p_1$,

$$\begin{aligned}\|\mathcal{F}_{\Omega,\alpha} f\|_{L^p(\mathbb{R}^n)} &\leq C \left\|(\mathcal{F}_{\Omega,\varepsilon_1} f)^{1+\kappa_1}\right\|_{L^p(\mathbb{R}^n)} \left(\|f\|_{L^q(\mathbb{R}^n)} \|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\right)^{-\kappa_1} \\ &\leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})} \|f\|_{L^q(\mathbb{R}^n)}. \end{aligned} \tag{3.8}$$

Thus, it remains to prove the boundedness of $\mathcal{F}_{\Omega,\alpha}$ for $2 < p < p_1$. Analogous to the proof in step 1, interpolating between (3.4) and (3.8) for $\alpha \in (\frac{n}{2}, n)$, we get the desired result.

3.2. Bilinear fractional integral. We will only prove (1) in Theorem 5, since the proof for other parts is the same. Let $q' = 1 + \frac{p_2}{p_1}$ and $q = 1 + \frac{p_1}{p_2}$, by Hölder's inequality we have

$$\widetilde{B}_{\Omega,\alpha}(f,g)(x) \leq \mathcal{F}_{\Omega,\alpha}(f^q)(x)^{1/q}\ \mathcal{L}_{\Omega,\alpha}(g^{q'})(x)^{1/q'}.$$

By Hölder's inequality again, we have

$$\left\|\widetilde{B}_{\Omega,\alpha}(f,g)\right\|^p_{L^p(\mathbb{R}^n)} \leq \|\mathcal{F}_{\Omega,\alpha}(f^q)\|^{p/q}_{L^p(\mathbb{R}^n)} \left\|\mathcal{L}_{\Omega,\alpha}(g^{q'})\right\|^{p/q'}_{L^p(\mathbb{R}^n)}.$$

By Theorem 4, we have a constant $C > 0$ such that

$$\begin{aligned}&\|\mathcal{F}_{\Omega,\alpha}(f^q)\|^{p/q}_{L^p(\mathbb{R}^n)} \left\|\mathcal{L}_{\Omega,\alpha}(g^{q'})\right\|^{p/q'}_{L^p(\mathbb{R}^n)} \\ &\leq C^p \|\Omega\|^p_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})} \|f^q\|^{p/q}_{L^\sigma(\mathbb{R}^n)} \left\|g^{q'}\right\|^{p/q'}_{L^\sigma(\mathbb{R}^n)},\end{aligned}$$

where

$$r > \frac{n-1}{n}\sigma' \text{ and } 1/p = 1/\sigma - \frac{\alpha}{n}.$$

Noting that

$$\|f^q\|^{p/q}_{L^\sigma(\mathbb{R}^n)} \left\|g^{q'}\right\|^{p/q'}_{L^\sigma(\mathbb{R}^n)} = \|f\|^p_{L^{\sigma q}(\mathbb{R}^n)} \|g\|^p_{L^{\sigma q'}(\mathbb{R}^n)},$$

and

$$1/\sigma = \frac{p_1 + p_2}{p_1 p_2},$$

we easily see that

$$\left\|\widetilde{B}_{\Omega,\alpha}(f,g)\right\|^p_{L^p(\mathbb{R}^n)} \leq C \|f\|^p_{L^{p_1}(\mathbb{R}^n)} \|g\|^p_{L^{p_2}(\mathbb{R}^n)}.$$

The theorem is proved. □

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References

[1] M. Bartl and D. Fan, On hyper-singular integral operators with variable kernels, *J. Math. Anal. Appl.* **328** (2007), 730–742.

[2] A. P. Calderón and A. Zygmund, On a problem of Mihlin, *Trans. Amer. Math. Soc.* **78** (1955), 209–224.

[3] A. P. Calderón and A. Zygmund, On singular integrals with variable kernels, *Appl. Anal.* **7** (1978), 221–238.

[4] J. Chen, Y. Ding and D. Fan, On a hyper Hilbert transform, *Chinese Ann. Math. Ser. B.* **24** (2003), 475–484.

[5] J. Chen, Y. Ding and D. Fan, Littlewood–Paley operators with variable kernels, *Sci. China Ser. A.* **49** (2006), 639–650.

[6] J. Chen, D. Fan and Y. Ying, Certain operators with rough singular kernels, *Canad. J. Math.* **55** (2003), 504–532.

[7] J. Chen and D. Fan, Rough bilinear fractional integrals with variable kernels, *Front. Math. China.* **5** (2010), 369–378.

[8] Y. Chen, Y. Ding and R. Li, The boundedness for commutator of fractional integral operator with rough variable kernel, *Potential Anal.* **38** (2013), 119–142.

[9] M. Chirst, J. Duoandikoetxea and J. L. Rubio de Francia, Maximal operators related to the Radon transform and the Calderon–Zygmund method of rotations, *Duke Math. J.* **53** (1986), 189–208.

[10] Y. Ding, J. Chen and D. Fan, A class of integral operators with variable kernels on Hardy spaces, *Chinese Ann. Math. Ser. A.* **23** (2002), 289–296.

[11] Y. Ding, C. Lin and S. Shao, On the Marcinkiewicz integral with variable kernels, *Indiana Univ. Math. J.* **53** (2004), 805–821.

[12] C. Lin, Y. Lin, X. Tao and X. Yu, The boundedness of Marcinkiewicz integral with variable kernel, *Illinois J. Math.* **53** (2009), 197–217.

[13] B. Muckenhoupt and R. L. Wheeden, Weighted norm inequalities for singular and fractional integrals, *Trans. Amer. Math. Soc.* **161** (1971), 249–258.

[14] E. M. Stein and G. L. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, *Princeton University Press*, *Princeton, N.J.*, 1971.

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