Publ. Math. Debrecen
90/1-2 (2017), 55–90
DOI: 10.5486/PMD.2017.7461

The groups $\mathbf{K_1}(\mathbb{S}_n, \mathfrak{p})$ of the algebra of one-sided inverses of a polynomial algebra

By VLADIMIR V. BAVULA (Sheffield)

Abstract. The algebra $\mathbb{S}_n$ of one-sided inverses of a polynomial algebra P_n in n variables is obtained from P_n by adding commuting, *left* (but not two-sided) inverses of the canonical generators of the algebra P_n. The algebra $\mathbb{S}_n$ is a noncommutative, non-Noetherian algebra of classical Krull dimension $2n$ and of global dimension n, and is not a domain. If the ground field K has characteristic zero, then the algebra $\mathbb{S}_n$ is canonically isomorphic to the algebra $K\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \int_1, \dots, \int_n \rangle$ of scalar integro-differential operators. It is proved that $\mathrm{K}_1(\mathbb{S}_n) \simeq K^*$. The main idea is to show that the group $\mathrm{GL}_\infty(\mathbb{S}_n)$ is generated by K^*, the group of elementary matrices $E_\infty(\mathbb{S}_n)$ and $(n-2)2^{n-1}+1$ explicit (tricky) matrices, and then to prove that all the matrices are elementary. For each nonzero idempotent prime ideal $\mathfrak{p}$ of height m of the algebra $\mathbb{S}_n$, it is proved that

$$\mathrm{K}_1(\mathbb{S}_n, \mathfrak{p}) \simeq \begin{cases} K^*, & \text{if } m = 1, \\ \mathbb{Z}^{\frac{m(m-1)}{2}} \times K^{*m} & \text{if } m > 1. \end{cases}$$

1. Introduction

Throughout, ring means an associative ring with 1; module means a left module; $\mathbb{N} := \{0, 1, \dots\}$ is the set of natural numbers; K is a field, and K^* is its group of units; $P_n := K[x_1, \dots, x_n]$ is a polynomial algebra over K; $\partial_1 := \frac{\partial}{\partial x_1}, \dots, \partial_n := \frac{\partial}{\partial x_n}$ are the usual partial derivatives (K-linear derivations) of P_n; $\mathrm{End}_K(P_n)$ is the algebra of all K-linear maps from P_n to P_n, and $\mathrm{Aut}_K(P_n)$ is its group of units (i.e. the group of all the invertible linear maps from P_n to P_n);

Mathematics Subject Classification: 19B28, 16S32.
Key words and phrases: the algebra of one-sided inverses of a polynomial algebra, K_1-group, the current groups.

the subalgebra $A_n := K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n\rangle$ of $\mathrm{End}_K(P_n)$ is called the n-th *Weyl* algebra.

Definition ([5]). The *algebra* $\mathbb{S}_n = \mathbb{S}_n(K)$ *of one-sided inverses* of P_n is the algebra generated over a field (or a ring) K by $2n$ elements $x_1, \dots, x_n, y_1, \dots, y_n$ that satisfy the defining relations:

$$y_1x_1 = 1, \dots, y_nx_n = 1, \quad [x_i, y_j] = [x_i, x_j] = [y_i, y_j] = 0 \quad \text{for all} \quad i \neq j,$$

where $[a, b] := ab - ba$ is the algebra commutator of elements a and b.

By the very definition, the algebra $\mathbb{S}_n$ is obtained from the polynomial algebra P_n by adding commuting, left (but not two-sided) inverses of its canonical generators. The algebra $\mathbb{S}_1 = K\langle x, y \,|\, yx = 1\rangle$ is a well-known primitive algebra [12, p. 35, Example 2]. Over the field $\mathbb{C}$ of complex numbers, the completion of the algebra $\mathbb{S}_1$ is the *Toeplitz algebra*, which is the C^*-algebra generated by a unilateral shift on the Hilbert space $l^2(\mathbb{N})$ (note that $y = x^*$). The Toeplitz algebra is the universal C^*-algebra generated by a proper isometry. If $\mathrm{char}(K) = 0$, then the algebra $\mathbb{S}_n$ is isomorphic to the algebra $K\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \int_1, \dots, \int_n\rangle$ of scalar integro-differential operators $\left(\text{via } x_i \mapsto \int_i,\ y_i \mapsto \frac{\partial}{\partial x_i}\right)$.

In [7], it is proved that $\mathrm{K}_1(\mathbb{S}_1) \simeq K^*$. The first aim of the paper is to prove that

$$\mathrm{K}_1(\mathbb{S}_n) \simeq K^* \quad \text{for all} \quad n \geq 1.$$

See Theorem 3.5.

The algebra $\mathbb{S}_n$ was studied in detail in [5]: its Gelfand–Kirillov dimension is $2n$, its classical Krull dimension $\mathrm{cl.Kdim}(\mathbb{S}_n) = 2n$, and its weak and global dimensions are n. The algebra $\mathbb{S}_n$ is neither left nor right Noetherian, as was shown by Jacobson [11], when $n = 1$ (see also Baer [1]). Moreover, it contains infinite direct sums of left and right ideals. It is an experimental fact that the algebra $\mathbb{S}_n \simeq \mathbb{S}_1^{\otimes n}$ has properties that are a mixture of properties of the Weyl algebra $A_n \simeq A_1^{\otimes n}$ (in characteristic zero) and the polynomial algebra $P_{2n} \simeq P_2^{\otimes n}$, which is not surprising when we look at their defining relations:

$$P_2 = K\langle x, y\rangle : yx - xy = 0;$$
$$A_1 = K\langle x, y\rangle : yx - xy = 1;$$
$$\mathbb{S}_1 = K\langle x, y\rangle : yx = 1.$$

The group $G_n := \mathrm{Aut}_{K-\mathrm{alg}}(\mathbb{S}_n)$ of K-algebra automorphisms of $\mathbb{S}_n$, and the group $\mathbb{S}_n^*$ of units of the algebra $\mathbb{S}_n$ were determined in the series of three papers [6], [7] and [8], and their explicit generators were found (both groups are huge). The group G_1 was found by Gerritzen [10].

Theorem 1.1.

(1) [6] *$G_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n)$, where S_n is the symmetric group, $\mathbb{T}^n \simeq K^{*n}$ is the n-dimensional algebraic torus, and $\mathrm{Inn}(\mathbb{S}_n)$ is the group of inner automorphisms of $\mathbb{S}_n$.*

(2) [7], [9] *$\mathbb{S}_n^* = K^* \times (1 + \mathfrak{a}_n)^*$, where $\mathfrak{a}_n$ is the ideal generated by all the height one prime ideals of $\mathbb{S}_n$.*

(3) [8] *The centre of the group $\mathbb{S}_n^*$ is K^*, and the centre of the group $(1 + \mathfrak{a}_n)^*$ is $\{1\}$.*

(4) [8] *The map $(1 + \mathfrak{a}_n)^* \to \mathrm{Inn}(\mathbb{S}_n)$, $u \mapsto \omega_u$, is a group isomorphism ($\omega_u(a) = uau^{-1}$).*

The structure of the proof of Theorem 3.5. The idea of the proof that $\mathrm{K}_1(\mathbb{S}_n) \simeq K^*$ (Theorem 3.5) is to use the fact that the group $\mathrm{GL}_\infty(\mathbb{S}_{n-1})$ is canonically isomorphic to the congruence subgroup $(1 + \mathfrak{p}_n)^*$ of $\mathbb{S}_n^* = K^* \times (1 + \mathfrak{a}_n)^*$, $(1 + \mathfrak{p}_n)^* \subseteq (1 + \mathfrak{a}_n)^*$, where $\mathfrak{p}_n$ is an (arbitrary) height one prime ideal of the algebra $\mathbb{S}_n$. The group $\mathbb{S}_n^*$ is huge, e.g.

$$\mathbb{S}_n^* \supset (1 + \mathfrak{a}_n)^* \supset \underbrace{\mathrm{GL}_\infty(K) \ltimes \cdots \ltimes \mathrm{GL}_\infty(K)}_{2^n - 1 \text{ times}}, \tag{1}$$

the iterated semi-direct product being a small part of the group $\mathbb{S}_n^*$. The key ingredients in finding the groups G_n, $\mathrm{Inn}(\mathbb{S}_n)$ and $\mathbb{S}_n^*$ (and their explicit generators) are the Fredholm operators and their indices, the current subgroups, and the K_1-theory. This explains why it is possible to recover the group $\mathrm{GL}_\infty(\mathbb{S}_{n-1})$ in $\mathbb{S}_n^*$ (this is not straightforward), to find its explicit generators. We prove in Theorem 3.3, Lemma 3.2, and (34) that

the group $\mathrm{GL}_\infty(\mathbb{S}_n)$ is generated by K^*, the group of elementary matrices $E_\infty(\mathbb{S}_n)$, and $(n-2)2^{n-1} + 1$ matrices $\begin{pmatrix} \theta_{ij}(J) & 0 \\ 0 & 1 \end{pmatrix}$ (Lemma 3.6), where, see (16),

$$\theta_{ij}(J) := \big(1 + (y_i - 1) \prod_{k \in J \setminus i} (1 - x_k y_k)\big)\big(1 + (x_j - 1) \prod_{l \in J \setminus j} (1 - x_l y_l)\big) \in (1 + \mathfrak{a}_n)^*,$$

J is a subset of $\{1, \dots, n\}$ with $|J| \geq 2$, i is the largest number in J, and $j \in J \setminus i$.

The final and the most difficult part of the proof is to show that

(Theorem 3.4) *all the above matrices* $\begin{pmatrix} \theta_{ij}(J) & 0 \\ 0 & 1 \end{pmatrix}$ *are elementary, i.e. the units* $\theta_{ij}(J)$ *are elementary when regarded as matrices via the inclusion* $\mathrm{GL}_1(\mathbb{S}_n) \subseteq \mathrm{GL}_\infty(\mathbb{S}_n)$.

We spend all of Section 4 to prove this fact.

(Theorem 5.7) *Let* $\mathfrak{p}$ *be a nonzero idempotent prime ideal of the algebra* $\mathbb{S}_n$ *and* $m = \mathrm{ht}(\mathfrak{p})$ *be its height. Then*

$$\mathrm{K}_1(\mathbb{S}_n, \mathfrak{p}) \simeq \begin{cases} K^*, & \text{if } m = 1, \\ \mathbb{Z}^{\binom{m}{2}} \times K^{*m} & \text{if } m > 1. \end{cases}$$

Let $\Theta_{n,s}$, $s = 1, \ldots, n-1$, denote the finitely generated subgroup of the group $(1+\mathfrak{a}_n)^*$, generated by the elements $\theta_{ij}(J)$, where J is a subset of $\{1, \ldots, n\}$ with $|J| = s + 1 \geq 2$, and i and j are two distinct elements of the set J. These, the so-called *current subgroups*, were introduced in [7] and [8], and they are the core (the non-obvious part) of the groups G_n, $\mathrm{Inn}(\mathbb{S}_n)$ and $\mathbb{S}_n^*$, and the key for determining the groups $\mathrm{GL}_\infty(\mathbb{S}_n)$, $\mathrm{K}_1(\mathbb{S}_n)$, $\mathrm{GL}_\infty(\mathbb{S}_n, \mathfrak{p})$ and $\mathrm{K}_1(\mathbb{S}_n, \mathfrak{p})$, as this paper demonstrates.

The paper is organized as follows. In Section 2, some necessary results and constructions are collected for the algebra $\mathbb{S}_n$ and the group $(1+\mathfrak{a}_n)^*$. In Section 3, the groups $\mathrm{K}_1(\mathbb{S}_n)$, $\mathrm{GL}_\infty(\mathbb{S}_n)$ and their explicit generators are found. In Section 4, Theorem 3.4 is proved. In Section 5, the groups $\mathrm{GL}_\infty(\mathbb{S}_n, \mathfrak{p})$, $\mathrm{K}_1(\mathbb{S}_n, \mathfrak{p})$ and explicit generators for them are found, and Theorem 5.7 is proved.

The structure of the proof of Theorem 5.7. The line of proof of Theorem 5.7 follows that of Theorem 3.5 (but there are new moments): first, we prove that the group $\mathrm{GL}_\infty(\mathbb{S}_n, \mathfrak{p})$ is generated by the group $E_\infty(\mathbb{S}_n, \mathfrak{p})$ of $\mathfrak{p}$-elementary matrices, some explicit 'diagonal' matrices, and *some* of the matrices $\begin{pmatrix} \theta_{ij}(J) & 0 \\ 0 & 1 \end{pmatrix}$ (Theorem 5.2, Lemma 5.4). Then an 'obvious' normal subgroup $\mathcal{E}(\mathbb{S}_n, \mathfrak{p})$ of $\mathrm{GL}_\infty(\mathbb{S}_n, \mathfrak{p})$ is introduced, and we prove that

$$\mathrm{GL}_\infty(\mathbb{S}_n, \mathfrak{p})/\mathcal{E}(\mathbb{S}_n, \mathfrak{p}) \simeq \begin{cases} K^*, & \text{if } m = 1, \\ \mathbb{Z}^{\binom{m}{2}} \times K^{*m} & \text{if } m > 1. \end{cases}$$

This gives the inclusion $E_\infty(\mathbb{S}_n, \mathfrak{p}) \subseteq \mathcal{E}(\mathbb{S}_n, \mathfrak{p})$. The key moment in proving that the opposite inclusion holds is (surprisingly) the fact that $\mathrm{K}_1(\mathbb{S}_n) \simeq K^*$. The new moment is that not all the 'diagonal' matrices and not all the matrices

$\begin{pmatrix} \theta_{ij}(J) & 0 \\ 0 & 1 \end{pmatrix}$ that form a part of the generating set for the group $\mathrm{GL}_\infty(\mathbb{S}_n, \mathfrak{p})$ are $\mathfrak{p}$-elementary.

A canonical form is found (Theorem 5.7) for each element $a \in \mathrm{GL}_\infty(\mathbb{S}_n, \mathfrak{p})$. Using it, an effective criterion (Corollary 5.10) is given for an element $a \in \mathrm{GL}_\infty(\mathbb{S}_n, \mathfrak{p})$ to be a product of $\mathfrak{p}$-elementary matrices, i.e. $a \in E_\infty(\mathbb{S}_n, \mathfrak{p})$.

2. The groups $\mathbb{S}_n^*$ and $(1 + \mathfrak{a}_n)^*$ and their subgroups

In this section, we collect some results without proofs on the algebras $\mathbb{S}_n$ from [5] and [8] that will be used in this paper, their proofs can be found in [5] and [8]. Several important subgroups of the group $(1 + \mathfrak{a}_n)^*$ are considered. The most interesting of these are the current subgroups $\Theta_{n,s}$, $s = 1, \ldots, n-1$. They encapsulate the most difficult parts of the groups $\mathbb{S}_n^*$ and G_n.

The algebra of one-sided inverses of a polynomial algebra. Clearly, $\mathbb{S}_n = \mathbb{S}_1(1) \otimes \cdots \otimes \mathbb{S}_1(n) \simeq \mathbb{S}_1^{\otimes n}$, where $\mathbb{S}_1(i) := K\langle x_i, y_i \,|\, y_i x_i = 1\rangle \simeq \mathbb{S}_1$, and $\mathbb{S}_n = \bigoplus_{\alpha,\beta \in \mathbb{N}^n} K x^\alpha y^\beta$, where $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $y^\beta := y_1^{\beta_1} \cdots y_n^{\beta_n}$ and $\beta = (\beta_1, \ldots, \beta_n)$. In particular, the algebra $\mathbb{S}_n$ contains two polynomial subalgebras P_n and $Q_n := K[y_1, \ldots, y_n]$, and is equal, as a vector space, to their tensor product $P_n \otimes Q_n$.

When $n = 1$, we usually drop the subscript '1' if this does not lead to confusion. So, $\mathbb{S}_1 = K\langle x, y \,|\, yx = 1\rangle = \bigoplus_{i,j \geq 0} K x^i y^j$. For each natural number $d \geq 1$, let $M_d(K) := \bigoplus_{i,j=0}^{d-1} K E_{ij}$ be the algebra of d-dimensional matrices, where $\{E_{ij}\}$ are the matrix units, and

$$M_\infty(K) := \varinjlim M_d(K) = \bigoplus_{i,j \in \mathbb{N}} K E_{ij}$$

is the algebra (without 1) of infinite dimensional matrices. The algebra $\mathbb{S}_1$ contains the ideal $F := \bigoplus_{i,j \in \mathbb{N}} K E_{ij}$, where

$$E_{ij} := x^i y^j - x^{i+1} y^{j+1}, \quad i, j \geq 0. \tag{2}$$

For all natural numbers i, j, k, and l, $E_{ij} E_{kl} = \delta_{jk} E_{il}$, where δ_{jk} is the Kronecker delta function. The ideal F is an algebra (without 1) isomorphic to the algebra (without 1) $M_\infty(K)$ via $E_{ij} \mapsto E_{ij}$. For all $i, j \geq 0$,

$$x E_{ij} = E_{i+1,j}, \quad y E_{ij} = E_{i-1,j} \quad (E_{-1,j} := 0), \tag{3}$$

$$E_{ij} x = E_{i,j-1}, \quad E_{ij} y = E_{i,j+1} \quad (E_{i,-1} := 0). \tag{4}$$

The algebra

$$\mathbb{S}_1 = K \oplus xK[x] \oplus yK[y] \oplus F \tag{5}$$

is a direct sum of vector spaces. Then

$$\mathbb{S}_1/F \simeq K[x, x^{-1}] =: L_1, \quad x \mapsto x, \quad y \mapsto x^{-1}, \tag{6}$$

since $yx = 1$, $xy = 1 - E_{00}$ and $E_{00} \in F$.

The algebra $\mathbb{S}_n = \bigotimes_{i=1}^n \mathbb{S}_1(i)$ contains the ideal

$$F_n := F^{\otimes n} = \bigoplus_{\alpha,\beta \in \mathbb{N}^n} KE_{\alpha\beta},$$

where

$$E_{\alpha\beta} := \prod_{i=1}^n E_{\alpha_i\beta_i}(i), \quad E_{\alpha_i\beta_i}(i) := x_i^{\alpha_i} y_i^{\beta_i} - x_i^{\alpha_i+1} y_i^{\beta_i+1}.$$

Note that $E_{\alpha\beta}E_{\gamma\rho} = \delta_{\beta\gamma}E_{\alpha\rho}$ for all elements $\alpha, \beta, \gamma, \rho \in \mathbb{N}^n$, where $\delta_{\beta\gamma}$ is the Kronecker delta function; $F_n = \bigotimes_{i=1}^n F(i)$ and $F(i) := \bigoplus_{s,t\in\mathbb{N}} KE_{st}(i)$.

- *The algebra $\mathbb{S}_n$ is central, prime and catenary. Every nonzero ideal of $\mathbb{S}_n$ is an essential left and right submodule of $\mathbb{S}_n$.*
- *The ideals of $\mathbb{S}_n$ commute ($IJ = JI$); and the set of ideals of $\mathbb{S}_n$ satisfy the a.c.c.*
- $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{ab}$ *for all idempotent ideals $\mathfrak{a}$ and $\mathfrak{b}$ of the algebra $\mathbb{S}_n$.*
- *The classical Krull dimension* $\mathrm{cl.Kdim}(\mathbb{S}_n)$ *of $\mathbb{S}_n$ is $2n$.*
- *Let I be an ideal of $\mathbb{S}_n$. Then the factor algebra $\mathbb{S}_n/I$ is left (or right) Noetherian iff the ideal I contains all the height one prime ideals of the algebra $\mathbb{S}_n$.*

The set of height one prime ideals of $\mathbb{S}_n$. Consider the ideals of the algebra $\mathbb{S}_n$:

$$\mathfrak{p}_1 := F \otimes \mathbb{S}_{n-1},\ \mathfrak{p}_2 := \mathbb{S}_1 \otimes F \otimes \mathbb{S}_{n-2}, \ldots,\ \mathfrak{p}_n := \mathbb{S}_{n-1} \otimes F.$$

Then $\mathbb{S}_n/\mathfrak{p}_i \simeq \mathbb{S}_{n-1} \otimes (\mathbb{S}_1/F) \simeq \mathbb{S}_{n-1} \otimes K[x_i, x_i^{-1}]$, and $\bigcap_{i=1}^n \mathfrak{p}_i = \prod_{i=1}^n \mathfrak{p}_i = F^{\otimes n} = F_n$. Clearly, $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$ for all $i \neq j$.

The set $\mathcal{H}_1$ of height one prime ideals of the algebra $\mathbb{S}_n$ is $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.

Let $\mathfrak{a}_n := \mathfrak{p}_1 + \cdots + \mathfrak{p}_n$. Then the factor algebra

$$\mathbb{S}_n/\mathfrak{a}_n \simeq (\mathbb{S}_1/F)^{\otimes n} \simeq \bigotimes_{i=1}^n K[x_i, x_i^{-1}] = K[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}] =: L_n \tag{7}$$

is a skew Laurent polynomial algebra in n variables, and so $\mathfrak{a}_n$ is a prime ideal of height and co-height n of the algebra $\mathbb{S}_n$.

Proposition 2.1 ([5]). *The polynomial algebra P_n is the only (up to isomorphism) faithful simple $\mathbb{S}_n$-module.*

In more detail, ${}_{\mathbb{S}_n}P_n \simeq \mathbb{S}_n/(\sum_{i=0}^n \mathbb{S}_n y_i) = \bigoplus_{\alpha\in\mathbb{N}^n} Kx^\alpha\overline{1}$, $\overline{1} := 1 + \sum_{i=1}^n \mathbb{S}_n y_i$; and the action of the canonical generators of the algebra $\mathbb{S}_n$ on the polynomial algebra P_n is given by the rule:

$$x_i * x^\alpha = x^{\alpha+e_i}, \quad y_i * x^\alpha = \begin{cases} x^{\alpha-e_i} & \text{if } \alpha_i > 0, \\ 0 & \text{if } \alpha_i = 0, \end{cases} \quad \text{and} \quad E_{\beta\gamma} * x^\alpha = \delta_{\gamma\alpha}x^\beta,$$

where the set $e_1 := (1, 0, \ldots, 0), \ldots, e_n := (0, \ldots, 0, 1)$ is the canonical basis for the free $\mathbb{Z}$-module $\mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z}e_i$. We identify the algebra $\mathbb{S}_n$ with its image in the algebra $\mathrm{End}_K(P_n)$ of all the K-linear maps from the vector space P_n to itself, i.e. $\mathbb{S}_n \subset \mathrm{End}_K(P_n)$.

For each non-empty subset I of the set $\{1, \ldots, n\}$, let $\mathbb{S}_I := \bigotimes_{i\in I} \mathbb{S}_1(i) \simeq \mathbb{S}_{|I|}$, where $|I|$ is the number of elements in the set I, $F_I := \bigotimes_{i\in I} F(i) \simeq M_\infty(K)$, $\mathfrak{a}_I$ is the ideal of the algebra $\mathbb{S}_I$ generated by the vector space $\bigoplus_{i\in I} F(i)$, i.e. $\mathfrak{a}_I := \sum_{i\in I} F(i) \otimes \mathbb{S}_{I\setminus i}$. The factor algebra $L_I := \mathbb{S}_I/\mathfrak{a}_I \simeq K[x_i, x_i^{-1}]_{i\in I}$ is a Laurent polynomial algebra. For elements $\alpha = (\alpha_i)_{i\in I}, \beta = (\beta_i)_{i\in I} \in \mathbb{N}^I$, let $E_{\alpha\beta}(I) := \prod_{i\in I} E_{\alpha_i\beta_i}(i)$. Then $E_{\alpha\beta}(I)E_{\xi\rho}(I) = \delta_{\beta\xi}E_{\alpha\rho}(I)$ for all $\alpha, \beta, \xi, \rho \in \mathbb{N}^I$.

The G_n-invariant normal subgroups $(1+\mathfrak{a}_{n,s})^*$ of $(1+\mathfrak{a}_n)^*$. Let $G_n := \mathrm{Aut}_{K-\mathrm{alg}}(\mathbb{S}_n)$. We will use often the following obvious lemma.

Lemma 2.2 ([6]). *Let R be a ring and $I_1, \ldots, I_n$ be ideals of the ring R such that $I_iI_j = 0$ for all $i \neq j$. Let $a = 1+a_1+\cdots+a_n \in R$, where $a_1 \in I_1, \ldots, a_n \in I_n$. The element a is a unit of the ring R iff all the elements $1 + a_i$ are units; and, in this case, $a^{-1} = (1 + a_i)^{-1}(1 + a_2)^{-1} \cdots (1 + a_n)^{-1}$.*

Let R be a ring, R^* be its group of units, I be an ideal of R such that $I \neq R$, and let $(1 + I)^*$ be the group of units of the multiplicative monoid $1 + I$. Then $R^* \cap (1 + I) = (1 + I)^*$, and $(1 + I)^*$ is a normal subgroup of R^*.

For each subset I of the set $\{1, \ldots, n\}$, let $\mathfrak{p}_I := \bigcap_{i\in I} \mathfrak{p}_i$, and $\mathfrak{p}_\emptyset := \mathbb{S}_n$. Each $\mathfrak{p}_I$ is an ideal of the algebra $\mathbb{S}_n$ and $\mathfrak{p}_I = \prod_{i\in I} \mathfrak{p}_i$. The complement to the subset I is denoted by CI. For a one-element subset $\{i\}$, we write Ci rather than $C\{i\}$. In particular, $\mathfrak{p}_{Ci} := \mathfrak{p}_{C\{i\}} = \bigcap_{j\neq i} \mathfrak{p}_j$.

For each number $s = 1, \ldots, n$, let $\mathfrak{a}_{n,s} := \sum_{|I|=s} \mathfrak{p}_I$. By the very definition, the ideals $\mathfrak{a}_{n,s}$ are G_n-invariant ideals (since the set $\mathcal{H}_1$ of all the height one prime ideals of the algebra $\mathbb{S}_n$ is $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$, see [6], and $\mathcal{H}_1$ is a G_n-orbit). We have a strictly descending chain of G_n-invariant ideals of the algebra $\mathbb{S}_n$:

$$\mathfrak{a}_n = \mathfrak{a}_{n,1} \supset \mathfrak{a}_{n,2} \supset \cdots \supset \mathfrak{a}_{n,s} \supset \cdots \supset \mathfrak{a}_{n,n} = F_n \supset \mathfrak{a}_{n,n+1} := 0.$$

These are also ideals of the subalgebra $K+\mathfrak{a}_n$ of $\mathbb{S}_n$. Each set $\mathfrak{a}_{n,s}$ is an ideal of the algebra $K+\mathfrak{a}_{n,t}$ for all $t\leq s$, and the group of units of the algebra $K+\mathfrak{a}_{n,s}$ is the direct product of its two subgroups

$$(K+\mathfrak{a}_{n,s})^* = K^* \times (1+\mathfrak{a}_{n,s})^*, \quad s=1,\ldots,n.$$

The groups $(K+\mathfrak{a}_{n,s})^*$ and $(1+\mathfrak{a}_{n,s})^*$ are G_n-invariant. There is the descending chain of G_n-invariant (hence normal) subgroups of $(1+\mathfrak{a}_n)^*$:

$$\begin{aligned}(1+\mathfrak{a}_n)^* &= (1+\mathfrak{a}_{n,1})^* \supset \cdots \supset (1+\mathfrak{a}_{n,s})^* \supset \cdots \supset (1+\mathfrak{a}_{n,n})^* \\ &= (1+F_n)^* \supset (1+\mathfrak{a}_{n,n+1})^* = \{1\}.\end{aligned}$$

For each number $s=1,\ldots,n$, the factor algebra

$$(K+\mathfrak{a}_{n,s})/\mathfrak{a}_{n,s+1} = K \bigoplus \bigoplus_{|I|=s} \overline{\mathfrak{p}}_I$$

contains the idempotent ideals $\overline{\mathfrak{p}}_I := (\mathfrak{p}_I+\mathfrak{a}_{n,s+1})/\mathfrak{a}_{n,s+1}$ such that $\overline{\mathfrak{p}}_I\overline{\mathfrak{p}}_J=0$ for all $I\neq J$ such that $|I|=|J|=s$.

Recall that for a Laurent polynomial algebra $L=K[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$, $\mathrm{K}_1(L)\simeq L^*$, [14], [2], [13],

$$\mathrm{GL}_\infty(L) = U(L) \ltimes E_\infty(L), \tag{8}$$

where $E_\infty(L)$ is the subgroup of $\mathrm{GL}_\infty(L)$ generated by all the *elementary matrices* $\{1+aE_{ij} \,|\, a\in L, i,j\in\mathbb{N}, i\neq j\}$, and $U(L):=\{\mu(u):=uE_{00}+1-E_{00} \,|\, u\in L^*\}\simeq L^*$, $\mu(u)\leftrightarrow u$. The group $E_\infty(L)$ is a normal subgroup of $\mathrm{GL}_\infty(L)$. This is true for an arbitrary coefficient ring.

By Lemma 2.2 and (8), the group of units of the algebra $(K+\mathfrak{a}_{n,s})/\mathfrak{a}_{n,s+1} =: K+\mathfrak{a}_{n,s}/\mathfrak{a}_{n,s+1}$ is the direct product of groups,

$$\begin{aligned}(K+\mathfrak{a}_{n,s}/\mathfrak{a}_{n,s+1})^* &= K^* \times \prod_{|I|=s} (1+\overline{\mathfrak{p}}_I)^* \\ &\simeq K^* \times \prod_{|I|=s} \mathrm{GL}_\infty(L_{CI}) \simeq K^* \times \prod_{|I|=s} U(L_{CI}) \ltimes E_\infty(L_{CI}),\end{aligned}$$

since $(1+\overline{\mathfrak{p}}_I)^* \simeq (1+M_\infty(L_{CI}))^* = \mathrm{GL}_\infty(L_{CI})$, where $L_{CI} := \mathbb{S}_{CI}/\mathfrak{a}_{CI} = \bigotimes_{i\in CI} K[x_i,x_i^{-1}]$ is the Laurent polynomial algebra. In more detail, for each non-empty subset I of $\{1,\ldots,n\}$, let $\mathbb{Z}^I := \bigoplus_{i\in I}\mathbb{Z}e_i$. It is a subgroup of $\mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z}e_i$. Similarly, $\mathbb{N}^I := \bigoplus_{i\in I}\mathbb{N}e_i$. By (8),

$$(1+\overline{\mathfrak{p}}_I)^* = U(L_{CI}) \ltimes E_\infty(L_{CI}) = (U_I(K)\times \mathbb{X}_{CI}) \ltimes E_\infty(L_{CI}), \tag{9}$$

where

$$\begin{aligned} U(L_{CI}) &:= \{\mu_I(u) := uE_{00}(I) + 1 - E_{00}(I) \,|\, u \in L^*_{CI}\} \simeq L^*_{CI},\ \mu_I(u) \leftrightarrow u, \\ L^*_{CI} &= \{\lambda x^\alpha \,|\, \lambda \in K^*, \alpha \in \mathbb{Z}^{CI}\}, \\ U_I(K) &:= \{\mu_I(\lambda) := \lambda E_{00}(I) + 1 - E_{00}(I) \,|\, \lambda \in K^*\} \simeq K^*,\ \mu_I(\lambda) \leftrightarrow \lambda, \\ \mathbb{X}_{CI} &:= \{\mu_I(x^\alpha) := x^\alpha E_{00}(I) + 1 - E_{00}(I) \,|\, \alpha \in \mathbb{Z}^{CI}\} \simeq \mathbb{Z}^{CI} \\ &\simeq \mathbb{Z}^{n-s},\ \mu_I(x^\alpha) \leftrightarrow \alpha, \\ E_\infty(L_{CI}) &:= \langle 1 + aE_{\alpha\beta}(I) \,|\, a \in L_{CI}, \alpha, \beta \in \mathbb{N}^I, \alpha \neq \beta \rangle. \end{aligned}$$

The algebra epimorphism $\psi_{n,s} : K + \mathfrak{a}_{n,s} \to (K + \mathfrak{a}_{n,s})/\mathfrak{a}_{n,s+1}$, $a \mapsto a + \mathfrak{a}_{n,s+1}$, yields the group homomorphism of their groups of units $(K + \mathfrak{a}_{n,s})^* \to (K + \mathfrak{a}_{n,s}/\mathfrak{a}_{n,s+1})^*$ and whose kernel is $(1 + \mathfrak{a}_{n,s+1})^*$. As a result, we have an exact sequence of group homomorphisms:

$$1 \to (1+\mathfrak{a}_{n,s+1})^* \to (1+\mathfrak{a}_{n,s})^* \stackrel{\psi_{n,s}}{\to} \prod_{|I|=s} (1+\overline{\mathfrak{p}}_I)^* \simeq \prod_{|I|=s} \mathrm{GL}_\infty(L_{CI}) \to \mathcal{Z}_{n,s} \to 1. \quad (10)$$

For $s = n$, the map $\psi_{n,n}$ is the identity map, and so $\mathcal{Z}_{n,n} = \{1\}$. Intuitively, the group $\mathcal{Z}_{n,s}$ represents 'relations' that determine the image $\mathrm{im}(\psi_{n,s})$ as a subgroup of $\prod_{|I|=s}(1+\overline{\mathfrak{p}}_I)^*$. The group $\mathcal{Z}_{n,s}$ is a free abelian group of rank $\binom{n}{s+1}$, [8]. So, the image of the map $\psi_{n,s}$ is large. Note that $\mathfrak{a}_{n,s+1}$ and $\mathfrak{p}_I$ (where $|I| = s$) are ideals of the algebra $K + \mathfrak{a}_{n,s}$. The groups $(1+\mathfrak{a}_{n,s+1})^*$ and $(1+\mathfrak{p}_I)^*$ (where $|I| = s$) are normal subgroups of $(1 + \mathfrak{a}_{n,s})^*$. Thus the subgroup $\Upsilon_{n,s}$ of $(1 + \mathfrak{a}_{n,s})^*$ generated by these normal subgroups is a normal subgroup of $(1 + \mathfrak{a}_{n,s})^*$. As a subset of $(1 + \mathfrak{a}_{n,s})^*$, the group $\Upsilon_{n,s}$ is equal to the product of the groups $(1 + \mathfrak{a}_{n,s+1})^*$, $(1 + \mathfrak{p}_I)^*$, $|I| = s$, in *arbitrary* order (by their normality), i.e.

$$\Upsilon_{n,s} = \prod_{|I|=s} (1 + \mathfrak{p}_I)^* \cdot (1 + \mathfrak{a}_{n,s+1})^*. \quad (11)$$

By Theorem 1.1, the group $\Upsilon_{n,s}$ is a G_n-invariant (hence, normal) subgroup of $\mathbb{S}^*_n$. The factor group $(1 + \mathfrak{a}_{n,s})^*/\Upsilon_{n,s}$ is a free abelian group of rank $\binom{n}{s+1}s$, [8].

By (9), the direct product of groups $\prod_{|I|=s}(1 + \overline{\mathfrak{p}}_I)^* = \mathbb{X}_{n,s} \ltimes \overline{\Gamma}_{n,s}$ is the semi-direct product of its two subgroups

$$\mathbb{X}_{n,s} := \prod_{|I|=s} \mathbb{X}_{CI} \simeq \mathbb{Z}^{\binom{n}{s}(n-s)} \quad \text{and} \quad \overline{\Gamma}_{n,s} := \prod_{|I|=s} U_I(K) \ltimes E_\infty(L_{CI}). \quad (12)$$

For each subset I of $\{1, \ldots, n\}$ such that $|I| = s$, $U_I(K) \ltimes E_\infty(\mathbb{S}_{CI})$ is a subgroup of $(1 + \mathfrak{p}_I)^*$, where

$$U_I(K) := \{\mu_I(\lambda) \,|\, \lambda \in K^*\} \simeq K^*,$$

$$E_\infty(\mathbb{S}_{CI}) := \langle 1 + aE_{\alpha\beta}(I) \,|\, a \in \mathbb{S}_{CI}, \alpha \neq \beta \in \mathbb{N}^I \rangle, \tag{13}$$

where $\mu_I(\lambda) := \lambda E_{00}(I) + 1 - E_{00}(I)$. Clearly,

$$\psi_{n,s}|_{U_I(K)} : U_I(K) \simeq U_I(K), \;\; \mu_I(\lambda) \mapsto \mu_I(\lambda),$$

and $\psi_{n,s}(U_I(K) \ltimes E_\infty(\mathbb{S}_{CI})) = U_I(K) \ltimes E_\infty(L_{CI})$ for all subsets I with $|I| = s$. The subgroup of $(1 + \mathfrak{a}_{n,s})^*$,

$$\Gamma_{n,s} := \psi_{n,s}^{-1}(\overline{\Gamma}_{n,s}) = {}^{set} \prod_{|I|=s} (U_I(K) \ltimes E_\infty(\mathbb{S}_{CI})) \cdot (1 + \mathfrak{a}_{n,s+1})^*, \tag{14}$$

is a normal subgroup as it is the pre-image of a normal subgroup. We added the upper script 'set' to indicate that this is a product of subgroups but not a direct product, in general. It is obvious that $\psi_{n,s}(\Gamma_{n,s}) = \overline{\Gamma}_{n,s}$, and $\Gamma_{n,s} \subseteq \Upsilon_{n,s}$. In fact, $\Gamma_{n,s} = \Upsilon_{n,s}$, [8]. Let $\Delta_{n,s} := (1 + \mathfrak{a}_{n,s})^*/\Gamma_{n,s}$. The group homomorphism $\psi_{n,s}$ (see (10)) induces the group monomorphism

$$\overline{\psi}_{n,s} : \Delta_{n,s} \to \prod_{|I|=s} (1 + \overline{\mathfrak{p}}_I)^*)/\overline{\Gamma}_{n,s} \simeq \mathbb{X}_{n,s} \simeq \mathbb{Z}^{\binom{n}{s}(n-s)}.$$

This means that the group $\Delta_{n,s}$ is a free abelian group of rank $\leq \binom{n}{s}(n-s)$. In fact, the rank is equal to $\binom{n}{s+1}s$, [8].

For each subset I with $|I| = s$, consider the free abelian group $\mathbb{X}'_{CI} := \bigoplus_{j \in CI} \mathbb{Z}(j, I) \simeq \mathbb{Z}^{n-s}$, where $\{(j, I) \,|\, j \in CI\}$ is its free basis. Let

$$\mathbb{X}'_{n,s} := \bigoplus_{|I|=s} \mathbb{X}'_{CI} = \bigoplus_{|I|=s} \bigoplus_{j \in CI} \mathbb{Z}(j, I) \simeq \mathbb{Z}^{\binom{n}{s}(n-s)}.$$

For each subset I, consider the isomorphism of abelian groups

$$\mathbb{X}_{CI} \to \mathbb{X}'_{CI}, \quad \mu_I(x_j) := x_j E_{00}(I) + 1 - E_{00}(I) \mapsto (j, I).$$

These isomorphisms yield the group isomorphism

$$\mathbb{X}_{n,s} \to \mathbb{X}'_{n,s}, \;\; \mu_I(x_j) \mapsto (j, I). \tag{15}$$

Each element a of the group $\mathbb{X}_{n,s}$ is a unique product $a = \prod_{|I|=s} \prod_{j \in CI} \mu_I(x_j)^{n(j,I)}$ where $n(j, I) \in \mathbb{Z}$. Each element a' of the group $\mathbb{X}'_{n,s}$ is a unique sum $a' = \sum_{|I|=s} \sum_{j \in CI} n(j, I) \cdot (j, I)$ where $n(j, I) \in \mathbb{Z}$. The map (15) sends a to a'. To make computations more readable, we set $e_I := E_{00}(I)$. Then $e_I e_J = e_{I \cup J}$.

The current groups $\Theta_{n,s}$, $s = 1, \dots, n-1$. The current groups $\Theta_{n,s}$ are the most important subgroups of the group $(1+\mathfrak{a}_n)^*$. They are finitely generated groups, and generators are given explicitly. The generators of the groups $\Theta_{n,s}$ are units of the algebra $\mathbb{S}_n$ but they are defined as a product of two *non-units*. As a result, the groups $\Theta_{n,s}$ capture the most delicate phenomena regarding the structure and properties of the groups $\mathbb{S}_n^*$ and G_n.

For each non-empty subset I of $\{1, \dots, n\}$ with $s := |I| < n$ and an element $i \in CI$, let

$$X(i, I) := \mu_I(x_i) = x_i E_{00}(I) + 1 - E_{00}(I)$$

and

$$Y(i, I) := \mu_I(y_i) = y_i E_{00}(I) + 1 - E_{00}(I).$$

Then $Y(i,I)X(i,I) = 1$, $\ker Y(i,I) = P_{C(I\cup i)}$, and $P_n = \operatorname{im} X(i,I) \bigoplus P_{C(I\cup i)}$, where $P_{C(I\cup i)} := K[x_j]_{j\in C(I\cup i)}$. Recall that $\mathbb{S}_n \subset \mathrm{End}_K(P_n)$. As an element of the algebra $\mathrm{End}_K(P_n)$, the map $X(i,I)$ is injective (but not bijective), and the map $Y(i,I)$ is surjective (but not bijective).

Definition. For each subset J of $\{1, \dots, n\}$ with $|J| = s+1 \geq 2$, and for two distinct elements i and j of the set J, let

$$\theta_{ij}(J) := Y(i, J\backslash i)X(j, J\backslash j) \in (1 + \mathfrak{p}_{J\backslash i} + \mathfrak{p}_{J\backslash j})^* \subseteq (1 + \mathfrak{a}_{n,s})^*. \tag{16}$$

The *current group* $\Theta_{n,s}$ is the subgroup of $(1+\mathfrak{a}_{n,s})^*$ generated by all the elements $\theta_{ij}(J)$ (for all the possible choices of J, i, and j).

The unit $\theta_{ij}(I)$ is the product in $\mathrm{End}_K(P_n)$ of an injective map and a surjective map, none of which is a bijection.

$$\theta_{ij}(J) = \theta_{ji}(J)^{-1}. \tag{17}$$

Suppose that i, j, and k are distinct elements of the set J (hence $|J| \geq 3$). Then

$$\theta_{ij}(J)\theta_{jk}(J) = \theta_{ik}(J). \tag{18}$$

For each number $s = 1, \dots, n-1$, the free abelian group $\mathbb{X}'_{n,s}$ admits the decomposition $\mathbb{X}'_{n,s} = \bigoplus_{|J|=s+1} \bigoplus_{j\cup I=J} \mathbb{Z}(j, I)$, and using it we define a character (a homomorphism) χ'_J, for each subset J with $|J| = s+1$:

$$\chi'_J : \mathbb{X}'_{n,s} \to \mathbb{Z}, \qquad \sum_{|J'|=s+1} \sum_{j\cup I=J'} n_{j,I}(j, I) \mapsto \sum_{j\cup I=J} n_{j,I}.$$

Let $\max(J)$ be the largest number in the set J. The group $\mathbb{X}'_{n,s}$ is the direct sum

$$\mathbb{X}'_{n,s} = \mathbb{K}'_{n,s} \bigoplus \mathbb{Y}'_{n,s} \tag{19}$$

of its free abelian subgroups,

$$\begin{aligned}
\mathbb{K}'_{n,s} &= \bigcap_{|J|=s+1} \ker(\chi'_J) \\
&= \bigoplus_{|J|=s+1} \bigoplus_{j\in J\setminus \max(J)} \mathbb{Z}(-(\max(J), J\setminus \max(J)) + (j, J\setminus j)) \simeq \mathbb{Z}^{\binom{n}{s+1}s}, \\
\mathbb{Y}'_{n,s} &= \bigoplus_{|J|=s+1} \mathbb{Z}(\max(J), J\setminus \max(J)) \simeq \mathbb{Z}^{\binom{n}{s+1}}.
\end{aligned}$$

The same decompositions hold if, instead of $\max(J)$, we choose any element of the set J. Consider the group homomorphism $\psi'_{n,s} : (1+\mathfrak{a}_{n,s})^* \to \mathbb{X}'_{n,s}$ defined as the composition of the following group homomorphisms:

$$\psi'_{n,s} : (1+\mathfrak{a}_{n,s})^* \to (1+\mathfrak{a}_{n,s})^*/\Gamma_{n,s} \stackrel{\overline{\psi}_{n,s}}{\to} \prod_{|I|=s} (1+\overline{\mathfrak{p}}_I)^*/\overline{\Gamma}_{n,s} \simeq \mathbb{X}_{n,s} \simeq \mathbb{X}'_{n,s}.$$

Then

$$\psi'_{n,s}(\theta_{ij}(J)) = -(i, J\setminus i) + (j, J\setminus j). \tag{20}$$

It follows that

$$\psi'_{n,s}(\Theta_{n,s}) = \mathbb{K}'_{n,s}, \tag{21}$$

since, by (20), $\psi'_{n,s}(\Theta_{n,s}) \supseteq \mathbb{K}'_{n,s}$ (as the free basis for $\mathbb{K}'_{n,s}$, introduced above, belongs to the set $\psi'_{n,s}(\Theta_{n,s})$); again, by (20), $\psi'_{n,s}(\Theta_{n,s}) \subseteq \bigcap_{|J|=s+1} \ker(\chi'_J) = \mathbb{K}'_{n,s}$.

Let $H, H_1, \ldots, H_m$ be subsets (usually subgroups) of a group H. We say that H is the *product* of $H_1, \ldots, H_m$, and write $H = {}^{set}\prod_{i=1}^m H_i = H_1 \cdots H_m$, if each element h of H is a product $h = h_1 \cdots h_m$, where $h_i \in H_i$. We add the subscript 'set' (sometime) in order to distinguish it from the direct product of groups. We say that H is the *exact product* of $H_1, \ldots, H_m$, and write $H = {}^{exact}\prod_{i=1}^m H_i = H_1 \times_{ex} \cdots \times_{ex} H_m$, if each element h of H is a *unique* product $h = h_1 \cdots h_m$, where $h_i \in H_i$. The order in the definition of the exact product is important. A *semi-direct product* of groups $H_1, \ldots, H_m$ is denoted by

$$H_1 \ltimes (H_2 \ltimes (\cdots \ltimes H_m)) = H_1 \ltimes H_2 \ltimes \cdots \ltimes H_m = {}^{semi}\prod_{i=1}^m H_i.$$

The subgroup of $(1 + \mathfrak{a}_{n,s})^*$ generated by the groups $\Theta_{n,s}$ and $\Gamma_{n,s}$ is equal to their product $\Theta_{n,s}\Gamma_{n,s}$, by the normality of $\Gamma_{n,s}$. The subgroup $\Gamma_{n,s}$ of the group $\Theta_{n,s}\Gamma_{n,s}$ is a normal subgroup, hence the intersection $\Theta_{n,s} \cap \Gamma_{n,s}$ is a normal subgroup of $\Theta_{n,s}$.

Lemma 2.3 ([8]). *For each number $s = 1, \ldots, n-1$, the group $\Theta_{n,s}\Gamma_{n,s}$ is the semi-direct product*

$$\Theta_{n,s}\Gamma_{n,s} = {}^{semi}\prod_{|J|=s+1} \prod_{j \in J \setminus \max(J)} \langle \theta_{\max(J),j}(J) \rangle \ltimes \Gamma_{n,s},$$

where the order in the double product is arbitrary. Each element $a \in \Theta_{n,s}\Gamma_{n,s}$ is a unique product

$$a = \prod_{|J|=s+1} \prod_{j \in J \setminus \max(J)} \theta_{\max(J),j}(J)^{n(j,J)} \cdot \gamma,$$

where $n(j, J) \in \mathbb{Z}$ and $\gamma \in \Gamma_{n,s}$.

For each number $s = 1, \ldots, n-1$, consider the subset of $(1 + \mathfrak{a}_{n,s})^*$,

$$\Theta'_{n,s} := {}^{exact}\prod_{|J|=s+1} \prod_{j \in J \setminus \max(J)} \langle \theta_{\max(J),j}(J) \rangle, \tag{22}$$

which is the exact product of cyclic groups (each of them is isomorphic to $\mathbb{Z}$), since each element u of $\Theta'_{n,s}$ is a *unique* product

$$u = \prod_{|J|=s+1} \prod_{j \in J \setminus \max(J)} \theta_{\max(J),j}(J)^{n(j,J)},$$

where $n(j, J) \in \mathbb{Z}$ (Lemma 2.3).

By Lemma 2.3, $\Theta_{n,s}/\Theta_{n,s} \cap \Gamma_{n,s} \simeq \Theta_{n,s}\Gamma_{n,s}/\Gamma_{n,s} \simeq \mathbb{K}'_{n,s} \simeq \mathbb{Z}^{\binom{n}{s+1}s}$, and so the commutant of the current group $\Theta_{n,s}$ belongs to the group $\Gamma_{n,s}$, i.e.

$$[\Theta_{n,s}, \Theta_{n,s}] \subseteq \Gamma_{n,s}. \tag{23}$$

Recall that the *commutant* $[G, G]$ of a group G is the subgroup of G, generated by all *group commutators* $[a, b] := aba^{-1}b^{-1}$, where $a, b \in G$. The commutant is a normal subgroup. The next theorem is the key point in finding explicit generators for the groups $\mathbb{S}_n^*$ and G_n.

Theorem 2.4 ([8]). *$\psi'_{n,s}((1+\mathfrak{a}_{n,s})^*) = \psi'_{n,s}(\Theta_{n,s})$ for $s = 1, \dots, n-1$.*

For each number $s = 1, \dots, n-1$, consider the following subsets of the group $(1+\mathfrak{a}_{n,s})^*$,

$$\mathbb{E}_{n,s} := \prod_{|I|=s} U_I(K) \ltimes E_\infty(\mathbb{S}_{CI}) \text{ and } \mathbb{P}_{n,s} := \prod_{|I|=s} (1+\mathfrak{p}_i)^*. \tag{24}$$

These are products of subgroups $(1+\mathfrak{a}_{n,s})^*$ in arbitrary order, but which is fixed for each s.

Theorem 2.5 ([8]).

(1) $(1+\mathfrak{a}_n)^* = \Theta_{n,1}\Gamma_{n,1} = \Theta_{n,1}\mathbb{E}_{n,1}\Theta_{n,2}\mathbb{E}_{n,2}\cdots\Theta_{n,n-1}\mathbb{E}_{n,n-1}$. *Moreover, for* $s = 1, \dots, n-1$, $(1+\mathfrak{a}_{n,s})^* = \Theta_{n,s}\Gamma_{n,s} = \Theta_{n,s}\mathbb{E}_{n,s}\Theta_{n,s+1}\mathbb{E}_{n,s+1}\cdots\Theta_{n,n-1}\mathbb{E}_{n,n-1}$.

(2) $(1+\mathfrak{a}_n)^* = \Theta_{n,1}\Upsilon_{n,1} = \Theta_{n,1}\mathbb{P}_{n,1}\Theta_{n,2}\mathbb{P}_{n,2}\cdots\Theta_{n,n-1}\mathbb{P}_{n,n-1}$. *Moreover, for* $s = 1, \dots, n-1$, $(1+\mathfrak{a}_{n,s})^* = \Theta_{n,s}\Upsilon_{n,s} = \Theta_{n,s}\mathbb{P}_{n,s}\Theta_{n,s+1}\mathbb{P}_{n,s+1}\cdots\Theta_{n,n-1}\mathbb{P}_{n,n-1}$.

Theorem 2.6 ([8]).

(1) $(1+\mathfrak{a}_n)^* = \Theta'_{n,1}\mathbb{E}_{n,1}\Theta'_{n,2}\mathbb{E}_{n,2}\cdots\Theta'_{n,n-1}\mathbb{E}_{n,n-1}$. *Moreover, for* $s = 1, \dots, n-1$, $(1+\mathfrak{a}_{n,s})^* = \Theta'_{n,s}\mathbb{E}_{n,s}\Theta'_{n,s+1}\mathbb{E}_{n,s+1}\cdots\Theta'_{n,n-1}\mathbb{E}_{n,n-1}$.

(2) $(1+\mathfrak{a}_n)^* = \Theta'_{n,1}\mathbb{P}_{n,1}\Theta'_{n,2}\mathbb{P}_{n,2}\cdots\Theta'_{n,n-1}\mathbb{P}_{n,n-1}$. *Moreover, for* $s = 1, \dots, n-1$, $(1+\mathfrak{a}_{n,s})^* = \Theta'_{n,s}\mathbb{P}_{n,s}\Theta'_{n,s+1}\mathbb{P}_{n,s+1}\cdots\Theta'_{n,n-1}\mathbb{P}_{n,n-1}$.

3. The groups $\mathbf{K_1}(\mathbb{S}_n)$ and $\mathbf{GL_\infty}(\mathbb{S}_n)$, and their generators

In this section, explicit generators are found for the group $\mathrm{GL}_\infty(\mathbb{S}_n)$ (Theorem 3.3, Theorem 3.5. (1)), and it is proved that $\mathrm{K}_1(\mathbb{S}_n) \simeq K^*$ (Theorem 3.5. (2)) modulo Theorem 3.4, which is proved in Section 4.

The subgroup $(1+\mathfrak{p}_n)^*$ of the group $\mathbb{S}_n^*$ is canonically isomorphic to the group $\mathrm{GL}_\infty(\mathbb{S}_{n-1})$ via the isomorphism $1 + \sum a_{ij}E_{ij}(n) \mapsto 1 + \sum a_{ij}E_{ij}$, where $a_{ij} \in \mathbb{S}_{n-1} = \bigotimes_{i=1}^{n-1} \mathbb{S}_1(i)$. It is convenient to identify the groups $(1+\mathfrak{p}_n)^*$ and $\mathrm{GL}_\infty(\mathbb{S}_{n-1})$, and to identify the matrix units $E_{ij}(n)$ and E_{ij}, i.e. $(1+\mathfrak{p}_n)^* = \mathrm{GL}_\infty(\mathbb{S}_{n-1})$ and $E_{ij}(n) = E_{ij}$. The group $(1+\mathfrak{p}_n)^*$ contains the descending chain of normal subgroups

$$\begin{aligned}(1+\mathfrak{p}_n)^* &= (1+\mathfrak{p}_n)_1^* \supset \cdots \supset (1+\mathfrak{p}_n)_s^* \supset \cdots \supset (1+\mathfrak{p}_n)_n^* \\ &= (1+F_n)^* \supset (1+\mathfrak{p}_n)_{n+1}^* = \{1\},\end{aligned}$$

where $(1+\mathfrak{p}_n)_s^* := (1+\mathfrak{p}_n)^* \cap (1+\mathfrak{a}_{n,s})^*$. The following lemma describes the normal subgroups $(1+\mathfrak{p}_n)_s^*$.

Lemma 3.1.

$$(1+\mathfrak{p}_n)^*_s = \begin{cases} (1+\sum_{|I|=s, n\in I}\mathfrak{p}_I)^* & \text{if } s=1,\dots,n-1,\\ (1+F_n)^* & \text{if } s=n. \end{cases}$$

PROOF. As the case $s=n$ is obvious, we assume that $s\neq n$. The ideal $\mathfrak{a}_{n,s}=\sum_{|I|=s}\mathfrak{p}_I$ of the algebra $\mathbb{S}_n$ is the sum of idempotent ideals $\mathfrak{p}_I$. Therefore, $\mathfrak{a}^2_{n,s}=\mathfrak{a}_{n,s}$. By [5, Corollary 7.4. (3)], $\mathfrak{a}\cap\mathfrak{b}=\mathfrak{a}\mathfrak{b}$ for all idempotent ideals $\mathfrak{a}$ and $\mathfrak{b}$ of the algebra $\mathbb{S}_n$. Since the ideals $\mathfrak{p}_n$ and $\mathfrak{a}_{n,s}$ of the algebra $\mathbb{S}_n$ are idempotent,

$$\mathfrak{p}_n\cap\mathfrak{a}_{n,s}=\mathfrak{p}_n\mathfrak{a}_{n,s}=\sum_{|I|=s}\mathfrak{p}_n\mathfrak{p}_I=\sum_{|I|=s,n\in I}\mathfrak{p}_I. \tag{25}$$

Thus $(1+\mathfrak{p}_n)^*_s=(1+\mathfrak{p}_n)^*\cap(1+\mathfrak{a}_{n,s})^*=(1+\mathfrak{p}_n\cap\mathfrak{a}_{n,s})^*=(1+\sum_{|I|=s,n\in I}\mathfrak{p}_I)^*$. $\square$

For each number $s=1,\dots,n-1$, consider the following subset of $\mathbb{E}_{n,s}$,

$$\widetilde{\mathbb{E}}_{n,s}=\prod_{|I|=s,n\in I}U_I(K)\ltimes E_\infty(\mathbb{S}_{CI}),$$

where the groups $U_I=U_I(K)$ and $E_\infty(\mathbb{S}_{CI})$ are defined in (13). This is the product of the subgroups $U_I(K)\ltimes E_\infty(\mathbb{S}_{CI})$ of $(1+\mathfrak{p}_n)^*_s$ in arbitrary order, but which is assumed to be fixed. Notice that $\widetilde{\mathbb{E}}_{n,1}=U_n(K)\ltimes E_\infty(\mathbb{S}_{n-1})$, where $U_n(K)=\{\mu_n(\lambda)=\lambda e_n+1-e_n=\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}\mid\lambda\in K^*\}$, and $E_\infty(\mathbb{S}_{n-1})$ is the subgroup of $\mathrm{GL}_\infty(\mathbb{S}_{n-1})$ generated by all the elementary matrices.

Consider the element $\mu_I(\lambda)=\lambda e_I+1-e_I\in U_I$, where $|I|=s$ and $n\in I$. Then

$$\mu_I(\lambda)=e_n(1+(\lambda-1)e_{I\setminus n})+1-e_n=\begin{pmatrix}1+(\lambda-1)e_{I\setminus n} & 0\\ 0 & 1\end{pmatrix}\in\mathrm{GL}_\infty(\mathbb{S}_{n-1}). \tag{26}$$

Lemma 3.2. *$E_\infty(\mathbb{S}_{n-1})\supseteq\widetilde{\mathbb{E}}_{n,s}$ for all $s=2,\dots,n-1$.*

PROOF. It is sufficient to show that the group $E_\infty(\mathbb{S}_{n-1})$ of elementary matrices contains the groups $E_\infty(\mathbb{S}_{CI})$ and $U_I(K)$, where $|I|=s$ and $n\in I$. The group $E_\infty(\mathbb{S}_{CI})$ is generated by the elementary matrices $u=1+aE_{\alpha\beta}(I)$, where $a\in\mathbb{S}_{CI}$, $\alpha=(\alpha_i)_{i\in I}$, $\beta=(\beta_i)_{i\in I}\in\mathbb{N}^I$ and $\alpha\neq\beta$. If $\alpha_n\neq\beta_n$, then $u=1+(a\prod_{i\in I,i\neq n}E_{\alpha_i\beta_i}(i))E_{\alpha_n\beta_n}(n)\in E_\infty(\mathbb{S}_{n-1})$. If $\alpha_n=\beta_n$, then choose an element $\gamma\in\mathbb{N}^I$ such that $\gamma_n\neq\alpha_n$, and so $\gamma\neq\alpha$ and $\gamma\neq\beta$. Since the elements

$1 + E_{\alpha\gamma}$ and $1 + aE_{\gamma\beta}$ belong to the group $E_\infty(\mathbb{S}_{n-1})$ (by the previous case), so does their group commutator

$$E_\infty(\mathbb{S}_{n-1}) \ni [1 + E_{\alpha\gamma}, 1 + aE_{\gamma\beta}] = 1 + aE_{\alpha\beta} = u.$$

Therefore, $E_\infty(\mathbb{S}_{CI}) \subseteq E_\infty(\mathbb{S}_{n-1})$.

It remains to show that $U_I(K) \subseteq E_\infty(\mathbb{S}_{n-1})$, i.e. $\mu_I(\lambda) = 1 + \lambda E_{00}(I) \in E_\infty(\mathbb{S}_{n-1})$ for all scalars $\lambda \in K\backslash\{-1\}$. Notice that $n \in I$ and $|I| = s \geq 2$. Choose an element, say $m \in I$, distinct from n. In the subgroup $\mathrm{GL}_\infty(\mathbb{S}_1(m))$ of $\mathrm{GL}_\infty(\mathbb{S}_{n-1})$, we have for all scalars $\lambda \in K\backslash\{-1\}$ the equality:

$$\begin{pmatrix} 1 & 0 \\ -\frac{y_m}{1+\lambda} & 1 \end{pmatrix} \begin{pmatrix} 1 & \lambda x_m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y_m & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{\lambda}{1+\lambda} x_m \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1+\lambda & 0 \\ 0 & \frac{1}{1+\lambda} \end{pmatrix} \begin{pmatrix} 1 - \frac{\lambda E_{00}(m)}{1+\lambda} & 0 \\ 0 & 1 \end{pmatrix}. \quad (27)$$

This can be checked by direct multiplication, using the equalities $y_m x_m = 1$ and $x_m y_m = 1 - E_{00}(m)$ that hold in the algebra $\mathbb{S}_1(m)$. The first five matrices in the equality belong to the group $E_\infty(\mathbb{S}_1(m))$. Therefore, the last matrix $c = \begin{pmatrix} 1 - \frac{\lambda E_{00}(m)}{1+\lambda} & 0 \\ 0 & 1 \end{pmatrix}$ belongs to the group $E_\infty(\mathbb{S}_1(m))$. The idempotent

$$e := \begin{cases} \prod_{i \in I\backslash\{n,m\}} E_{00}(i) & \text{if } |I| > 2, \\ 1 & \text{if } |I| = 2, \end{cases}$$

determines the group monomorphism

$$\tau_e : \mathrm{GL}_\infty(\mathbb{S}_1(m)) = (1 + \sum_{i,j \in \mathbb{N}} \mathbb{S}_1(m) E_{ij}(m))^* \to \mathrm{GL}_\infty(\mathbb{S}_{n-1})$$
$$= (1 + \mathfrak{p}_n)^*, \quad u \mapsto eu + 1 - e, \quad (28)$$

that maps the group $E_\infty(\mathbb{S}_1(m))$ into the group $E_\infty(\mathbb{S}_{n-1})$. Therefore,

$$\tau_e(c) = e(E_{00}(n)(1 - \frac{\lambda}{1+\lambda} E_{00}(m)) + 1 - eE_{00}(n)) + 1 - e$$
$$= 1 - \frac{\lambda}{1+\lambda} E_{00}(I) = \mu_I(-\frac{\lambda}{1+\lambda}) \in E_\infty(\mathbb{S}_{n-1}) \cap U_K(I).$$

Since the map $\varphi : K\backslash\{-1\} \to K\backslash\{-1\}$, $\lambda \mapsto -\frac{\lambda}{1+\lambda}$, is a bijection ($\varphi^{-1} = \varphi$), all the elements $\mu_I(\lambda)$ belong to the group $E_\infty(\mathbb{S}_{n-1})$. The proof of the lemma is complete. □

By (10), there is the group monomorphism

$$\varphi_{n,s} : \frac{(1+\mathfrak{p}_n)^*_s}{(1+\mathfrak{p}_n)^*_{s+1}} \to \frac{(1+\mathfrak{a}_{n,s})^*}{(1+\mathfrak{a}_{n,s+1})^*} \to \prod_{|I|=s} (1+\overline{\mathfrak{p}}_I)^* = \prod_{|I|=s, n\in I} (1+\overline{\mathfrak{p}}_I)^* \times \prod_{|I'|=s, n\notin I'} (1+\overline{\mathfrak{p}}_{I'})^*,$$

which is the composition of two group monomorphisms. By Lemma 3.1,

$$\mathrm{im}(\varphi_{n,s}) \subseteq \prod_{|I|=s, n\in I} (1+\overline{\mathfrak{p}}_I)^*. \tag{29}$$

Recall that $(1+\overline{\mathfrak{p}}_I)^* = (\mathbb{X}_{CI} \times U_I) \ltimes E_\infty(L_{CI})$. Since $\varphi_{n,s}(\widetilde{\mathbb{E}}_{n,s}(1+\mathfrak{p}_n)^*_{s+1}) = \prod_{|I|=s,n\in I} U_I \ltimes E_\infty(L_{CI})$, we see that

$$\varphi^{-1}_{n,s}(\overline{\Gamma}_{n,s}) = \varphi^{-1}_{n,s}(\mathrm{im}(\varphi_{n,s}) \cap \overline{\Gamma}_{n,s}) = \varphi^{-1}_{n,s}\left(\prod_{|I|=s,n\in I} U_I \ltimes E_\infty(L_{CI})\right) = \widetilde{\mathbb{E}}_{n,s}(1+\mathfrak{p}_n)^*_{s+1},$$

and so there is the group monomorphism

$$\overline{\varphi}_{n,s} : (1+\mathfrak{p}_n)^*_s/\widetilde{\mathbb{E}}_{n,s}(1+\mathfrak{p}_n)^*_{s+1} \to (1+\mathfrak{a}_{n,s})^*/\Gamma_{n,s} \simeq \mathbb{X}_{n,s} \simeq \mathbb{X}'_{n,s} = \prod_{|I|=s,n\in I} \mathbb{X}'_{CI} \times \prod_{|I'|=s,n\notin I'} \mathbb{X}'_{CI'}.$$

Notice that the group $\widetilde{\mathbb{E}}_{n,s}(1+\mathfrak{p}_n)^*_{s+1}$ is a normal subgroup of $(1+\mathfrak{p}_n)^*_s$. For each number $s = 2, \ldots, n-1$, in the set $\Theta'_{n,s}$ consider the exact product of cyclic groups (the order is arbitrary)

$$\widetilde{\Theta}_{n,s} := \prod_{|J|=s+1, n\in J} \prod_{j\in J\setminus\{n,m(J)\}} \langle \theta_{m(J),j}(J) \rangle, \tag{30}$$

where $m(J)$ is the largest element of the set $J\setminus n$. Instead of the element $m(J)$, we can choose an arbitrary element of the set $J\setminus n$. By (29), $\mathrm{im}(\overline{\varphi}_{n,s}) \subseteq \prod_{|I|=s,n\in I} \mathbb{X}'_{CI}$. Recall that $\mathrm{im}(\psi'_{n,s}) = \psi'_{n,s}(\Theta_{n,s}) = \mathbb{K}_{n,s} = \bigcap_{|J|=s+1} \ker(\chi'_J)$, by Theorem 2.4 and (21). The following argument is the key moment in the proof of Theorem 3.3,

$$\mathrm{im}(\overline{\varphi}_{n,s}) \subseteq \mathrm{im}(\psi'_{n,s}) \bigcap \prod_{|I|=s,n\in I} \mathbb{X}'_{CI} = \bigcap_{|J|=s+1} \ker(\chi'_J) \bigcap \prod_{|I|=s,n\in I} \mathbb{X}'_{CI} =$$

$$= \begin{cases} 0 & \text{if } s=1, \\ \prod_{|J|=s+1, n\in J} \prod_{j\in J\setminus\{n,m(J)\}} \mathbb{Z}(-(m(J), J\backslash m(J))+(j, J\backslash j)) & \text{if } s=2,\dots,n-1, \end{cases}$$

$$\stackrel{\text{by(20)}}{=} \begin{cases} 0 & \text{if } s=1, \\ \psi'_{n,s}(\widetilde{\Theta}_{n,s}) & \text{if } s=2,\dots,n-1, \end{cases}$$

$$= \begin{cases} 0 & \text{if } s=1, \\ \overline{\varphi}'_{n,s}(\widetilde{\Theta}_{n,s}\widetilde{\mathbb{E}}_{n,s}(1+\mathfrak{p}_n)^*_{s+1}) & \text{if } s=2,\dots,n-1. \end{cases}$$

It follows that

$$(1+\mathfrak{p}_n)^*_s = \begin{cases} \widetilde{\mathbb{E}}_{n,1}(1+\mathfrak{p}_n)^*_2 & \text{if } s=1, \\ \widetilde{\Theta}_{n,s} \times_{ex} \widetilde{\mathbb{E}}_{n,s}(1+\mathfrak{p}_n)^*_{s+1} & \text{if } s=2,\dots,n-1. \end{cases} \tag{31}$$

Theorem 3.3. *The group* $\mathrm{GL}_\infty(\mathbb{S}_{n-1}) = (1+\mathfrak{p}_n)^*$ *is equal to* $\widetilde{\mathbb{E}}_{n,1}\widetilde{\Theta}_{n,2}\widetilde{\mathbb{E}}_{n,2}\cdots\widetilde{\Theta}_{n,n-1}\widetilde{\mathbb{E}}_{n,n-1}$. *Moreover,*

$$(1+\mathfrak{p}_n)^*_s = \begin{cases} \widetilde{\mathbb{E}}_{n,1}\widetilde{\Theta}_{n,2}\widetilde{\mathbb{E}}_{n,2}\cdots\widetilde{\Theta}_{n,n-1}\widetilde{\mathbb{E}}_{n,n-1} & \text{if } s=1, \\ \widetilde{\Theta}_{n,s}\widetilde{\mathbb{E}}_{n,s}\cdots\widetilde{\Theta}_{n,n-1}\widetilde{\mathbb{E}}_{n,n-1} & \text{if } s=2,\dots,n-1, \\ (1+F_n)^* & \text{if } s=n. \end{cases}$$

Proof. By [8, Proposition 3.10], we have the inclusion $(1+\mathfrak{p}_n)^*_n = (1+F_n)^* \subseteq \widetilde{\mathbb{E}}_{n,n-1}$. Now, the theorem follows from (31). □

For each subset J of the set $\{1,\dots,n\}$ such that $n\in J$ and $|J|\geq 3$, and for each pair of distinct elements i and j of the set $J\backslash n$, the unit $\theta_{ij}(J)\in\mathbb{S}^*_n$ can be written as follows

$$\begin{aligned}\theta_{ij}(J) &= (y_i e_{J\backslash i} e_n + 1 - e_n + e_n(1-e_{J\backslash i}))(x_j e_{J\backslash j} e_n + 1 - e_n + e_n(1-e_{J\backslash j})) \\ &= e_n(y_i e_{J\backslash i} + 1 - e_{J\backslash i})(x_j e_{J\backslash j} + 1 - e_{J\backslash j}) + 1 - e_n = e_n\theta_{ij}(J\backslash n) + 1 - e_n,\end{aligned}$$

where $e_n := E_{00}(n)$, $e_{J\backslash i} := \prod_{k\in J\backslash i} E_{00}(k)$ and $e_{J\backslash j} := \prod_{k\in J\backslash j} E_{00}(k)$. Therefore, the unit $\theta_{ij}(J)$, as an element of the group $\mathrm{GL}_\infty(\mathbb{S}_{n-1})$, is the matrix

$$\theta_{ij}(J) = \begin{pmatrix} \theta_{ij}(J\backslash n) & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_\infty(\mathbb{S}_{n-1}), \tag{32}$$

where $\theta_{ij}(J\backslash n)\in\mathbb{S}^*_{n-1}$.

The determinant $\overline{\det}$ on $\mathrm{GL}_\infty(\mathbb{S}_{n-1})$. The algebra epimorphism $\mathbb{S}_{n-1}\to\mathbb{S}_{n-1}/\mathfrak{a}_{n-1} = L_{n-1}$, $a\mapsto\overline{a} := a+\mathfrak{a}_{n-1}$, yields the group homomorphisms $\mathrm{GL}_\infty(\mathbb{S}_{n-1})\to\mathrm{GL}_\infty(L_{n-1})$, $u\mapsto\overline{u}$, and $\overline{\det}: \mathrm{GL}_\infty(\mathbb{S}_{n-1})\to\mathrm{GL}_\infty(L_{n-1}) \stackrel{\det}{\to}$

L^*_{n-1}. Clearly, $\overline{\det}(E_\infty(\mathbb{S}_{n-1})) = 1$, $\overline{\det}(\widetilde{\Theta}_{n,s}) = 1$ for all $s = 2, \dots, n-1$, and $\overline{\det}(U_n(K)) = K^*$, since $\overline{\det}(\mu_n(\lambda)) = \lambda$ for all $\lambda \in K^*$. By Theorem 3.3 and Lemma 3.2, $\mathrm{GL}_\infty(\mathbb{S}_{n-1}) = U_n(K)\widetilde{\Theta}_{n,2}\cdots\widetilde{\Theta}_{n,n-1}E_\infty(\mathbb{S}_{n-1})$, since $E_\infty(\mathbb{S}_{n-1})$ is a normal subgroup of $\mathrm{GL}_\infty(\mathbb{S}_{n-1})$. It follows that the image of the map $\overline{\det}$ is K^*, i.e. we have the group epimorphism

$$\overline{\det} : \mathrm{GL}_\infty(\mathbb{S}_{n-1}) \to K^*, \;\; u \mapsto \det(\overline{u}), \tag{33}$$

and

$$\begin{aligned} \mathrm{GL}_\infty(\mathbb{S}_{n-1}) \;&= U_n(K) \ltimes \ker(\overline{\det}) \\ \mathrm{SL}_\infty(\mathbb{S}_{n-1}) := \ker(\overline{\det}) &= \widetilde{\Theta}_{n,2}\cdots\widetilde{\Theta}_{n,n-1}E_\infty(\mathbb{S}_{n-1}). \end{aligned} \tag{34}$$

Theorem 3.4. *$\widetilde{\Theta}_{n,s} \subseteq E_\infty(\mathbb{S}_{n-1})$ for all $s = 2, \dots, n-1$.*

The proof of Theorem 3.4 is not easy and is given in Section 4.

Theorem 3.5.

(1) *$\mathrm{GL}_\infty(\mathbb{S}_{n-1}) = U_n(K) \ltimes E_\infty(\mathbb{S}_{n-1})$ and $\mathrm{SL}_\infty(\mathbb{S}_{n-1}) = E_\infty(\mathbb{S}_{n-1})$, where $U_n(K) = \{\mu_n(\lambda) := 1 + (\lambda - 1)E_{00}(n) \,|\, \lambda \in K^*\}$. So, each element $a \in \mathrm{GL}_\infty(\mathbb{S}_{n-1})$ is the unique product $a = \mu_n(\lambda)e$, where $\lambda = \overline{\det}(a)$ and $e := \mu_n(\overline{\det}(a))^{-1}a \in E_\infty(\mathbb{S}_{n-1})$.*

(2) *$\mathrm{K}_1(\mathbb{S}_n) \simeq K^*$ for all $n \geq 1$.*

Proof. The theorem follows from Theorem 3.4 and (34). □

The number of generators $\theta_{\max(J),j}(J)$ in the block $\widetilde{\Theta}_{n+1,2}\cdots\widetilde{\Theta}_{n+1,n}$ for the group $\mathrm{GL}_\infty(\mathbb{S}_n) = U_{n+1}(K) \ltimes \widetilde{\Theta}_{n+1,2}\cdots\widetilde{\Theta}_{n+1,n}E_\infty(\mathbb{S}_n)$ is $\sum_{s=2}^n \binom{n}{s}(s-1) = (n-2)2^{n-1} + 1$ as the next lemma shows.

Lemma 3.6. *For each natural number $n \geq 2$, $\sum_{s=2}^n \binom{n}{s}(s-1) = (n-2)2^{n-1} + 1$.*

Proof. Taking the derivative of the polynomial $(1+x)^n = \sum_{s=0}^n \binom{n}{s}x^s$, we have the equality $n(1+x)^{n-1} = \sum_{s=1}^n \binom{n}{s}sx^{s-1}$. Then taking the difference of both equalities at $x = 1$, we obtain the result: $\sum_{s=2}^n \binom{n}{s}(s-1)-1 = n2^{n-1}-2^n = (n-2)2^{n-1}$. □

4. Proof of Theorem 3.4

The whole section is a proof of Theorem 3.4. The proof is constructive (but slightly technical) and split into a series of lemmas that produce more and more

sophisticated elementary matrices in $E_\infty(\mathbb{S}_{n-1})$. These elementary matrices are used to show that the elements of the sets $\widetilde{\Theta}_{n,s}$ are elementary matrices (Propositions 4.6 and 4.8).

Lemma 4.1. *Let D be a division ring, and let $\Lambda = D \oplus De$ be a ring over D such that $e^2 = e$ and $de = ed$ for all $d \in D$. Then*

(1) *the group of units Λ^* of the ring Λ is the semi-direct product $D^* \ltimes \Gamma$ of the group of units D^* of the ring D and the subgroup $\Gamma := \{1+\lambda e \,|\, \lambda \in D\backslash\{-1\}\}$ of Λ^*.*

(2) $(1+\lambda e)^{-1} = 1 - \frac{\lambda}{1+\lambda} e$ *for all elements* $\lambda \in D\backslash\{-1\}$.

(3) *The map* $\phi : D\backslash\{-1\} \to D\backslash\{-1\}$, $\lambda \mapsto -\frac{\lambda}{1+\lambda}$, *is a bijection with* $\phi^{-1} = \phi$.

(4) $(1-2e)^{-1} = 1 - 2e$.

Proof. Straightforward. □

We are interested in the rings Λ and their groups of units, since the algebra $K + M_\infty(\mathbb{S}_{n-1})$ of infinite dimensional matrices over the algebra $\mathbb{S}_{n-1}$ contains plenty of them, and as a result, the group $\mathrm{GL}_\infty(\mathbb{S}_{n-1})$ contains their groups of units.

Lemma 4.2. *Let $\mathbb{S}_1(\Lambda) = \Lambda\langle x, y \,|\, yx = 1\rangle$ be the algebra $\mathbb{S}_1$ over the ring Λ from Lemma 4.1. Then, for each element $\lambda \in D\backslash\{-1\}$,*

$$\begin{pmatrix} 1 & 0 \\ -\frac{y}{1+\lambda e} & 1 \end{pmatrix} \begin{pmatrix} 1 & \lambda e x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{\lambda e}{1+\lambda e} x \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1+\lambda e & 0 \\ 0 & \frac{1}{1+\lambda e} \end{pmatrix} \begin{pmatrix} 1 - \frac{\lambda}{1+\lambda} e E_{00} & 0 \\ 0 & 1 \end{pmatrix}, \quad (35)$$

where $E_{00} := 1 - xy$ (the element $1 + \lambda e$ is a unit of the algebra $\mathbb{S}_1(\Lambda)$, by Lemma 4.1).

Proof. The RHS of the equality (35) is the product of four matrices, say $A_1 \cdots A_4$.

$$A_1 A_2 A_3 = \begin{pmatrix} 1 & \lambda e x \\ -\frac{y}{1+\lambda e} & 1 - \frac{\lambda e}{1+\lambda e} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} 1+\lambda exy & \lambda e x \\ 0 & \frac{1}{1+\lambda e} \end{pmatrix},$$

$$A_1 \cdots A_4 = \begin{pmatrix} 1+\lambda exy & 0 \\ 0 & \frac{1}{1+\lambda e} \end{pmatrix},$$

since $(1+\lambda exy)\left(-\frac{\lambda e}{1+\lambda e}x\right)+\lambda ex = -\frac{\lambda e}{1+\lambda e}(1+\lambda e)x+\lambda ex = 0$. Now,

$$A_1 \cdots A_4 = \begin{pmatrix} 1+\lambda e & 0 \\ 0 & \frac{1}{1+\lambda e} \end{pmatrix} \begin{pmatrix} 1-\frac{\lambda e}{1+\lambda e}E_{00} & 0 \\ 0 & 1 \end{pmatrix}, \tag{36}$$

since $(1+\lambda e)\left(1-\frac{\lambda e}{1+\lambda e}E_{00}\right) = 1+\lambda e(1-E_{00}) = 1+\lambda exy$. Finally, the equality (35) follows from Lemma 4.1. (2), $\frac{\lambda e}{1+\lambda e} = \lambda e\left(1-\frac{\lambda}{1+\lambda}e\right) = \lambda\left(1-\frac{\lambda}{1+\lambda}\right)e = \frac{\lambda}{1+\lambda}e$. □

For each ring R and a natural number $m \geq 1$, $E_n(R)$ is the subgroup of $\mathrm{GL}_n(R)$ generated by all elementary matrices.

Lemma 4.3.

(1) $\begin{pmatrix} y & 0 \\ E_{00} & x \end{pmatrix} \in E_2(\mathbb{S}_1)$, *where* $E_{00} := 1-xy$.

(2) $\begin{pmatrix} x & E_{00} \\ 0 & y \end{pmatrix} = \begin{pmatrix} y & 0 \\ E_{00} & x \end{pmatrix}^{-1} \in E_2(\mathbb{S}_1)$.

Proof. (1) Using the equalities $yx = 1$ and $E_{00}x = 0$, we can easily check that

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1-x & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ E_{00} & x \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix} = \begin{pmatrix} 1-2E_{00} & 0 \\ 0 & 1 \end{pmatrix}. \tag{37}$$

By (27), the RHS is an element of the group $E_2(\mathbb{S}_1)$, since $\frac{-2}{1+(-2)} = 2$, and so statement 1 holds. (2) It is obvious. □

Let R be a ring and u be its unit. The 2×2 matrix $\begin{pmatrix} y & 0 \\ uE_{00} & x \end{pmatrix} \in M_2(\mathbb{S}_1(R))$ is invertible, where $E_{00} := 1-xy$. Moreover,

$$\begin{pmatrix} y & 0 \\ uE_{00} & x \end{pmatrix}^{-1} = \begin{pmatrix} x & u^{-1}E_{00} \\ 0 & y \end{pmatrix}. \tag{38}$$

Lemma 4.4. *Let the ring Λ be as in Lemma 4.1. Then, for each element $\lambda \in D\backslash\{-1\}$,*

$$\begin{pmatrix} y & 0 \\ (1+\lambda e)E_{00} & x \end{pmatrix} \in E_2(\mathbb{S}_1(\Lambda))$$

and

$$\begin{pmatrix} x & (1+\lambda e)^{-1}E_{00} \\ 0 & y \end{pmatrix} = \begin{pmatrix} y & 0 \\ (1+\lambda e)E_{00} & x \end{pmatrix}^{-1} \in E_2(\mathbb{S}_1(\Lambda)),$$

where $(1+\lambda e)^{-1} = 1-\frac{\lambda}{1+\lambda}e$ (by Lemma 4.1. (2)).

PROOF. It suffices to prove the first inclusion, since then the equality and the second inclusion follow from (38). Using the equalities $yx = 1$ and $E_{00}x = 0$, we can check that

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1-x & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ (1+\lambda e)E_{00} & x \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1-(2+\lambda e)E_{00} & 0 \\ 0 & 1 \end{pmatrix}, \tag{39}$$

$$\begin{pmatrix} 1-(2+\lambda e)E_{00} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-2E_{00} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1+\lambda e E_{00} & 0 \\ 0 & 1 \end{pmatrix}.$$

By (37), $\left(\begin{smallmatrix} 1-2E_{00} & 0 \\ 0 & 1 \end{smallmatrix}\right) \in E_2(\mathbb{S}_1)$, and then by (35), $\left(\begin{smallmatrix} 1+\lambda e E_{00} & 0 \\ 0 & 1 \end{smallmatrix}\right) \in E_2(\mathbb{S}_1(\Lambda))$, since $\lambda \in D\backslash\{-1\}$. Therefore, $\left(\begin{smallmatrix} y & 0 \\ (1+\lambda e)E_{00} & x \end{smallmatrix}\right) \in E_2(\mathbb{S}_1(\Lambda))$, by (39). □

Lemma 4.5. $\begin{pmatrix} 1+(y_2-1)x_1y_1 & 0 \\ e_2y_1 & x_2 \end{pmatrix} \in E_2(\mathbb{S}_2)$, *where* $e_2 := E_{00}(2) = 1-x_2y_2$.

PROOF. The statement follows from the equality

$$\begin{pmatrix} 1 & 0 \\ -x_2y_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & (y_2-1)x_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -(y_2-1)x_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & (y_2-1)(1-x_2)x_1 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1+(y_2-1)x_1y_1 & 0 \\ e_2y_1 & x_2 \end{pmatrix}, \tag{40}$$

which can be checked directly using the equalities $y_ix_i = 1$, $x_iy_i = 1-e_i$, $y_ie_i = 0$ and $e_ix_i = 0$, where $e_i := E_{00}(i)$. The RHS of the equality (40) is the product of five matrices $A_1 \cdots A_5$.

$$A_1A_2A_3 = \begin{pmatrix} 1 & (y_2-1)x_1 \\ -x_2y_1 & 1-x_2(y_2-1) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y_1 & 1 \end{pmatrix} = \begin{pmatrix} 1+(y_2-1)x_1y_1 & (y_2-1)x_1 \\ e_2y_1 & 1-x_2(y_2-1) \end{pmatrix},$$

since $-x_2y_1 + (1-x_2y_2)y_1 + x_2y_1 = e_2y_1$. Now,

$$A_1 \cdots A_4 = \begin{pmatrix} 1+(y_2-1)x_1y_1 & -(y_2-1)^2x_1 \\ e_2y_1 & 1-(x_2+e_2)(y_2-1) \end{pmatrix},$$

since $-y_1x_1e_2(y_2-1)+1-x_2(y_2-1) = 1-(x_2+e_2)(y_2-1)$. Finally, $A_1 \cdots A_5 = \begin{pmatrix} 1+(y_2-1)x_1y_1 & a \\ e_2y_1 & b \end{pmatrix}$, where

$$\begin{aligned} a &= (1+(y_2-1)x_1y_1)(y_2-1)(1-x_2)x_1 - (y_2-1)^2x_1 \\ &= (x_1+(y_2-1)x_1)(y_2-1)(1-x_2) - (y_2-1)^2x_1 \\ &= x_1(y_2-1)(y_2-1) - (y_2-1)^2x_1 = 0, \\ b &= 1-(x_2+e_2)(y_2-1)+e_2y_1(y_2-1)(1-x_2)x_1 \\ &= 1-x_2(y_2-1)-e_2(y_2-1)+e_2(y_2-1)-e_2(1-x_2) \\ &= 1-x_2y_2+x_2-e_2 = x_2. \end{aligned}$$ □

Proposition 4.6. $\begin{pmatrix}\theta_{12} & 0\\ 0 & 1\end{pmatrix} \in E_2(\mathbb{S}_2)$, *where* $\theta_{12} = \theta_{12}(\{1,2\}) = (1+(y_1-1)e_2)(1+(x_2-1)e_1)$, $e_1 = E_{00}(1)$ *and* $e_2 = E_{00}(2)$.

Proof. By Lemma 4.3, $\begin{pmatrix}x_2 & e_2\\ 0 & y_2\end{pmatrix} \in E_2(\mathbb{S}_1(2)) \subseteq E_2(\mathbb{S}_2)$. Then, by Lemma 4.5,

$$E_2(\mathbb{S}_2) \ni \begin{pmatrix}x_2 & e_2\\ 0 & y_2\end{pmatrix}\begin{pmatrix}1+(y_2-1)x_1y_1 & 0\\ e_2y_1 & x_2\end{pmatrix} = \begin{pmatrix}\theta_{12} & 0\\ 0 & 1\end{pmatrix}. \tag{41}$$

Indeed, let a be the $(1,1)$-entry of the product, then

$$\begin{aligned} a &= x_2(1+(y_2-1)x_1y_1) + e_2^2y_1 = x_2 + (x_2y_2-x_2)x_1y_1 + e_2y_1 \\ &= x_2e_1 + (1-e_2)(1-e_1) + e_2y_1 = 1 + (x_2-1)e_1 + (y_1-1)e_2 + e_1e_2 \\ &= (1+(y_1-1)e_2)(1+(x_2-1)e_1) = \theta_{12}, \end{aligned}$$

since $(y_1-1)e_2 \cdot (x_2-1)e_1 = (y_1-1)e_1 \cdot e_2(x_2-1) = (-e_1)\cdot(-e_2) = e_1e_2$. □

Lemma 4.7. *Let* $J = \{1,\dots,m\}$, *where* $m \geq 3$, *and let* $I = J\backslash\{1,2\}$. *Then*

$$\begin{pmatrix}1+(y_2-1)x_1y_1e_I & 0\\ e_2y_1e_I & 1+(x_2-1)e_I\end{pmatrix} \in E_2(\mathbb{S}_2(K\oplus Ke_I)),$$

where $e_2 := E_{00}(2)$ *and* $e_I := \prod_{k\in I} E_{00}(k)$.

Proof. The statement follows from the equality

$$\begin{aligned} &\begin{pmatrix}1 & 0\\ -x_2y_2e_I & 1\end{pmatrix}\begin{pmatrix}1 & (y_2-1)x_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ y_1e_I & 1\end{pmatrix}\begin{pmatrix}1 & -(y_2-1)x_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & (y_2-1)(1-x_2)x_1e_I\\ 0 & 1\end{pmatrix} \\ &= \begin{pmatrix}1+(y_2-1)x_1y_1e_I & 0\\ e_2y_1e_I & 1+(x_2-1)e_I\end{pmatrix}. \end{aligned} \tag{42}$$

The equality can be written shortly as $A_1 \cdots A_5 = A$. Then

$$A_2A_3A_4 = \begin{pmatrix} 1+(y_2-1)x_1y_1e_I & (y_2-1)x_1 \\ y_1e_I & 1 \end{pmatrix} \begin{pmatrix} 1 & -(y_2-1)x_1 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1+(y_2-1)x_1y_1e_I & -(y_2-1)^2x_1e_I \\ y_1e_I & 1-(y_2-1)e_I \end{pmatrix},$$

where we have used the fact that $y_1x_1 = 1$. Then

$$A_1 \cdots A_4 = \begin{pmatrix} 1+(y_2-1)x_1y_1e_I & -(y_2-1)^2x_1e_I \\ e_2y_1e_I & 1-(x_2+e_2)(y_2-1)e_I \end{pmatrix}.$$

In more detail, let (α, β) be the second row of the product. Using the fact that $y_1x_1 = 1$ and $e_I^2 = e_I$, we see that

$$\begin{aligned} \alpha &= -x_2y_1e_I(1+(y_2-1)x_1y_1e_I) + y_1e_I = (-x_2(y_1+(y_2-1)y_1)+y_1)e_I \\ &= (1-x_2y_2)y_1e_I = e_2y_1e_I, \\ \beta &= x_2y_1e_I(y_2-1)^2x_1e_I + 1 - (y_2-1)e_I = 1 + (x_2y_2 - x_2 - 1)(y_2-1)e_I \\ &= 1-(x_2+e_2)(y_2-1)e_I. \end{aligned}$$

Finally, $A_1 \cdots A_5 = \begin{pmatrix} 1+(y_2-1)x_1y_1e_I & a' \\ e_2y_1e_I & b' \end{pmatrix}$, where (below, we use the fact that $a = 0$ and $b = x_2$, see the proof of Lemma 4.5)

$$\begin{aligned} a' &= (1+(y_2-1)x_1y_1e_I)(y_2-1)(x_2-1)x_1e_I - (y_2-1)^2x_1e_I \\ &= \big((1+(y_2-1)x_1y_1)(y_2-1)(x_2-1)x_1 - (y_2-1)^2x_1\big)e_I = a \cdot e_I = 0 \cdot e_I = 0, \\ \beta &= 1-(x_2+e_2)(y_2-1)e_I + e_2y_1(y_2-1)(1-x_2)x_1e_I \\ &= 1 + \big(-1+1-(x_2+e_2)(y_2-1) + e_2y_1(y_2-1)(1-x_2)x_1\big)e_I \\ &= 1 + (-1+b)e_I = 1 + (x_2-1)e_I. \end{aligned}$$

The proof of the lemma is complete. □

Let $J = \{1, 2, \ldots, m\}$ and $m \geq 3$. By multiplying out, the element $\theta_{12}(J) = (1+(y_1-1)e_{J\setminus 1})(1+(x_2-1)e_{J\setminus 2}) \in \mathbb{S}_m^*$ can be written as the sum

$$\theta_{12}(J) = x_2e_1e_I + (1-e_1e_I)(1-e_2e_I) + y_1e_2e_I, \tag{43}$$

where $I := J\setminus\{1,2\}$.

Proposition 4.8. *Let $J = \{1, 2, \dots, m\}$, and $m \geq 3$. Then* $\begin{pmatrix} \theta_{12}(J) & 0 \\ 0 & 1 \end{pmatrix} \in E_2(\mathbb{S}_m)$.

Proof. We keep the notation of Lemma 4.7. By Lemma 4.3. (2) and Lemma 4.7, the product of the following two elementary matrices is also an elementary matrix,

$$E_2(\mathbb{S}_2) \ni \begin{pmatrix} x_2 & e_2 \\ 0 & y_2 \end{pmatrix} \begin{pmatrix} 1 + (y_2 - 1)x_1 y_1 e_I & 0 \\ e_2 y_1 e_I & 1 + (x_2 - 1)e_I \end{pmatrix} = \begin{pmatrix} \theta_{12}(J) + (x_2 - 1)(1 - e_I) & e_2(1 - e_I) \\ 0 & e_I + (1 - e_I)y_2 \end{pmatrix}. \quad (44)$$

Indeed, the LHS is a matrix of type $\begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}$ (since $y_2 e_2 = 0$), where

$$\begin{aligned}
\alpha &= x_2(1 + (y_2 - 1)x_1 y_1 e_I) + e_2 y_1 e_I = x_2 + (1 - e_2 - x_2)(1 - e_1)e_I + e_2 y_1 e_I \\
&= x_2 - x_2(1 - e_1)e_I + (1 - e_1)(1 - e_2)e_I + y_1 e_2 e_I \\
&= x_2(1 - e_I) + \big(x_2 e_1 e_I + (1 - e_2 e_I)(1 - e_1 e_I) + y_1 e_2 e_I\big) \\
&\quad + (1 - e_1)(1 - e_2)e_I - (1 - e_1 e_I)(1 - e_2 e_I) \\
&\overset{\text{by (43)}}{=} x_2(1 - e_I) + \theta_{12}(J) + e_I - e_1 e_I - e_2 e_I + e_J - 1 + e_1 e_I + e_2 e_I - e_J \\
&= \theta_{12}(J) + (x_2 - 1)(1 - e_I), \\
\beta &= y_2(1 + (x_2 - 1)e_I) = y_2 + (1 - y_2)e_I = e_I + (1 - e_I)y_2, \\
\gamma &= e_2(1 + (x_2 - 1)e_I) = e_2(1 - e_I),
\end{aligned}$$

since $e_2 x_2 = 0$. By (43),

$$\theta_{12}(J)(1 - e_I) = 1 - e_I. \quad (45)$$

Using (45), the RHS of (44) is equal to the product of two matrices

$$\begin{pmatrix} \theta_{12}(J) + (x_2 - 1)(1 - e_I) & e_2(1 - e_I) \\ 0 & e_I + (1 - e_I)y_2 \end{pmatrix} = \begin{pmatrix} \theta_{12}(J) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 + (x_2 - 1)(1 - e_I) & e_2(1 - e_I) \\ 0 & e_I + (1 - e_I)y_2 \end{pmatrix}.$$

In order to finish the proof of the proposition, it suffices to show that the last matrix is elementary. This follows from the next two equalities, as the last two matrices in the equality (47) belong to the group $E_2(\mathbb{S}_m)$, by Lemma 3.2.

$$\begin{pmatrix} 1 & -(x_2-1+2e_2)(1-e_I) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1+(x_2-1)(1-e_I) & e_2(1-e_I) \\ 0 & e_I+(1-e_I)y_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x_2 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & (1-y_2)(1-e_I) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1-(x_2-1)e_I & 1 \end{pmatrix} = \begin{pmatrix} 1-2e_2(1-e_I) & 0 \\ 0 & 1 \end{pmatrix}, \quad (46)$$

$$\begin{pmatrix} 1-2e_2(1-e_I) & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-2e_2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1-2e_2e_I & 0 \\ 0 & 1 \end{pmatrix}. \quad (47)$$

The equality (47) is obvious, and the equality (46) can be written in the form $A_1 \cdots A_5 = A$. Using the identities $e_2x_2 = 0$, $y_2x_2 = 1$, $e_I^2 = e_I$ and $(1-e_I)^2 = 1-e_I$, we see that

$$\begin{aligned} A_2A_3A_4 &= \begin{pmatrix} 1+(x_2-1)(1-e_I) & e_2(1-e_I) \\ 1+(x_2-1)e_I & e_I+(1-e_I)y_2 \end{pmatrix}\begin{pmatrix} 1 & (1-y_2)(1-e_I) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1+(x_2-1)(1-e_I) & (x_2-1+2e_2)(1-e_I) \\ 1+(x_2-1)e_I & 1 \end{pmatrix}. \end{aligned}$$

In more detail, let $(u,v)^t$ be the second column of the product of the two matrices in the middle. Then

$$\begin{aligned} u &= (1+(x_2-1)(1-e_I))(1-y_2)(1-e_I) + e_2(1-e_I) = (x_2(1-y_2)+e_2)(1-e_I) \\ &= (x_2-(1-e_2)+e_2)(1-e_I) = (x_2-1+2e_2)(1-e_I), \\ v &= (1+(x_2-1)e_I)(1-y_2)(1-e_I) + e_I + (1-e_I)y_2 \\ &= (1-y_2)(1-e_I) + e_I + (1-e_I)y_2 = 1. \end{aligned}$$

Finally,

$$A_2 \cdots A_5 = \begin{pmatrix} 1-2e_2(1-e_I) & (x_2-1+2e_2)(1-e_I) \\ 0 & 1 \end{pmatrix},$$

since $1+(x_2-1)(1-e_I)-(x_2-1+2e_2)(1-e_I)(1+(x_2-1)e_I) = 1+(x_2-1-x_2+1-2e_2)(1-e_I) = 1-2e_2(1-e_I)$. Now, (46) is obvious. The proof of the proposition is complete. □

Proof of Theorem 3.4. Notice that $\mathbb{S}_{n-1} \simeq \mathbb{S}_1^{\otimes(n-1)}$, and the symmetric group S_{n-1} is a subgroup of the group of automorphisms of the algebra $\mathbb{S}_{n-1}$ (it acts by permuting the tensor components). Then, the matrix $\begin{pmatrix} \theta_{ij}(J) & 0 \\ 0 & 1 \end{pmatrix}$ (where $J \subseteq \{1,\dots,n-1\}$ with $|J| \geq 2$) is elementary by Proposition 4.6 (when $|J| = 2$) and Proposition 4.8 (when $|J| > 2$). Now, Theorem 3.4 is obvious. □

5. The groups $\mathbf{K_1}(\mathbb{S}_n, \mathfrak{p})$ and $\mathbf{GL_\infty}(\mathbb{S}_n, \mathfrak{p})$, and their generators

In this section, explicit generators are found for the group $\mathrm{GL}_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$, where $\mathfrak{p}$ is an arbitrary nonzero idempotent prime ideal of the algebra $\mathbb{S}_{n-1}$, and it is proved that $\mathrm{K}_1(\mathbb{S}_{n-1}, \mathfrak{p}) \simeq \mathbb{Z}^{\binom{m}{2}} \times K^{*m}$ (Theorem 5.7), where m is the height of the ideal $\mathfrak{p}$.

For a ring A and an ideal $\mathfrak{a}$ of A, the normal subgroup of $\mathrm{GL}_\infty(A)$,

$$\mathrm{GL}_\infty(A, \mathfrak{a}) := \ker(\mathrm{GL}_\infty(A) \to \mathrm{GL}_\infty(A/\mathfrak{a})),$$

is called the *congruence group* of level $\mathfrak{a}$. The normal subgroup $E_\infty(A, \mathfrak{a})$ of $E_\infty(A)$ which is generated by all the $\mathfrak{a}$-*elementary matrices* $(1 + aE_{ij},\ a \in \mathfrak{a},\ i \neq j)$ is a normal subgroup of $\mathrm{GL}_\infty(A)$. Moreover, $[\mathrm{GL}_\infty(A), \mathrm{GL}_\infty(A, \mathfrak{a})] = E_\infty(A, \mathfrak{a})$ [2], and so the K_1-group

$$\mathrm{K}_1(A, \mathfrak{a}) := \mathrm{GL}_\infty(A, \mathfrak{a})/E_\infty(A, \mathfrak{a})$$

is abelian. Let $E'_\infty(A, \mathfrak{a})$ be the subgroup of $E_\infty(A)$ generated by all the $\mathfrak{a}$-elementary matrices. Then $E'_\infty(A, \mathfrak{a}) \subseteq E_\infty(A, \mathfrak{a}) \subseteq E_\infty(A)$.

We keep the notation of the previous sections. Recall that we identified the groups $(1 + \mathfrak{p}_n)^*$ and $\mathrm{GL}_\infty(\mathbb{S}_{n-1})$. Each nonzero idempotent prime ideal $\mathfrak{p}$ of the algebra $\mathbb{S}_{n-1}$ is a *unique* sum (up to order) of distinct height one prime ideals $\mathfrak{p} = \mathfrak{p}_{i_1} + \cdots + \mathfrak{p}_{i_m}$ and $\mathrm{ht}(\mathfrak{p}) = m$, where ht stands for the *height* of an ideal, [5, Corollary 4.8]. The set $\mathrm{supp}(\mathfrak{p}) := \{i_1, \ldots, i_m\}$ is called the *support* of the idempotent prime ideal $\mathfrak{p}$. The group $\mathrm{GL}_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$ can be identified with the subgroup $(1 + \mathfrak{p}\mathfrak{p}_n)^*$ of the group $(1 + \mathfrak{a}_n)^*$. The group $(1 + \mathfrak{p}\mathfrak{p}_n)^*$ contains the descending chain of normal subgroups

$$\begin{aligned}(1 + \mathfrak{p}\mathfrak{p}_n)^* &= (1 + \mathfrak{p}\mathfrak{p}_n)^*_1 \supset \cdots \supset (1 + \mathfrak{p}\mathfrak{p}_n)^*_s \supset \cdots \supset (1 + \mathfrak{p}\mathfrak{p}_n)^*_n \\ &= (1 + F_n)^* \supset (1 + \mathfrak{p}\mathfrak{p}_n)^*_{n+1} = \{1\},\end{aligned}$$

where $(1 + \mathfrak{p}\mathfrak{p}_n)^*_s := (1 + \mathfrak{p}\mathfrak{p}_n)^* \cap (1 + \mathfrak{a}_{n,s})^*$. Moreover, the groups $(1 + \mathfrak{p}\mathfrak{p}_n)^*_s$ are normal subgroups of the group $(1 + \mathfrak{a}_n)^*$. The following lemma describes the normal subgroups $(1 + \mathfrak{p}\mathfrak{p}_n)^*_s$.

Lemma 5.1. *Let $\mathfrak{p} = \mathfrak{p}_{i_1} + \cdots + \mathfrak{p}_{i_m}$, where $i_1, \ldots, i_m$ are distinct elements of the set $\{1, \ldots, n\}$. Then*

$$(1 + \mathfrak{p}\mathfrak{p}_n)^*_s = \begin{cases} (1 + \sum_{|I|=s, I \in \mathcal{J}(\mathfrak{p})} \mathfrak{p}_I)^* & \text{if } s = 2, \ldots, n-1, \\ (1 + F_n)^* & \text{if } s = n, \end{cases}$$

where $\mathcal{J}(\mathfrak{p}) := \{J \subseteq \{1, \ldots, n\} \mid n \in J, J \cap \mathrm{supp}(\mathfrak{p}) \neq \emptyset\}$. In particular, $(1 + \mathfrak{p}\mathfrak{p}_n)^*_1 = (1 + \mathfrak{p}\mathfrak{p}_n)^*_2 = (1 + \mathfrak{p}\mathfrak{p}_n)^*$.

PROOF. The case $s = n$ is obvious. So, we assume that $s \neq n$. Since the ideals $\mathfrak{pp}_n$ and $\mathfrak{a}_{n,s}$ of the algebra $\mathbb{S}_n$ are idempotent ideals,

$$\mathfrak{pp}_n \cap \mathfrak{a}_{n,s} = \mathfrak{pp}_n \mathfrak{a}_{n,s} = \sum_{\nu=1}^{m} \mathfrak{p}_{i_\nu} \mathfrak{p}_n \mathfrak{a}_{n,s} = \sum_{|I|=s, I\in \mathcal{J}(\mathfrak{p})} \mathfrak{p}_I.$$

Therefore, $(1 + \mathfrak{pp}_n)^*_s = (1 + \mathfrak{pp}_n)^* \cap (1 + \mathfrak{a}_{n,s})^* = (1 + \mathfrak{pp}_n \cap \mathfrak{a}_{n,s})^* = (1 + \sum_{|I|=s, I\in \mathcal{J}(\mathfrak{p})} \mathfrak{p}_I)^*$. □

By (10), there is a group monomorphism

$$\varphi_{n,s} : \frac{(1 + \mathfrak{pp}_n)^*_s}{(1 + \mathfrak{pp}_n)^*_{s+1}} \to \frac{(1 + \mathfrak{a}_{n,s})^*}{(1 + \mathfrak{a}_{n,s+1})^*} \to \prod_{|I|=s} (1 + \overline{\mathfrak{p}}_I)^*$$
$$= \prod_{|I|=s, I\in \mathcal{J}(\mathfrak{p})} (1 + \overline{\mathfrak{p}}_I)^* \times \prod_{|I'|=s, I'\notin \mathcal{J}(\mathfrak{p})} (1 + \overline{\mathfrak{p}}_{I'})^*,$$

which is the composition of two group monomorphisms. By Lemma 5.1,

$$\mathrm{im}(\varphi_{n,s}) \subseteq \prod_{|I|=s, I\in \mathcal{J}(\mathfrak{p})} (1 + \overline{\mathfrak{p}}_I)^*. \tag{48}$$

For each number $s = 2, \ldots, n-1$, consider the following subset of the group $(1 + \mathfrak{pp}_n)^*$,

$$\widetilde{\mathbb{E}}_{n,s}(\mathfrak{p}) := \prod_{|I|=s, I\in \mathcal{J}(\mathfrak{p})} U_I \ltimes E_\infty(\mathbb{S}_{CI}).$$

It is a product of subgroups of $(1 + \mathfrak{pp}_n)^*_s$ in arbitrary order, but which is assumed to be fixed for each s.

Recall that $(1 + \overline{\mathfrak{p}}_I)^* = (\mathbb{X}_{CI} \times U_I) \ltimes E_\infty(L_{CI})$. Since $\varphi_{n,s}(\widetilde{\mathbb{E}}_{n,s}(\mathfrak{p})(1 + \mathfrak{pp}_n)^*_{s+1}) = \prod_{|I|=s, I\in \mathcal{J}(\mathfrak{p})} U_I \ltimes E_\infty(L_{CI})$, we see that there is the group monomorphism

$$\overline{\varphi}_{n,s} : \frac{(1 + \mathfrak{pp}_n)^*_s}{\widetilde{\mathbb{E}}_{n,s}(\mathfrak{p})(1 + \mathfrak{pp}_n)^*_{s+1}} \to \frac{(1 + \mathfrak{a}_{n,s})^*}{\Gamma_{n,s}} \simeq \mathbb{X}_{n,s} \simeq \mathbb{X}'_{n,s}$$
$$= \prod_{|I|=s, I\in \mathcal{J}(\mathfrak{p})} \mathbb{X}'_{CI} \times \prod_{|I'|=s, I'\in \mathcal{J}(\mathfrak{p})} \mathbb{X}'_{CI'}.$$

Notice that the group $\widetilde{\mathbb{E}}_{n,s}(\mathfrak{p})(1 + \mathfrak{pp}_n)^*_{s+1}$ is a normal subgroup of $(1 + \mathfrak{pp}_n)^*_s$. For each number $s = 2, \ldots, n-1$, in the set $\Theta'_{n,s}$ consider the exact product of cyclic groups (the order is arbitrary)

$$\widetilde{\Theta}_{n,s}(\mathfrak{p}) = \widetilde{\Theta}^{[1]}_{n,s}(\mathfrak{p}) \times_{ex} \widetilde{\Theta}^{[2]}_{n,s}(\mathfrak{p}), \tag{49}$$

$$\widetilde{\Theta}^{[1]}_{n,s}(\mathfrak{p}) :=^{\text{exact}} \prod_{i\in \mathrm{supp}(\mathfrak{p})} \prod_{|J'|=s+1, n\in J', J'\cap \mathrm{supp}(\mathfrak{p})=\{i\}} \prod_{j'\in J\setminus\{n,i,m'(J')\}} \langle \theta_{m'(J'),j'}(J')\rangle,$$

$$\widetilde{\Theta}^{[2]}_{n,s}(\mathfrak{p}) :=^{\text{exact}} \prod_{|J|=s+1, n\in J, J\cap \mathrm{supp}(\mathfrak{p})\geq 2} \prod_{j\in J\setminus\{n,m(J)\}} \langle \theta_{m(J),j}(J)\rangle,$$

where $m'(J')$ is the largest element of the set $J'\setminus\{n,i\}$, and $m(J)$ is the largest element of the set $J\setminus n$. Notice that $\widetilde{\Theta}_{n,2}(\mathfrak{p}) = \widetilde{\Theta}^{[2]}_{n,2}(\mathfrak{p})$ as the set $\widetilde{\Theta}^{[1]}_{n,2}(\mathfrak{p})$ is an empty set.

By (48), $\mathrm{im}(\overline{\varphi}_{n,s}) \subseteq \prod_{|I|=s, I\in\mathcal{J}(\mathfrak{p})} \mathbb{X}'_{CI}$, and

$$\begin{aligned}
\prod_{|I|=s, I\in\mathcal{J}(\mathfrak{p})} \mathbb{X}'_{CI} &= \prod_{|I|=s, I\in\mathcal{J}(\mathfrak{p})} \prod_{i\in CI} \mathbb{Z}(i, I) \\
&= \prod_{|J|=s+1, n\in J, J\cap\mathrm{supp}(\mathfrak{p})\neq\emptyset} \prod_{j\in J\setminus n, (J\setminus n)\cap\mathrm{supp}(\mathfrak{p})\neq\emptyset} \mathbb{Z}(j, J\setminus j) \\
&= \prod_{i\in\mathrm{supp}(\mathfrak{p})} \prod_{|J'|=s+1, n\in J', J'\cap\mathrm{supp}(\mathfrak{p})=\{i\}} \prod_{j'\in J'\setminus\{n,i\}} \mathbb{Z}(j', J'\setminus j') \\
&\quad \times \prod_{|J|=s+1, n\in J, |J\cap\mathrm{supp}(\mathfrak{p})|\geq 2} \prod_{j\in J\setminus n} \mathbb{Z}(j, J\setminus j).
\end{aligned}$$

Recall that $\mathrm{im}(\psi'_{n,s}) = \psi'_{n,s}(\Theta_{n,s}) = \mathbb{K}_{n,s} = \bigcap_{|J|=s+1} \ker(\chi'_J)$, by Theorem 2.4 and (21). The following argument is the key moment in the proof of Theorem 5.2. For each number $s = 2, \ldots, n-1$,

$$\begin{aligned}
\mathrm{im}(\overline{\varphi}_{n,s}) &\subseteq \mathrm{im}(\psi'_{n,s}) \bigcap \prod_{|I|=s, I\in\mathcal{J}(\mathfrak{p})} \mathbb{X}'_{CI} = \bigcap_{|I|=s+1} \ker(\chi'_J) \bigcap \prod_{|I|=s, I\in\mathcal{J}(\mathfrak{p})} \mathbb{X}'_{CI} \\
&= \prod_{i\in\mathrm{supp}(\mathfrak{p})} \prod_{|J'|=s+1, n\in J', J'\cap\mathrm{supp}(\mathfrak{p})=\{i\}} \\
&\quad \times \prod_{j'\in J\setminus\{n,i,m'(J')\}} \mathbb{Z}(-(m'(J'), J'\setminus m'(J')) + (j', J\setminus j')) \\
&\quad \times \prod_{|J|=s+1, n\in J, |J\cap\mathrm{supp}(\mathfrak{p})|\geq 2} \prod_{j\in J\setminus\{n,m(J)\}} \mathbb{Z}(-(m(J), J\setminus m(J)) + (j, J\setminus j)) \\
&\overset{\text{by (20)}}{=} \psi'_{n,s}(\widetilde{\Theta}_{n,s}(\mathfrak{p})) = \overline{\varphi}'_{n,s}(\widetilde{\Theta}_{n,s}(\mathfrak{p}) \widetilde{\mathbb{E}}_{n,s}(\mathfrak{p})(1+\mathfrak{p}\mathfrak{p}_n)^*_{s+1}).
\end{aligned}$$

The first equality above follows from the decomposition of the abelian group $\prod_{|I|=s, I\in\mathcal{J}(\mathfrak{p})} \mathbb{X}'_{CI}$ above, and the definition of the homomorphisms χ'_J. It follows that

$$(1+\mathfrak{p}\mathfrak{p}_n)^*_s = \widetilde{\Theta}_{n,s}(\mathfrak{p}) \times_{ex} \widetilde{\mathbb{E}}_{n,s}(\mathfrak{p})(1+\mathfrak{p}\mathfrak{p}_n)^*_{s+1}, \quad s = 2, \ldots, n-1. \tag{50}$$

Theorem 5.2. *Let $\mathfrak{p}$ be a nonzero idempotent prime ideal of the algebra $\mathbb{S}_{n-1}$. Then the group $\mathrm{GL}_\infty(\mathbb{S}_{n-1},\mathfrak{p}) = (1+\mathfrak{p}\mathfrak{p}_n)^*$ is equal to $\widetilde{\Theta}_{n,2}(\mathfrak{p})\widetilde{\mathbb{E}}_{n,2}(\mathfrak{p})\cdots\widetilde{\Theta}_{n,n-1}(\mathfrak{p})\widetilde{\mathbb{E}}_{n,n-1}(\mathfrak{p})$. Moreover,*

$$(1+\mathfrak{p}\mathfrak{p}_n)^*_s = \begin{cases} \widetilde{\Theta}_{n,2}(\mathfrak{p})\widetilde{\mathbb{E}}_{n,2}(\mathfrak{p})\cdots\widetilde{\Theta}_{n,n-1}(\mathfrak{p})\widetilde{\mathbb{E}}_{n,n-1}(\mathfrak{p}) & \text{if } s=1,\\ \widetilde{\Theta}_{n,s}(\mathfrak{p})\widetilde{\mathbb{E}}_{n,s}(\mathfrak{p})\cdots\widetilde{\Theta}_{n,n-1}(\mathfrak{p})\widetilde{\mathbb{E}}_{n,n-1}(\mathfrak{p}) & \text{if } s=2,\dots,n-1,\\ (1+F_n)^* & \text{if } s=n. \end{cases}$$

Proof. By [8, Proposition 3.10], we have the inclusion $(1+\mathfrak{p}\mathfrak{p}_n)^*_n = (1+F_n)^* \subseteq \widetilde{\mathbb{E}}_{n,n-1}(\mathfrak{p})$. Now, the theorem follows from (50). □

Lemma 5.3. *Let $\mathbb{S}_1(\Lambda)$ be the algebra $\mathbb{S}_1$ over the ring Λ from Lemma 4.1. Then, for each element $\lambda \in D\backslash\{-1\}$,*

$$\begin{pmatrix} 1 & 0\\ -\frac{ey}{1+\lambda e} & 1\end{pmatrix}\begin{pmatrix} 1 & \lambda ex\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0\\ ey & 1\end{pmatrix}\begin{pmatrix} 1 & -\frac{\lambda e}{1+\lambda e}x\\ 0 & 1\end{pmatrix}$$
$$= \begin{pmatrix} 1+\lambda e & 0\\ 0 & \frac{1}{1+\lambda e}\end{pmatrix}\begin{pmatrix} 1-\frac{\lambda}{1+\lambda}eE_{00} & 0\\ 0 & 1\end{pmatrix}, \quad (51)$$

where $E_{00} := 1-xy$ and $\frac{1}{1+\lambda e} = 1-\frac{\lambda}{1+\lambda}e$, by Lemma 4.1. (2).

Proof. The RHS of the equality (51) is the product of four matrices, say $A_1\cdots A_4$.

$$A_1A_2A_3 = \begin{pmatrix} 1 & \lambda ex\\ -\frac{ey}{1+\lambda e} & \frac{1}{1+\lambda e}\end{pmatrix}\begin{pmatrix} 1 & 0\\ ey & 1\end{pmatrix} = \begin{pmatrix} 1+\lambda exy & \lambda ex\\ 0 & \frac{1}{1+\lambda e}\end{pmatrix},$$
$$A_1\cdots A_4 = \begin{pmatrix} 1+\lambda exy & 0\\ 0 & \frac{1}{1+\lambda e}\end{pmatrix},$$

since $(1+\lambda exy)(-\frac{\lambda e}{1+\lambda e}x)+\lambda ex = -\frac{\lambda e}{1+\lambda e}(1+\lambda e)x+\lambda ex = 0$. The product $A_1\cdots A_4$ coincides with the product '$A_1\cdots A_4$' in the proof of Lemma 4.2, and so the equality (51) follows from (36). □

Lemma 5.4. *$E'_\infty(\mathbb{S}_{n-1},\mathfrak{p}) \supseteq \widetilde{\mathbb{E}}_{n,s}$ for all $s=3,\dots,n-1$, and $E'_\infty(\mathbb{S}_{n-1},\mathfrak{p}) \supseteq E_\infty(\mathbb{S}_{CI})$ for all sets $I \in \mathcal{J}(\mathfrak{p})$ such that $|I|=2$.*

Proof. We have to show that the group $E'_\infty(\mathbb{S}_{n-1},\mathfrak{p})$ contains the groups $E_\infty(\mathbb{S}_{CI})$ for all subsets $I \in \mathcal{J}(\mathfrak{p})$ such that $|I| = 2,\dots,n-1$, and the groups U_I for all subsets $I \in \mathcal{J}(\mathfrak{p})$ such that $|I| = 3,\dots,n-1$. By Lemma 5.3, the groups U_I belong to the group $E'_\infty(\mathbb{S}_{n-1},\mathfrak{p})$. Indeed, by (26), each element of the group U_I is

a matrix $u = \begin{pmatrix} 1+\mu e_{I\setminus n} & 0 \\ 0 & 1 \end{pmatrix}$ for some scalar $\mu \in K\setminus\{-1\}$. Since $I \in \mathcal{J}(\mathfrak{p})$ and $|I| \geq 3$, we can choose a number $j \in I\setminus n$ such that $(I\setminus\{j,n\})\cap\mathrm{supp}(\mathfrak{p}) \neq 0$. Then $e_{I\setminus n} = e \cdot E_{00}(j)$, where $e = e_{I\setminus\{j,n\}} \in \mathfrak{p}$. By Lemma 5.3, the matrix u belongs to the group $E_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$, since the map $\varphi : K\setminus\{-1\} \to K\setminus\{-1\}$, $\lambda \mapsto -\frac{\lambda}{1+\lambda}$, is a bijection.

The group $E_\infty(\mathbb{S}_{CI})$ is generated by the elementary matrices $u = 1+aE_{\alpha\beta}(I)$, where $a \in \mathbb{S}_{CI}$, $\alpha = (\alpha_i)_{i\in I}$, $\beta = (\beta_i)_{i\in I} \in \mathbb{N}^I$ and $\alpha \neq \beta$. If $\alpha_n \neq \beta_n$, then $u = 1+(a\prod_{i\in I, i\neq n} E_{\alpha_i\beta_i}(i))E_{\alpha_n\beta_n}(n) \in E'_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$, since $I \in \mathcal{J}(\mathfrak{p})$. If $\alpha_n = \beta_n$, then choose an element $\gamma \in \mathbb{N}^I$ such that $\gamma_n \neq \alpha_n$, and so $\gamma \neq \alpha$ and $\gamma \neq \beta$. Since the elements $1 + E_{\alpha\gamma}$ and $1 + aE_{\gamma\beta}$ belong to the group $E'_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$ (by the previous case), so does their group commutator

$$E_\infty(\mathbb{S}_{n-1}, \mathfrak{p}) \ni [1 + E_{\alpha\gamma}, 1 + aE_{\gamma\beta}] = 1 + aE_{\alpha\beta} = u.$$

Therefore, $E_\infty(\mathbb{S}_{CI}) \subseteq E'_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$. □

Lemma 5.5. *Let $J = \{i, j, n\}$, where the numbers i, j and n are distinct. Let $I = \{k, n\}$, where $k \neq n$ and $\lambda \in K^*$. Then*

$$[\theta_{ij}(J), \mu_I(\lambda)] = \begin{cases} 1 & \text{if } k \neq i, k \neq j, \\ 1 + (\lambda^{-1} - 1)e_J = \mu_J(\lambda)^{-1} & \text{if } k = i, \\ 1 + (\lambda - 1)E_{11}(j)e_i e_n & \text{if } k = j. \end{cases}$$

Proof. Let c be the group commutator, $J' = \{i, j\}$, $\theta_{ij} = \theta_{ij}(J)$ and $\theta'_{ij} = \theta_{ij}(J')$. Since $\theta_{ij}^{\pm1} e_n = e_n\theta_{ij}^{\pm1} = \theta_{ij}'^{\pm1} e_n = e_n\theta_{ij}'^{\pm1}$ and $\theta_{ij}'^{-1} = \theta'_{ji}$, we see that

$$c = \theta_{ij}(1 + (\lambda - 1)e_k e_n)\theta_{ij}^{-1}\mu_I^{-1}(\lambda) = (1 + (\lambda - 1)\theta'_{ij}e_k\theta'_{ji}e_n)\mu_I^{-1}(\lambda).$$

If $k \neq i$ and $k \neq j$, then the elements θ'_{ij} and e_k commute, and we get $c = \mu_I(\lambda)\mu_I(\lambda)^{-1} = 1$.

If $k = i$, then $\theta'_{ij}e_i = x_j e_i$ and $e_i\theta'_{ji} = e_i y_j$, by (43), and so

$$\begin{aligned} c &= (1 + (\lambda - 1)x_j y_j e_i e_n)\mu_I(\lambda)^{-1} = (\mu_I(\lambda) - (\lambda - 1)e_J)\mu_I(\lambda)^{-1} \\ &= 1 - (\lambda - 1)e_J(1 + (\lambda^{-1} - 1)e_I) = 1 - \frac{\lambda - 1}{\lambda}e_J = 1 + (\lambda^{-1} - 1)e_J = \mu_J(\lambda)^{-1}. \end{aligned}$$

If $k = j$, then $\theta'_{ij}e_j = y_i e_j + E_{10}(j)e_i$ and $e_j\theta'_{ji} = x_i e_j + E_{01}(j)e_i$, by (43), and so

$$\begin{aligned} c &= (1 + (\lambda - 1)(y_i e_j + E_{10}(j)e_i)(x_i e_j + E_{01}(j)e_i)e_n)\mu_I(\lambda)^{-1} \\ &= (\mu_I(\lambda) + (\lambda - 1)E_{11}(j)e_i e_n)\mu_I(\lambda)^{-1} = 1 + (\lambda - 1)E_{11}(j)e_i e_n. \end{aligned}$$

□

Let A and B be subgroups/subsets of a group G. The *commutant* $[A, B]$ is the subgroup of G generated by all the group commutators $[a, b] = aba^{-1}b^{-1}$, where $a \in A$ and $b \in B$. For an element $g \in G$, let $\omega_g : x \mapsto gxg^{-1}$ be the inner automorphism of the group G determined by the element g. We can easily verify that for all elements $a_1, a_2, b_1, b_2 \in G$,

$$[a_1b_1, a_2b_2] = \omega_{a_1}([b_1, a_2])\omega_{a_1a_2}([b_1, b_2])[a_1, a_2]\omega_{a_2}([a_1, b_2]). \tag{52}$$

The normal subgroup $\mathcal{E}(\mathbb{S}_{n-1}, \mathfrak{p})$. Consider the subgroup

$$\mathcal{E}(\mathbb{S}_{n-1}, \mathfrak{p}) := \prod_{|I|=2, I\in\mathcal{J}(\mathfrak{p})} E_\infty(\mathbb{S}_{CI}) \cdot (1 + \mathfrak{p}\mathfrak{p}_n)^*_3$$

of the group $(1 + \mathfrak{p}\mathfrak{p}_n)^* = \mathrm{GL}_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$. By (50), the group $(1 + \mathfrak{p}\mathfrak{p}_n)^*$ is the exact product of sets,

$$(1 + \mathfrak{p}\mathfrak{p}_n)^* = \widetilde{\Theta}_{n,2}(\mathfrak{p}) \times_{ex}{}^{exact} \prod_{|I|=2, I\in\mathcal{J}(\mathfrak{p})} U_I \times_{ex} \mathcal{E}(\mathbb{S}_{n-1}, \mathfrak{p}). \tag{53}$$

By the very definition, the subgroup $\mathcal{E}(\mathbb{S}_{n-1}, \mathfrak{p})$ is a *normal* subgroup of $(1+\mathfrak{p}\mathfrak{p}_n)^*$ (see the definition of the map $\varphi_{n,s}$). There is the inclusion

$$[\widetilde{\Theta}_{n,2}(\mathfrak{p}), \widetilde{\Theta}_{n,2}(\mathfrak{p})] \subseteq (1 + \mathfrak{p}\mathfrak{p}_n)^*_3, \tag{54}$$

which is obvious due to the fact that the image of each element $\theta_{ij}(J)$ (where $|J| = 3$ and $J \in \mathcal{J}(\mathfrak{p})$) under the map $\varphi_{n,s}$ is the direct product of two 'diagonal' matrices with entries in (commutative) Laurent polynomial algebras, hence all the images commute.

Theorem 5.6. $\mathcal{E}(\mathbb{S}_{n-1}, \mathfrak{p}) = E_\infty(\mathbb{S}_{n-1}, \mathfrak{p}) = E'_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$.

Proof. Recall that $\mathrm{GL}_\infty(R, \mathfrak{a})/E_\infty(R, \mathfrak{a})$ is an abelian group for any ring R and ideal $\mathfrak{a}$ of R, [2]. By (53), Lemma 5.5 and (54), the factor group $(1 + \mathfrak{p}\mathfrak{p}_n)^*/\mathcal{E}(\mathbb{S}_{n-1}, \mathfrak{p})$ is abelian.

Let us show that $E'_\infty(\mathbb{S}_{n-1}, \mathfrak{p}) \subseteq \mathcal{E} := \mathcal{E}(\mathbb{S}_{n-1}, \mathfrak{p})$. We have to show that $1 + \mathfrak{p}E_{ij}(n) \subseteq \mathcal{S}$ for all $i \neq j$. Since

$$1 + \mathfrak{p}E_{ij}(n) = 1 + (\mathfrak{p}_{i_1} + \cdots + \mathfrak{p}_{i_m})E_{ij}(n) = \prod_{\nu=1}^{m}(1 + \mathfrak{p}_{i_\nu}E_{ij}(n)),$$

it suffices to show that $1 + \mathfrak{p}_{i_\nu}E_{ij}(n) \subseteq \mathcal{E}$ for all $\nu = 1, \ldots, m$ and $i \neq j$, but this is obvious, since

$$1 + \mathfrak{p}_{i_\nu}E_{ij}(n) \subseteq E_\infty(\mathbb{S}_{CI}) \subseteq \mathcal{E},$$

where $I = \{i_\nu, n\}$ (and so $|I| = 2$ and $I \in \mathcal{S}(\mathfrak{p})$), see the definition of $\mathcal{E}$.

To finish the proof of the theorem, it suffices to show that $\mathcal{E}(\mathbb{S}_{n-1}, \mathfrak{p}) \subseteq E'_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$ (namely, then it follows that the group $E'_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$ is a normal subgroup of $\mathrm{GL}_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$ as $\mathrm{GL}_\infty(\mathbb{S}_{n-1}, \mathfrak{p})/\mathcal{E}$ is an abelian group and $\mathcal{E} \subseteq E'_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$; hence $E_\infty(\mathbb{S}_{n-1}, \mathfrak{p}) = E'_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$ and $E'_\infty(\mathbb{S}_{n-1}, \mathfrak{p}) = \mathcal{E}$). By Theorem 5.2,

$$\mathcal{E}(\mathbb{S}_{n-1}, \mathfrak{p}) = \prod_{|I|=2, I \in \mathcal{J}(\mathfrak{p})} E_\infty(\mathbb{S}_{CI}) \cdot \widetilde{\Theta}_{n,3}(\mathfrak{p})\widetilde{\mathbb{E}}_{n,3}(\mathfrak{p}) \cdots \widetilde{\Theta}_{n,n-1}(\mathfrak{p})\widetilde{\mathbb{E}}_{n,n-1}(\mathfrak{p}).$$

By Lemma 5.4, the inclusion $\mathcal{E}(\mathbb{S}_{n-1}, \mathfrak{p}) \subseteq E'_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$ holds iff $\widetilde{\Theta}_{n,s}(\mathfrak{p}) \subseteq E'_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$ for all $s = 3, \ldots, n-1$ iff $\widetilde{\Theta}^{[1]}_{n,s}(\mathfrak{p}), \widetilde{\Theta}^{[2]}_{n,s}(\mathfrak{p}) \subseteq E'_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$ for all $s = 3, \ldots, n-1$.

Fix an element θ such that either $\theta \in \widetilde{\Theta}^{[1]}_{n,s}(\mathfrak{p})$ or $\theta \in \widetilde{\Theta}^{[2]}_{n,s}(\mathfrak{p})$, i.e. either $\theta = \theta_{m'(J'),j'}(J')$ or $\theta = \theta_{m(J),j}(J)$, see (49). In the second case, without loss of generality, we may assume that $m(J) \notin J \cap \mathrm{supp}(\mathfrak{p})$, by changing, if necessary, the order in the set J (or simply by taking a suitable element). In both cases, we can choose an element, say $k \in J \cap \mathrm{supp}(\mathfrak{p})$, such that $k \notin \{m'(J'), j'\}$ in the first case, and $k \notin \{m(J), j\}$ in the second case. In both cases, we can write $\theta = \theta_{ij}(J)$, where $k \in J \cap \mathrm{supp}(\mathfrak{p})$ and $k \notin \{i, j\}$. As we have seen in Section 3,

$$\theta_{ij}(J) = e_k \theta_{ij}(J \backslash k) + 1 - e_k.$$

By Theorem 3.5, $\theta_{ij}(J\backslash k) \in E_\infty(\bigotimes_{l=1, l\neq k}^{n-1} \mathbb{S}_1(l)) \subseteq \mathrm{GL}_\infty(\bigotimes_{l=1, l\neq k}^{n-1} \mathbb{S}_1(l))$. Under the algebra monomorphism

$$\mathrm{GL}_\infty\left(\bigotimes_{l=1, l\neq k}^{n-1} \mathbb{S}_1(l)\right) \to \mathrm{GL}_\infty(\mathbb{S}_{n-1}, \mathfrak{p}), \quad a \mapsto e_k a + 1 - e_k,$$

the group of elementary matrices $E_\infty(\bigotimes_{l=1, l\neq k}^{n-1} \mathbb{S}_1(l))$ is mapped into the group of $\mathfrak{p}$-elementary matrices $E'_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$, since $e_k \in \mathfrak{p}$. Therefore, $\theta \in E'_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$. The proof of the theorem is complete. □

Theorem 5.7. *Let $\mathfrak{p}$ be a nonzero idempotent prime ideal of the algebra $\mathbb{S}_{n-1}$, and $m = \mathrm{ht}(\mathfrak{p})$ be its height. Then (below is the direct product of groups)*

$$\mathrm{K}_1(\mathbb{S}_{n-1}, \mathfrak{p}) \simeq \prod_{\{i>j \,|\, i,j \in \mathrm{supp}(\mathfrak{p})\}} \langle \theta_{ij}(\{i, j, n\}) \rangle \times \prod_{k \in \mathrm{supp}(\mathfrak{p})} U_{\{k,n\}}$$

$$\simeq \begin{cases} K^*, & \text{if } m = 1, \\ \mathbb{Z}^{\binom{m}{2}} \times K^{*m} & \text{if } m > 1. \end{cases}$$

The group $\mathrm{GL}_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$ *is generated by the elements* $\theta_{ij} := \theta_{ij}(\{i, j, n\})$ *(where* $i > j$ *and* $i, j \in \mathrm{supp}(\mathfrak{p})$*), and the groups* $E_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$, $U_{\{k,n\}}$, *where* $k \in \mathrm{supp}(\mathfrak{p})$. *Moreover, each element* a *of the group* $\mathrm{GL}_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$ *is the unique product (the order is arbitrary)*

$$a = \prod_{\{i>j \,|\, i,j\in \mathrm{supp}(\mathfrak{p})\}} \theta_{ij}^{n_{ij}} \cdot \prod_{k\in \mathrm{supp}(\mathfrak{p})} \mu_{\{k,n\}}(\lambda_k) \cdot e, \tag{55}$$

where $n_{ij} \in \mathbb{Z}$, $\lambda_k \in K^*$ *and* $e \in E_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$.

Proof. The theorem follows from the equality (53) and Theorem 5.6. □

We can find effectively (in finitely many steps) the decomposition (55) (Corollary 5.9). For, we introduce several explicit group homomorphisms.

Definition. For each nonempty subset I of $\{1, \dots, n\}$ with $s = |I| < n$, and for each element $j \in CI$, define the group homomorphism $\det_I : (1+\mathfrak{a}_{n,s})^* \to L^*_{CI}$ as the composition of the group homomorphisms (see (10))

$$(1 + \mathfrak{a}_{n,s})^* \stackrel{\psi_{n,s}}{\to} \prod_{|J|=s} (1 + \overline{\mathfrak{p}}_J)^* \stackrel{\mathrm{pr}_I}{\to} (1 + \overline{\mathfrak{p}}_I)^* \simeq \mathrm{GL}_\infty(L_{CI}) \stackrel{\det}{\to} L^*_{CI},$$

where pr_I is the projection map. Define the group homomorphism $\deg_{n,I,j} : (1 + \mathfrak{a}_{n,s})^* \to \mathbb{Z}$ as the composition of the group homomorphisms $(1 + \mathfrak{a}_{n,s})^* \stackrel{\det_I}{\to} L^*_{CI} \stackrel{\deg_{x_j}}{\to} \mathbb{Z}$, where $\deg_{x_j}$ is the degree in x_j of monomial ($\deg_{x_j}(\lambda \prod_{i\in CI} x_i^{\alpha_i}) = \alpha_j$, where $\lambda \in K^*$ and $\alpha_i \in \mathbb{Z}$).

Lemma 5.8. *Let* $n \geq 3$ *and* $s = 1, \dots, n-1$. *Then for all subsets* I *and* J *of the set* $\{1, \dots, n\}$ *such that* $|I| = s$, $|J| = s + 1$ *and* $n \in J$,

$$\deg_{n,I,i}(\theta_{m(J),j}(J)) = \begin{cases} -1 & \text{if } I = J\backslash m(J), i = m(J), \\ 1 & \text{if } I = J\backslash j, i = j, \\ 0 & \text{otherwise,} \end{cases}$$

where $i \in CI$ *and* $j \in J\backslash m(J)$.

Proof. The result follows at once from the equality $\theta_{m(J),j} = (1 + (y_{m(J)} - 1)e_{J\backslash m(J)})(1 + (x_j - 1)e_{J\backslash j})$. □

Corollary 5.9. *Given a product decomposition* (55) *for an element* $a \in \mathrm{GL}_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$, *we have*

$$n_{ij} = \deg_{n,\{i,n\},j}(a),$$

$$\lambda_k = \det_{\{k,n\}} \left(a \cdot \prod_{\{i>j \mid i,j \in \mathrm{supp}(\mathfrak{p})\}} \theta_{ij}^{-n_{ij}} \right),$$

$$e = \left(\prod_{\{i>j \mid i,j \in \mathrm{supp}(\mathfrak{p})\}} \theta_{ij}^{n_{ij}} \cdot \prod_{k \in \mathrm{supp}(\mathfrak{p})} \mu_{\{k,n\}}(\lambda_k) \right)^{-1} a.$$

PROOF. By Lemma 5.8, $\deg_{n,\{i,n\},j}(a) = n_{ij} \deg_{n,\{i,n\},j}(\theta_{ij}) = n_{ij}$. Similarly,

$$\det_{\{k,n\}} \left(a \cdot \prod_{\{i>j \mid i,j \in \mathrm{supp}(\mathfrak{p})\}} \theta_{ij}^{-n_{ij}} \right) = \det_{\{k,n\}}(\mu_{\{k,n\}}(\lambda_k)) = \lambda_k.$$

The rest is obvious. □

Corollary 5.9 gives an effective criterion of whether an element $a \in \mathrm{GL}_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$ is a product of $\mathfrak{p}$-elementary matrices.

Corollary 5.10. *Let $a \in \mathrm{GL}_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$. Then $a \in E_\infty(\mathbb{S}_{n-1}, \mathfrak{p})$ iff all $n_{ij} = 0$ and $\lambda_k = 1$, iff $\deg_{n,\{i,n\},j}(a) = 1$ for all $i > j$ such that $i, j \in \mathrm{supp}(\mathfrak{p})$, and $\det_{\{k,n\}}(a) = 1$ for all $k \in \mathrm{supp}(\mathfrak{p})$.*

References

[1] R. BAER, Inverses and zero-divisors, *Bull. Amer. Math. Soc.* **48** (1942), 630–638.

[2] H. BASS, Algebraic K-Theory, *W. A. Benjamin, Inc., New York – Amsterdam*, 1968.

[3] V. V. BAVULA, The Jacobian algebras, *J. Pure Appl. Algebra* **213** (2009), 664–685.

[4] V. V. BAVULA, The group of automorphisms of the first Weyl algebra in prime characteristic and the restriction map, *Glasgow Math. J.* **5** (2009), 263–274.

[5] V. V. BAVULA, The algebra of one-sided inverses of a polynomial algebra, *J. Pure Appl. Algebra* **214** (2010), 1874–1897.

[6] V. V. BAVULA, The group of automorphisms of the algebra of one-sided inverses of a polynomial algebra, *Münster J. Math.* **6** (2013), 1–51.

[7] V. V. BAVULA, $\mathrm{K}_1(S_1)$ and the group of automorphisms of the algebra S_2 of one-sided inverses of a polynomial algebra in two variables, *J. K-Theory* **10** (2012), 583–601.

[8] V. V. BAVULA, The group of automorphisms of the algebra of one-sided inverses of a polynomial algebra, II, *Proc. Edinburgh Math. Soc. (2)* **58** (2015), 543–580.

[9] V. V. BAVULA, The group of automorphisms of the Jacobian algebra A_n, *J. Pure Appl. Algebra* **216** (2012), 535–564.

[10] L. GERRITZEN, Modules over the algebra of the noncommutative equation $yx = 1$, *Arch. Math. (Basel)* **75** (2000), 98–112.

[11] N. Jacobson, Some remarks on one-sided inverses, *Proc. Amer. Math. Soc.* **1** (1950), 352–355.

[12] N. Jacobson, Structure of Rings, Am. Math. Soc. Colloq., Vol. **37** (rev. ed.), *Am. Math. Soc.*, *Providence*, 1964.

[13] J. Milnor, Introduction to Algebraic K-Theory, Annals of Mathematics Studies, Vol. **72**, *Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo*, 1971.

[14] R. G. Swan, Algebraic K-Theory, Lecture Notes in Mathematics, Vol. **76**, *Springer-Verlag*, *Berlin – New York*, 1968.

VLADIMIR V. BAVULA
DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF SHEFFIELD
HICKS BUILDING
SHEFFIELD S3 7RH
UK

E-mail: v.bavula@sheffield.ac.uk

(Received August 10, 2015; revised February 29, 2016)