Publ. Math. Debrecen
90/1-2 (2017), 125–147
DOI: 10.5486/PMD.2017.7494

Vanishing generalized Orlicz–Morrey spaces and fractional maximal operator

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Abstract. We find sufficient conditions for the non-triviality of the generalized Orlicz–Morrey spaces $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$, and prove the boundedness of the fractional maximal operator and its commutators with BMO-coefficients in vanishing generalized Orlicz–Morrey spaces $V\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ including weak versions of these spaces. The main advance in comparison with the existing results is that we manage to obtain conditions for the boundedness not in integral terms but in less restrictive terms of supremal operators involving the Young functions $\Phi(u), \Psi(u)$ and the function $\varphi(x,r)$ defining the space. No kind of monotonicity condition on $\varphi(x,r)$ in r is imposed.

1. Introduction

1.1. Some background. As is well-known, Morrey spaces are widely used to investigate the local behavior of solutions to second-order elliptic partial differential equations (PDE). Recall that the classical Morrey spaces $\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ are defined by

$$\mathcal{M}^{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathcal{M}^{p,\lambda}} := \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,r))} < \infty \right\},$$

Mathematics Subject Classification: 42B20, 42B25, 42B35, 46E30.
Key words and phrases: vanishing generalized Orlicz–Morrey space, fractional maximal operator, commutator, BMO.
The research of V. S. Guliyev and F. Deringoz was partially supported by the grant of Ahi Evran University Scientific Research Project (PYO.FEN.4001.14.017). The research of V. S. Guliyev was partially supported by the grant of Science Development Foundation under the President of the Republic of Azerbaijan, Grant EIF-2013-9(15)-46/10/1, and by the grant of Presidium Azerbaijan National Academy of Science 2015.

where $0 \leq \lambda \leq n$, $1 \leq p < \infty$.

Here and everywhere in the sequel, $B(x,r)$ is the ball in $\mathbb{R}^n$ of radius r centered at x, and $|B(x,r)|$ is its Lebesgue measure.

By $W\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ we denote the weak Morrey space defined as the set of functions f in the local weak space $WL^p_{\rm loc}(\mathbb{R}^n)$, for which

$$\|f\|_{W\mathcal{M}^{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL^p(B(x,r))} < \infty.$$

The spaces $\mathcal{M}^{p,\varphi}$ defined by the norm

$$\|f\|_{\mathcal{M}^{p,\varphi}} := \sup_{x\in\mathbb{R}^n,\, r>0} \frac{1}{\varphi(x,r)} \|f\|_{L^p(B(x,r))} \tag{1.1}$$

with a function φ positive on $\mathbb{R}^n \times (0,\infty)$ are known as generalized Morrey spaces.

Orlicz space was first introduced by Orlicz in [34], [35] as a generalization of Lebesgue spaces L^p. Since then, this space has been one of important functional frames in the mathematical analysis, and especially in real and harmonic analysis. Orlicz space is also an appropriate substitute for L^1 space when L^1 space does not work. For example, the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy \tag{1.2}$$

is bounded on L^p for $1 < p < \infty$, but not on L^1, yet using Orlicz spaces, we can investigate the boundedness of the maximal operator near $p = 1$, see [23], [24] and [6] for more precise statements.

1.2. On vanishing generalized Orlicz–Morrey spaces and the goal of the paper. A natural step in the theory of functions spaces was to study generalized Orlicz–Morrey spaces

$$\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n),$$

where the "Morrey-type measuring" of regularity of functions is realized with respect to the Orlicz norm over balls instead of the Lebesgue one. Such spaces were studied in [30] first, see also [9], [10], [20], [21], [31], [32], [42]. The most general spaces of such a type, Musielak–Orlicz–Morrey spaces, unifying the classical and variable exponent approaches, were an object of study in [29], where potential operators in such spaces were considered.

The generalized Orlicz–Morrey spaces we work with are precisely defined in Section 2.5. The weakest restrictions on the functions Φ and φ, defining the space $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$, for the boundedness of the maximal and singular operators, were provided in [10], up to the authors' knowledge, together with estimates in weak generalized Orlicz–Morrey spaces.

Our definition of generalized Orlicz–Morrey spaces introduced in [10] and used here is different from that of the papers [30], [42], and other papers.

Morrey and Orlicz–Morrey spaces are not separable due to the L^∞-norm with respect to r and x. The closure of nice functions in the Morrey or Orlicz–Morrey norm gives a subspace of the corresponding space. Such spaces corresponding to the classical Morrey space $\mathcal{M}^{p,\lambda}$, known under the name of vanishing Morrey space, appeared in connection with PDE in [46], [47], and they were also used in [38]. The vanishing generalized Morrey spaces were introduced and studied in [40], see also a study of commutators of Hardy operators in such spaces in [37].

Vanishing generalized Orlicz–Morrey spaces, including their weak versions, appeared in [14], where the boundedness of the so called Φ-admissible singular operators and their commutators was studied. Then the boundedness of the (Φ, Ψ)-admissible potential operators and their commutators on these spaces was investigated by the same authors in [15].

The results obtained in [15] for the fractional maximal operator and its commutators provide boundedness conditions in terms of some integral inequality.

The main goal of this paper is to find sufficient conditions for the nontriviality of the generalized Orlicz–Morrey spaces, and to show that the boundedness of the fractional maximal operator and its commutators in vanishing generalized Orlicz–Morrey spaces may be obtained under weaker conditions, namely in terms of the so-called supremal operators. More precisely, we find sufficient conditions on general Young functions Φ, Ψ and functions φ_1, φ_2, which ensure the boundedness of the operators under consideration from one vanishing Orlicz–Morrey space $V\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to another $V\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$ including weak estimates.

1.3. Operators to consider. Let $0 < \alpha < n$. The fractional maximal operator M_α and the Riesz potential operator I_α are defined by

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |f(y)| dy, \qquad I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

If $\alpha = 0$, then $M \equiv M_0$ is the Hardy–Littlewood maximal operator defined in (1.2).

It is known that the operator M_α is of weak type $(p, np/(n-\alpha p))$ if $1 \le p \le n/\alpha$, and of strong type $(p, np/(n-\alpha p))$ if $1 < p \le n/\alpha$. Also the operator I_α is of weak type $(p, np/(n-\alpha p))$ if $1 \le p < n/\alpha$, and of strong type $(p, np/(n-\alpha p))$ if $1 < p < n/\alpha$.

The boundedness of I_α from Orlicz space $L^\Phi(\mathbb{R}^n)$ to the corresponding another Orlicz space $L^\Psi(\mathbb{R}^n)$ was studied by [33] and [45] under some restrictions involving the growths and certain monotonicity properties of Φ and Ψ. Moreover, in [6], there were given necessary and sufficient conditions for the boundedness of both the operators M_α and I_α from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$, and also from $L^\Phi(\mathbb{R}^n)$ to the weak Orlicz space $WL^\Psi(\mathbb{R}^n)$.

For the boundedness of the operators M_α and I_α in Morrey spaces $\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$, see [36] and [1].

It is well-known that commutators of classical operators of harmonic analysis play an important role in various topics of analysis and PDE, see, for instance, [3], [4], [7] and [8].

The commutators generated by $b \in L^1_{\rm loc}(\mathbb{R}^n)$ and the operators M_α and I_α are defined by

$$M_{b,\alpha}(f)(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |b(x) - b(y)||f(y)|dy, \tag{1.3}$$

$$[b, I_\alpha]f(x) = \int_{\mathbb{R}^n} \frac{b(x) - b(y)}{|x-y|^{n-\alpha}} f(y)dy, \tag{1.4}$$

respectively.

The fractional maximal and potential operators in Morrey-type spaces were studied in various papers. In Orlicz–Morrey spaces they were recently studied in [19], [20]. Commutators in Morrey spaces were studied in a less generality. We refer, for instance, to [38] and [41] in the case of the classical Morrey spaces.

In the case of generalized Morrey spaces, we refer to [43] (where some monotonicity assumptions were imposed on the function φ, but on the other hand, an anisotropic case was admitted), and to [17], [18] (where no monotonicity assumptions on φ were imposed, including an anisotropic case in [18]), where other references may be also found.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Preliminaries

2.1. On commutators in Lebesgue spaces. We recall that the space $BMO(\mathbb{R}^n) = \{b \in L^1_{\rm loc}(\mathbb{R}^n) : \|b\|_* < \infty\}$ is defined by the seminorm

$$\|b\|_* := \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}|dy,$$

where $b_{B(x,r)} = \frac{1}{|B(x,r)|}\int_{B(x,r)} b(y)dy$, and refer, for instance, to [44] for details on this space.

The following characterization of the $L^p \to L^q$-boundedness of the commutator $[b, I_\alpha]$ was given in [5] in terms of mean oscillation of b.

Theorem 2.1 ([5]). *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then $[b, I_\alpha]$ is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $b \in BMO(\mathbb{R}^n)$.*

The statement of Theorem 2.1 remains valid for $M_{b,\alpha}$ and for the operator

$$|b, I_\alpha|f(x) := \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|}{|x-y|^{n-\alpha}} f(y)dy.$$

Namely, the following theorem holds.

Theorem 2.2. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then $M_{b,\alpha}$ and $|b, I_\alpha|$ are bounded operators from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $b \in BMO(\mathbb{R}^n)$.*

Proof. The sufficiency in the case of the operator $|b, I_\alpha|$ is known even in a more general case, see [11, Remark 3].

For the operator $M_{b,\alpha}$, it was formulated in [28, Theorem 3.5.2], however, its proof contains the unjustified inequality (3.5.7), valid if on the right-hand side one replaces $b(x) - b(y)$ by $|b(x) - b(y)|$ inside the integral. By this reason, for completeness of the proof, we justify the main statement in Theorem 2.2 using other means, following known ideas though.

The sufficiency for $M_{b,\alpha}$ in fact follows immediately from the known inequality

$$M_{b,\alpha}(f)(x) \lesssim \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|}{|x-y|^{n-\alpha}} |f(y)|dy, \tag{2.1}$$

and the above reference [11, Remark 3].

As regards the necessity, in view of the above inequality, it suffices to prove it for $M_{b,\alpha}$. Suppose that $M_{b,\alpha}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Choose any ball $B = B(x,r)$ in $\mathbb{R}^n$,

$$\begin{aligned}
&\frac{1}{|B|}\int_B |b(y) - b_B|dy \\
&\quad = \frac{1}{|B|}\int_B \Big|\frac{1}{|B|}\int_B (b(y) - b(z))dz\Big|dy \le \frac{1}{|B|^2}\int_B\int_B |b(y) - b(z)|dydz \\
&\quad = \frac{1}{|B|^{1+\frac{\alpha}{n}}}\int_B \frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B |b(y) - b(z)|\chi_B(z)dzdy \le \frac{1}{|B|^{1+\frac{\alpha}{n}}}\int_B M_{b,\alpha}(\chi_B)(y)dy \\
&\quad \le \frac{1}{|B|^{1+\frac{\alpha}{n}}}|B|^{\frac{1}{q'}}\left(\int_B \big(M_{b,\alpha}(\chi_B)(y)\big)^q dy\right)^{\frac{1}{q}} \le \frac{C}{|B|^{1+\frac{\alpha}{n}}}|B|^{\frac{1}{q'}}\,|B|^{\frac{1}{p}} = C.
\end{aligned}$$

Thus $b \in BMO(\mathbb{R}^n)$. $\square$

2.2. On Young functions. We start with recalling the definition of Young functions.

Definition 2.3. A function $\Phi : [0, +\infty) \to [0, \infty]$ is called a Young function if it is convex, left-continuous, $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim_{r\to+\infty} \Phi(r) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0, +\infty)$ such that $\Phi(s) = +\infty$, then $\Phi(r) = +\infty$ for $r \geq s$. The set of Young functions such that

$$0 < \Phi(r) < +\infty \quad \text{for} \quad 0 < r < +\infty$$

will be denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0, +\infty)$ and bijective from $[0, +\infty)$ to itself.

For a Young function Φ and $0 \leq s \leq +\infty$, let

$$\Phi^{-1}(s) = \inf\{r \geq 0 : \Phi(r) > s\}.$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. We note that

$$\Phi(\Phi^{-1}(r)) \leq r \leq \Phi^{-1}(\Phi(r)) \quad \text{for} \quad 0 \leq r < +\infty. \tag{2.2}$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted also as $\Phi \in \Delta_2$, if

$$\Phi(2r) \leq k\Phi(r) \quad \text{for} \quad r > 0,$$

for some $k > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \leq \frac{1}{2k}\Phi(kr), \quad r \geq 0,$$

for some $k > 1$. The function $\Phi(r) = r$ satisfies the Δ_2-condition but does not satisfy the ∇_2-condition. If $1 < p < \infty$, then $\Phi(r) = r^p$ satisfies both conditions. The function $\Phi(r) = e^r - r - 1$ satisfies the ∇_2-condition but does not satisfy the Δ_2-condition. A Young function Φ is said to satisfy the Δ'-condition, denoted also by $\Phi \in \Delta'$, if

$$\Phi(rt) \leq c\Phi(r)\Phi(t), \quad r, t \geq 0,$$

for some positive constant c. Note that each element of Δ'-class is also an element of Δ_2-class.

For a Young function Φ, the complementary function $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\} & \text{if } r \in [0,\infty) \\ +\infty & \text{if } r = +\infty. \end{cases}$$

The complementary function $\widetilde{\Phi}$ is also a Young function and $\widetilde{\widetilde{\Phi}} = \Phi$. If $\Phi(r) = r$, then $\widetilde{\Phi}(r) = 0$ for $0 \le r \le 1$, and $\widetilde{\Phi}(r) = +\infty$ for $r > 1$. If $1 < p < \infty$, $1/p + 1/p' = 1$ and $\Phi(r) = r^p/p$, then $\widetilde{\Phi}(r) = r^{p'}/p'$. If $\Phi(r) = e^r - r - 1$, then $\widetilde{\Phi}(r) = (1+r)\log(1+r) - r$. Note that $\Phi \in \nabla_2$ if and only if $\widetilde{\Phi} \in \Delta_2$. It is known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \quad \text{for} \quad r \ge 0. \tag{2.3}$$

Note that Young functions satisfy the properties $\Phi(\alpha t) \le \alpha\Phi(t)$ for all $0 \le \alpha \le 1$ and $0 \le t < \infty$, and $\Phi(\beta t) \ge \beta\Phi(t)$ for all $\beta > 1$ and $0 \le t < \infty$.

Definition 2.4. Let Φ be a Young function. Let

$$a_\Phi := \inf_{t \in (0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}, \quad b_\Phi := \sup_{t \in (0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}.$$

Remark 2.5. It is known that $\Phi \in \Delta_2 \cap \nabla_2$ if and only if $1 < a_\Phi \le b_\Phi < \infty$, see [25].

2.3. On Orlicz spaces.

Definition 2.6 (Orlicz space). For a Young function Φ, the set

$$L^\Phi(\mathbb{R}^n) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < +\infty \text{ for some } k > 0 \right\}$$

is called Orlicz space. If $\Phi(r) = r^p$, $1 \le p < \infty$, then $L^\Phi(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\Phi(r) = 0$, $(0 \le r \le 1)$ and $\Phi(r) = \infty$, $(r > 1)$, then $L^\Phi(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. The space $L^\Phi_{\mathrm{loc}}(\mathbb{R}^n)$ endowed with the natural topology is defined as the set of all functions f such that $f\chi_B \in L^\Phi(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$. We refer to the books [25], [26], [39] for the theory of Orlicz spaces.

$L^\Phi(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L^\Phi} = \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1 \right\}.$$

We note that

$$\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\|f\|_{L^\Phi}}\right) dx \le 1.$$

For a measurable function f and $t > 0$, let

$$m(f,t) = |\{x \in \mathbb{R}^n : \ |f(x)| > t\}|.$$

Definition 2.7. The weak Orlicz space is defined as $WL^{\Phi}(\mathbb{R}^n) := \{f \in L^1_{\rm loc}(\mathbb{R}^n) : \|f\|_{WL^{\Phi}} < +\infty\}$, where

$$\|f\|_{WL^{\Phi}} = \inf\left\{\lambda > 0 \;:\; \sup_{t>0} \Phi(t) m\Big(\frac{f}{\lambda},\; t\Big) \;\le 1\right\}.$$

The following theorem is an analogue of Lebesgue differentiation theorem in Orlicz spaces.

Theorem 2.8 ([22])**.** *Suppose that Φ is a Young function and let $f \in L^{\Phi}(\mathbb{R}^n)$ be nonnegative. Then*

$$\liminf_{r\to 0+} \frac{\|f\chi_{B(x,r)}\|_{L^{\Phi}}}{\|\chi_{B(x,r)}\|_{L^{\Phi}}} \ge f(x) \textit{ for almost every } x \in \mathbb{R}^n.$$

If we, moreover, assume that our Φ satisfies the Δ' condition, then

$$\lim_{r\to 0+} \frac{\|f\chi_{B(x,r)}\|_{L^{\Phi}}}{\|\chi_{B(x,r)}\|_{L^{\Phi}}} = f(x) \textit{ for almost every } x \in \mathbb{R}^n.$$

For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following analogue of the Hölder inequality is known:

$$\|fg\|_{L^1} \le 2\|f\|_{L^{\Phi}}\|g\|_{L^{\tilde{\Phi}}}. \tag{2.4}$$

The following lemma is valid.

Lemma 2.9 ([2], [27])**.** *Let Φ be a Young function and B a set in $\mathbb{R}^n$ with finite Lebesgue measure. Then*

$$\|\chi_B\|_{WL^{\Phi}} = \|\chi_B\|_{L^{\Phi}} = \frac{1}{\Phi^{-1}\left(|B|^{-1}\right)}.$$

2.4. On boundedness of the fractional maximal operator and its commutators in Orlicz spaces. Necessary and sufficient conditions on (Φ, Ψ) for the boundedness of M_α and I_α from $L^{\Phi}(\mathbb{R}^n)$ to $L^{\Psi}(\mathbb{R}^n)$ and $L^{\Phi}(\mathbb{R}^n)$ to $WL^{\Psi}(\mathbb{R}^n)$ have been obtained in [6, Theorem 1 and 2]. To formulate the results from [6], we recall that, for functions Φ and Ψ from $[0, \infty)$ into $[0, \infty]$, the function Ψ is said to dominate Φ globally if there exists a positive constant c such that $\Phi(s) \le \Psi(cs)$ for all $s \ge 0$.

In the theorems below we also use the notation

$$\widetilde{\Psi_P}(s) = \int_0^s r^{P'-1}(\mathcal{B}_P^{-1}(r^{P'}))^{P'}\,dr, \tag{2.5}$$

where $1 < P \le \infty$ and $\widetilde{\Psi_P}(s)$ is the Young conjugate function to $\Psi_P(s)$, and

$$\Phi_P(s) = \int_0^s r^{P'-1}(\mathcal{A}_P^{-1}(r^{P'}))^{P'}\,dr, \tag{2.6}$$

where $\mathcal{B}_P^{-1}(s)$ and $\mathcal{A}_P^{-1}(s)$ are inverses to

$$\mathcal{B}_P(s) = \int_0^s \frac{\Psi(t)}{t^{1+P'}}\,dt \quad \text{and} \quad \mathcal{A}_P(s) = \int_0^s \frac{\widetilde{\Phi}(t)}{t^{1+P'}}\,dt,$$

respectively. These functions $\Psi_P(s)$ and $\Phi_P(s)$ are used below with $P = \frac{n}{\alpha}$. In the case $P = \infty$, the function Ψ_∞ is interpreted as $s\int_0^s \frac{\Psi(t)}{t^2}dt$, see [6].

Theorem 2.10 ([6])**.** *Let Φ and Ψ be Young functions and $0 \le \alpha < n$. Then*

(i) *The fractional maximal operator M_α is bounded from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$ if and only if*

$$\Phi \text{ dominates globally the function } Q, \tag{2.7}$$

whose inverse is given by

$$Q^{-1}(r) = r^{\alpha/n}\Psi^{-1}(r).$$

(ii) *It is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$ if and only if*

$$\int_0^1 \frac{\Psi(t)}{t^{1+n/(n-\alpha)}}dt < \infty \text{ and } \Phi \text{ dominates globally the function } \Psi_{n/\alpha}. \tag{2.8}$$

Theorem 2.11 ([6])**.** *Let Φ and Ψ Young functions and $0 < \alpha < n$. Then*

(i) *The Riesz potential I_α is bounded from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$ if and only if*

$$\int_0^1 \widetilde{\Phi}(t)/t^{1+n/(n-\alpha)}dt < \infty \text{ and } \Phi_{n/\alpha} \text{ dominates } \Psi \text{ globally.} \tag{2.9}$$

(ii) *The Riesz potential I_α is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$ if and only if*

$$\int_0^1 \widetilde{\Phi}(t)/t^{1+n/(n-\alpha)}dt < \infty, \ \int_0^1 \Psi(t)/t^{1+n/(n-\alpha)}dt < \infty, \tag{2.10}$$

$$\text{and } \Phi \text{ dominates } \Psi_{n/\alpha} \text{ globally and } \Phi_{n/\alpha} \text{ dominates } \Psi \text{ globally.} \tag{2.11}$$

Remark 2.12. Note that condition (2.9) implies the condition (2.7). For the proof of this fact, see [20, Lemma 12].

The following lemma was proved in [19]. In the case of L^p-spaces it was proved in [16].

Lemma 2.13. *Let Φ and Ψ be Young functions and $0 \le \alpha < n$, $f \in L^{\Phi}_{\mathrm{loc}}(\mathbb{R}^n)$ and $B = B(x,r)$. If Φ and Ψ satisfy the conditions (2.7), then*

$$\|M_\alpha f\|_{WL^\Psi(B)} \lesssim \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \sup_{t>2r} \Psi^{-1}\big(t^{-n}\big)\, \|f\|_{L^\Phi(B(x,t))}. \tag{2.12}$$

If Φ and Ψ satisfy the conditions (2.8), then

$$\|M_\alpha f\|_{L^\Psi(B)} \lesssim \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \sup_{t>2r} \Psi^{-1}\big(t^{-n}\big)\, \|f\|_{L^\Phi(B(x,t))}. \tag{2.13}$$

The known boundedness statement for the commutator operator $[b, I_\alpha]$ in Orlicz spaces runs as follows.

Theorem 2.14 ([13]). *Let $0 < \alpha < n$ and $b \in BMO(\mathbb{R}^n)$. Let Φ be a Young function and Ψ defined by its inverse $\Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha/n}$. If $\Phi, \Psi \in \Delta_2 \cap \nabla_2$, then $[b, I_\alpha]$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.*

Remark 2.15. Note that the operator $|b, I_\alpha|$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$ under the conditions of Theorem 2.14. The proof of this fact is similar to proof of Theorem 2.14.

In [12], it was proved that the commutator of the Hardy–Littlewood maximal operator M_b with $b \in BMO(\mathbb{R}^n)$ is bounded in $L^\Phi(\mathbb{R}^n)$ for any Young function Φ with $\Phi \in \Delta_2 \cap \nabla_2$. This result, together with the inequality (2.1) and Remark 2.15, implies the following theorem.

Theorem 2.16. *Let $0 \le \alpha < n$ and b, Φ and Ψ the same as in Theorem 2.14. If $\Phi, \Psi \in \Delta_2 \cap \nabla_2$, then $M_{b,\alpha}$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.*

2.5. Generalized Orlicz–Morrey spaces. Various versions of generalized Orlicz–Morrey spaces were introduced in [10], [32] and [42]. We used the definition of [10], which runs as follows.

Definition 2.17. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and Φ any Young function. We define the generalized Orlicz–Morrey space $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ as the space of functions $f \in L^{\Phi}_{\mathrm{loc}}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{\mathcal{M}^{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \frac{\|f\|_{L^\Phi(B(x,r))}}{\varphi(x,r)}.$$

Lemma 2.18. *Let Φ be a Young function, and φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$.*

(i) *If*

$$\sup_{t<r<\infty} \varphi(x,r)^{-1} = \infty \tag{2.14}$$

for all $t>0$ and for all $x\in\mathbb{R}^n$, then $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

(ii) *If $\Phi \in \Delta'$ and*

$$\sup_{0<r<t} \frac{1}{\varphi(x,r)\,\Phi^{-1}(|B(x,r)|^{-1})} = \infty \tag{2.15}$$

for all $t>0$ and for almost all $x\in\mathbb{R}^n$, then $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n) = \Theta$.

Proof. (i) Let (2.14) be satisfied and f be not equivalent to zero. Then $A = \|f\|_{L^\Phi(B(x_0,t_0))} > 0$ for some $x_0\in\mathbb{R}^n$ and $t_0>0$. Hence

$$\|f\|_{\mathcal{M}^{\Phi,\varphi}} \geq \sup_{t_0<r<\infty} \varphi(x_0,r)^{-1}\,\|f\|_{L^\Phi(B(x_0,r))} \geq A \sup_{t_0<r<\infty} \varphi(x_0,r)^{-1}.$$

Therefore, $\|f\|_{\mathcal{M}^{\Phi,\varphi}} = \infty$.
(ii) Let $f\in\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ and (2.15) be satisfied. Then by Theorem 2.8, for almost all $x\in\mathbb{R}^n$,

$$\lim_{r\to 0+} \frac{\|f\chi_{B(x,r)}\|_{L^\Phi}}{\|\chi_{B(x,r)}\|_{L^\Phi}} = |f(x)|. \tag{2.16}$$

We claim that $f(x)=0$ for all those x. Indeed, fix x and assume $|f(x)|>0$. Then by Lemma 2.9 and (2.16) there exists $t_0>0$ such that

$$\Phi^{-1}(|B(x,r)|^{-1})\,\|f\|_{L^\Phi(B(x,r))} \geq \frac{|f(x)|}{2}$$

for all $0<r\leq t_0$. Consequently,

$$\|f\|_{\mathcal{M}^{\Phi,\varphi}} \geq \sup_{0<r<t_0} \varphi(x,r)^{-1}\,\|f\|_{L^\Phi(B(x,r))} \geq \frac{|f(x)|}{2} \sup_{0<r<t_0} \frac{1}{\varphi(x,r)\,\Phi^{-1}(|B(x,r)|^{-1})}$$

Hence $\|f\|_{\mathcal{M}^{\Phi,\varphi}}=\infty$, so $f\notin\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$, and we have arrived at a contradiction. □

Remark 2.19. Let Φ be a Young function. We denote by Ω_Φ the sets of all positive measurable functions φ on $\mathbb{R}^n\times(0,\infty)$ such that for some $t_1,t_2>0$,

$$\sup_{x\in\mathbb{R}^n} \|\varphi(x,r)^{-1}\|_{L^\infty(t_1,\infty)} < \infty,$$

and

$$\sup_{x\in\mathbb{R}^n} \left\| \frac{1}{\varphi(x,r)\,\Phi^{-1}(|B(x,r)|^{-1})} \right\|_{L^\infty(0,t_2)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 2.18, we always assume that $\varphi \in \Omega_\Phi$ and $\Phi \in \Delta'$.

Also, we define the weak generalized Orlicz–Morrey space $WM^{\Phi,\varphi}(\mathbb{R}^n)$ as the set of functions $f \in WL^\Phi_{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM^{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \frac{\|f\|_{WL^\Phi(B(x,r))}}{\varphi(x,r)} < \infty.$$

According to these definitions, we recover the generalized Morrey space $\mathcal{M}^{p,\varphi}$ and the weak generalized Morrey space $W\mathcal{M}^{p,\varphi}$ under the choice $\Phi(r)=r^p, 1\le p<\infty$:

$$\mathcal{M}^{p,\varphi} = \mathcal{M}^{\Phi,\varphi}\Big|_{\Phi(r)=r^p},\ W\mathcal{M}^{p,\varphi} = W\mathcal{M}^{\Phi,\varphi}\Big|_{\Phi(r)=r^p}.$$

The following theorem was proved in [19]. Our results for vanishing spaces are based on this theorem.

Theorem 2.20. *Let $0 \le \alpha < n$, $\varphi_1 \in \Omega_\Phi$, $\varphi_2 \in \Omega_\Psi$, $\Phi, \Psi \in \Delta'$, and the following condition be satisfied:*

$$\sup_{t>r} \Psi^{-1}\big(t^{-n}\big) \operatorname*{ess\,inf}_{s>t} \varphi_1(x,s) \le C\,\varphi_2(x,r)\Psi^{-1}\big(r^{-n}\big), \tag{2.17}$$

where C does not depend on x and r. Then the operator M_α is bounded from $M^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M^{\Psi,\varphi_2}(\mathbb{R}^n)$ under the conditions (2.8)*, and from $M^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $WM^{\Psi,\varphi_2}(\mathbb{R}^n)$ under the conditions* (2.7)*.*

3. Vanishing generalized Orlicz–Morrey spaces

The vanishing generalized Orlicz–Morrey spaces were introduced in [9], see also [14], [15]. We used the definition of [9], which runs as follows.

Definition 3.1. The vanishing generalized Orlicz–Morrey space $V\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ and its weak version $VW\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ are defined as the spaces of functions $f \in \mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ and $f \in W\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n} \frac{\|f\|_{L^\Phi(B(x,r))}}{\varphi(x,r)} = 0 \quad \text{and} \quad \lim_{r\to 0}\sup_{x\in\mathbb{R}^n} \frac{\|f\|_{WL^\Phi(B(x,r))}}{\varphi(x,r)} = 0,$$

respectively.

Remark 3.2. Let Φ be a Young function. We denote by Ω_Φ^V the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0,\infty)$ such that

$$\inf_{x\in\mathbb{R}^n} \inf_{r>\delta} \varphi(x,r) > 0 \quad \text{for some } \delta > 0, \tag{3.1}$$

and

$$\lim_{r\to 0} \frac{1}{\Phi^{-1}(r^{-n}) \inf_{x\in\mathbb{R}^n} \varphi(x,r)} = 0. \tag{3.2}$$

Taking into account Lemma 2.18 for the non-triviality of the space $V\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$, we always assume that $\varphi \in \Omega_\Phi^V$ and $\Phi \in \Delta'$.

The spaces $V\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ and $VW\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ are closed subspaces of the Banach spaces $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ and $W\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$, respectively, which may be shown by standard means.

We will also use the notation

$$\mathfrak{A}_{\Phi,\varphi}(f;x,r) := \frac{\|f\|_{L^\Phi(B(x,r))}}{\varphi(x,r)} \quad \text{and} \quad \mathfrak{A}^W_{\Phi,\varphi}(f;x,r) := \frac{\|f\|_{WL^\Phi(B(x,r))}}{\varphi(x,r)}$$

for brevity, so that

$$V\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n) = \left\{ f \in \mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n) : \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi}(f;x,r) = 0 \right\},$$

and similarly for $VW\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$.

4. Boundedness of the fractional maximal operator in the spaces $V\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$

In this section, we give sufficient conditions for the boundedness of the fractional maximal operator M_α in vanishing generalized Orlicz–Morrey spaces $V\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$.

Theorem 4.1. *Let $0 \le \alpha < n$, Φ, Ψ be Young functions, $\varphi_1 \in \Omega_\Phi^V$, $\varphi_2 \in \Omega_\Psi^V$ and $\Phi, \Psi \in \Delta'$. If*

$$m_\delta := \sup_{t>\delta} \sup_{x\in\mathbb{R}^n} \varphi_1(x,t)\Psi^{-1}\big(t^{-n}\big) < \infty \tag{4.1}$$

for every $\delta > 0$, and

$$\frac{\sup_{t>r} \Psi^{-1}\big(t^{-n}\big)\varphi_1(x,t)}{\Psi^{-1}\big(r^{-n}\big)\varphi_2(x,r)} \le C_0, \tag{4.2}$$

where C_0 does not depend on x and r, then the fractional maximal operator M_α is bounded from $V\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $VW\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$ under the conditions (2.7)*, and from $V\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $V\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$ under the conditions* (2.8).

Proof. The norm inequalities follow from Theorem 2.20. Thus we only have to prove that

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{\Phi,\varphi_1}(f;x,r)=0 \implies \lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{\Psi,\varphi_2}(M_\alpha f;x,r)=0, \tag{4.3}$$

under the conditions (2.8), and

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{\Phi,\varphi_1}(f;x,r)=0 \implies \lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}^W_{\Psi,\varphi_2}(Mf;x,r)=0, \tag{4.4}$$

under the conditions (2.7).

In the proof of (4.3)–(4.4), we follow the ideas of [40], where the authors investigated the boundedness of some classical operators on vanishing generalized Morrey spaces, but base ourselves on Lemma 2.13.

We start with (4.3). We rewrite the inequality (2.13) in the form

$$\mathfrak{A}_{\Psi,\varphi_2}(M_\alpha f;x,r)\le C\frac{\sup_{t>r}\Psi^{-1}\big(t^{-n}\big)\,\|f\|_{L^\Phi(B(x,t))}}{\varphi_2(x,r)\Psi^{-1}\left(r^{-n}\right)}. \tag{4.5}$$

To show that $\sup\limits_{x\in\mathbb{R}^n}\mathfrak{A}_{\Psi,\varphi_2}(M_\alpha f;x,r)<\varepsilon$ for small r, we split the right-hand side of (4.5):

$$\mathfrak{A}_{\Psi,\varphi_2}(M_\alpha f;x,r)\le C[I_{\delta_0}(x,r)+J_{\delta_0}(x,r)], \tag{4.6}$$

where $\delta_0>0$ will be chosen as shown below (we may take $\delta_0<1$), and

$$I_{\delta_0}(x,r):=\frac{\sup_{r<t<\delta_0}\Psi^{-1}\big(t^{-n}\big)\|f\|_{L^\Phi(B(x,t))}}{\varphi_2(x,r)\Psi^{-1}\left(r^{-n}\right)},$$
$$J_{\delta_0}(x,r):=\frac{\sup_{t>\delta_0}\Psi^{-1}\big(t^{-n}\big)\|f\|_{L^\Phi(B(x,t))}}{\varphi_2(x,r)\Psi^{-1}\left(r^{-n}\right)},$$

and it is supposed that $r<\delta_0$. Now, we choose any fixed $\delta_0>0$ such that

$$\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{\Phi,\varphi_1}(f;x,t)<\frac{\varepsilon}{2CC_0},\quad \text{for all } 0<t<\delta_0,$$

where C and C_0 are constants from (4.6) and (4.2), which is possible since f is assumed to be in $V\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$. Then $\|f\|_{L^\Phi(B(x,t))}<\frac{\varepsilon}{2CC_0}\varphi_1(x,t)$, and we obtain the estimate of the first term uniform in $r\in(0,\delta_0)$:

$$\sup_{x\in\mathbb{R}^n}CI_{\delta_0}(x,r)<\frac{\varepsilon}{2},\quad 0<r<\delta_0,$$

by (4.2).

The estimation of the second term now may be made already by the choice of r sufficiently small thanks to the condition (3.2). We have

$$J_\delta(x,r) \le \frac{m_{\delta_0}\|f\|_{\mathcal{M}^{\Phi,\varphi_1}}}{\Psi^{-1}(r^{-n})\varphi_2(x,r)},$$

where m_{δ_0} is the constant from (4.1) with $\delta = \delta_0$. Then, by (3.2), it suffices to choose r small enough such that

$$\sup_{x\in\mathbb{R}^n} \frac{1}{\Psi^{-1}(r^{-n})\varphi_2(x,r)} \le \frac{\varepsilon}{2m_{\delta_0}\|f\|_{\mathcal{M}^{\Phi,\varphi_1}}},$$

which completes the proof of (4.3).

The proof of (4.4) is, line by line, similar to the proof of (4.3). □

Remark 4.2. The condition (4.1) may be omitted when $\varphi(x,r)$ does not depend on x, since (4.1) follows from (4.2) in this case.

If we take $\Phi(t) = t^p$, $\Psi(t) = t^q$ with $1 \le p,q < \infty$ at Theorem 4.1, we get the following new result.

Corollary 4.3. *Let $0 \le \alpha < n$, $1 \le p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $\varphi_1 \in \Omega_p^V$, $\varphi_2 \in \Omega_q^V$. Let also φ_1, φ_2 satisfy the conditions*

$$\lim_{r\to 0} \frac{r^{\frac{n}{p}}}{\inf_{x\in\mathbb{R}^n} \varphi_2(x,r)} = 0, \quad \sup_{t>\delta} \sup_{x\in\mathbb{R}^n} \varphi_1(x,t)t^{-\frac{n}{q}} < \infty$$

for every $\delta > 0$, and

$$\sup_{t>r} \frac{\varphi_1(x,t)}{t^{\frac{n}{q}}} \le C\frac{\varphi_2(x,r)}{r^{\frac{n}{q}}},$$

where C does not depend on x and r. Then the operator M_α is bounded from $V\mathcal{M}^{p,\varphi_1}(\mathbb{R}^n)$ to $V\mathcal{M}^{q,\varphi_2}(\mathbb{R}^n)$ when $p>1$, and from $V\mathcal{M}^{p,\varphi_1}(\mathbb{R}^n)$ to $VW\mathcal{M}^{q,\varphi_2}(\mathbb{R}^n)$ when $p \ge 1$.

Remark 4.4. Theorem 2.20 leads us to the corresponding mapping properties in the vanishing spaces stated in Theorem 4.1. Note that for vanishing spaces we have to impose the condition (4.2) more restrictive than the condition (2.17). Indeed, if the condition (4.2) holds, then

$$\sup_{t>r} \operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)\, \Psi^{-1}\big(t^{-n}\big) \le \sup_{t>r} \varphi_1(x,t)\Psi^{-1}\big(t^{-n}\big), \quad r \in (0,\infty),$$

so the condition (2.17) holds.

On the other hand, the functions

$$\varphi_1(x,t) = \frac{t^\beta}{\chi_{(1,\infty)}(t)}, \quad \varphi_2(x,t) = \Psi^{-1}\big(t^{-n}\big)(1+t^\beta), \quad \beta > 0 \tag{4.7}$$

with supremal regularity condition

$$\sup_{t>r} \Psi^{-1}\big(t^{-n}\big)t^\beta \lesssim \Psi^{-1}\big(r^{-n}\big)r^\beta$$

satisfy the condition (2.17), but do not satisfy the condition (4.2).

To compare, we formulate the following theorem (proved in [15]) and remark below.

Theorem 4.5. *Let $0 < \alpha < n$, Φ, Ψ be Young functions $\varphi_1 \in \Omega^V_\Phi, \varphi_2 \in \Omega^V_\Psi$ and $\Phi, \Psi \in \Delta'$. If*

$$c_\delta := \int_\delta^\infty \sup_{x\in\mathbb{R}^n} \varphi_1(x,t) \frac{\Psi^{-1}\big(t^{-n}\big)}{t} dt < \infty \tag{4.8}$$

for every $\delta > 0$, and

$$\int_r^\infty \varphi_1(x,t)\, \Psi^{-1}\big(t^{-n}\big) \frac{dt}{t} \le C_0 \varphi_2(x,r)\Psi^{-1}\big(r^{-n}\big), \tag{4.9}$$

where C_0 does not depend on $x \in \mathbb{R}^n$ and $r > 0$, then the operator I_α is bounded from $V\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $V\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$ under the conditions (2.10)–(2.11), *and from $V\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $VW\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$ under the conditions* (2.9).

Remark 4.6. Although fractional maximal function is pointwise dominated by the Riesz potential, and consequently, the results for the former could be derived from the results for the latter, we consider them separately, because we are able to study the fractional maximal operator under weaker assumptions than derived from the results for the potential operator. More precisely, under the nondecreasingness condition on $\varphi_1(x,r)$ in r, conditions (4.9) and (4.8) imply the conditions (4.2) and (4.1), respectively. Indeed, by (2.3) we have

$$\Psi^{-1}\big(s^{-n}\big) \approx \Psi^{-1}\big(s^{-n}\big)s^n \int_s^\infty \frac{dt}{t^{n+1}} \lesssim \int_s^\infty \Psi^{-1}\big(t^{-n}\big)\frac{dt}{t}.$$

This inequality implies that

$$\varphi_2(x,r)\Psi^{-1}\big(r^{-n}\big) \gtrsim \int_r^\infty \varphi_1(x,t)\, \Psi^{-1}\big(t^{-n}\big)\frac{dt}{t} \gtrsim \int_s^\infty \varphi_1(x,t)\Psi^{-1}\big(t^{-n}\big)\frac{dt}{t}$$

$$\gtrsim \varphi_1(x,s)\int_s^\infty \Psi^{-1}\big(t^{-n}\big)\frac{dt}{t} \gtrsim \varphi_1(x,s)\Psi^{-1}\big(s^{-n}\big),$$

where we took $s \in (r,\infty)$, so that

$$\sup_{s>r}\varphi_1(x,s)\Psi^{-1}\big(s^{-n}\big) \lesssim \varphi_2(x,r)\Psi^{-1}\big(r^{-n}\big).$$

The proof of implication "(4.8) $\Rightarrow$ (4.1)" can be made similarly.

Note that if we did not impose the monotonicity condition on $\varphi_1(x,r)$ in r, these implications could not be true. For example, for the function $\varphi_1(x,r) = \frac{\sum_{n=0}^\infty \chi_{[n,n+2^{-n}]}(r)}{\Psi^{-1}(r^{-n})}$, the implication "(4.9) $\Rightarrow$ (4.2)" is not true. Also note that the functions $\varphi_1(x,t)=\varphi_2(x,t)=\frac{1}{\Psi^{-1}\left(t^{-n}\right)}$ satisfy (4.2), but not (4.9).

5. Commutators of the fractional maximal operator in the spaces $V\mathcal{M}^{\Phi,\varphi}$

For a possibility to compare with our new results, we formulate the following theorem, which was proved in [15].

Theorem 5.1. *Let $0<\alpha<n$ and $b \in BMO(\mathbb{R}^n)$. Let Φ be a Young function and Ψ defined, via its inverse, by $\Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha/n}$ and $\Phi,\Psi \in \Delta' \cap \nabla_2$. Let also $\varphi_1 \in \Omega_\Phi^V,\ \varphi_2 \in \Omega_\Psi^V$,*

$$\int_r^\infty \left(1+\ln\frac{t}{r}\right)\varphi_1(x,t)\,\Psi^{-1}\left(t^{-n}\right)\frac{dt}{t} \le C_0\varphi_2(x,r)\Psi^{-1}\big(r^{-n}\big), \tag{5.1}$$

where C_0 does not depend on $x\in\mathbb{R}^n$ and $r>0$,

$$\lim_{r\to 0}\frac{\ln\frac{1}{r}}{\Psi^{-1}(r^{-n})\inf_{x\in\mathbb{R}^n}\varphi_2(x,r)} = 0 \tag{5.2}$$

and

$$c_\delta := \int_\delta^\infty (1+|\ln t|)\sup_{x\in\mathbb{R}^n}\varphi_1(x,t)\frac{\Psi^{-1}\big(t^{-n}\big)}{t}dt < \infty \tag{5.3}$$

for every $\delta>0$. Then the operator $[b,I_\alpha]$ is bounded from $V\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $V\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

We find it important to underline once again that the results of this section for the commutator $M_{b,\alpha}$ of the fractional maximal operator are obtained in supremal terms, i.e., under weaker assumptions than derived from Theorem 5.1. More precisely, the supremal condition (5.6) is weaker than the corresponding integral condition (5.1), see Remark 4.6.

The following lemma and theorem were proved in [19].

Lemma 5.2. *Let $0 \le \alpha < n$ and $b \in BMO(\mathbb{R}^n)$. Let also Φ be a Young function and Ψ defined, via its inverse, by $\Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha/n}$ and $\Phi, \Psi \in \Delta_2 \cap \nabla_2$. Then*

$$\|M_{b,\alpha} f\|_{L^\Psi(B(x_0,r))} \lesssim \frac{\|b\|_*}{\Psi^{-1}\big(r^{-n}\big)} \sup_{t>2r} \left(1+\ln\frac{t}{r}\right) \Psi^{-1}\big(t^{-n}\big) \|f\|_{L^\Phi(B(x_0,t))} \tag{5.4}$$

for any ball $B(x_0, r)$ and all $f \in L^\Phi_{\mathrm{loc}}(\mathbb{R}^n)$.

Theorem 5.3. *Let $0 \le \alpha < n$ and $b \in BMO(\mathbb{R}^n)$. Let also Φ be a Young function and Ψ defined, via its inverse, by $\Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha/n}$, $\Phi, \Psi \in \Delta' \cap \nabla_2$ and $\varphi_1 \in \Omega_\Phi$, $\varphi_2 \in \Omega_\Psi$. If*

$$\sup_{t>r} \left(1+\ln\frac{t}{r}\right) \operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)\, \Psi^{-1}\big(t^{-n}\big) \le C_0\, \varphi_2(x,r)\, \Psi^{-1}\big(r^{-n}\big), \tag{5.5}$$

then the operator $M_{b,\alpha}$ is bounded from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

These results lead us to the following theorem.

Theorem 5.4. *Let $0 \le \alpha < n$, $b \in BMO(\mathbb{R}^n)$, Φ be a Young function and Ψ defined, via its inverse, by $\Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha/n}$ and $\Phi, \Psi \in \Delta' \cap \nabla_2$. Let also $\varphi_1 \in \Omega^V_\Phi$, $\varphi_2 \in \Omega^V_\Psi$ satisfy the conditions (5.2) and*

$$\sup_{t>r} \left(1+\ln\frac{t}{r}\right) \varphi_1(x,t)\, \Psi^{-1}\big(t^{-n}\big) \le C_0\, \varphi_2(x,r)\, \Psi^{-1}\big(r^{-n}\big). \tag{5.6}$$

Suppose also that

$$\sup_{t>\delta} (1+|\ln t|)\, \Psi^{-1}\big(t^{-n}\big) \sup_{x\in\mathbb{R}^n} \varphi_1(x,t) < \infty \tag{5.7}$$

for every $\delta > 0$. Then the operator $M_{b,\alpha}$ is bounded from $V\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $V\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

Proof. The norm inequality have already been provided by Theorem 5.3, hence we only have to prove the implication

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\frac{\|f\|_{L^\Phi(B(x,r))}}{\varphi_1(x,r)}=0 \implies \lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\frac{\|M_{b,\alpha}f\|_{L^\Psi(B(x,r))}}{\varphi_2(x,r)}=0. \tag{5.8}$$

To check whether

$$\sup_{x\in\mathbb{R}^n}\frac{\|M_{b,\alpha}f\|_{L^\Psi(B(x,r))}}{\varphi_2(x,r)}<\varepsilon \quad \text{for small } r,$$

we use the estimate (5.4):

$$\frac{\|M_{b,\alpha}f\|_{L^\Psi(B(x,r))}}{\varphi_2(x,r)}\lesssim\frac{\|b\|_*}{\varphi_2(x,r)\Psi^{-1}\big(r^{-n}\big)}\sup_{r<t<\infty}\Big(1+\ln\frac{t}{r}\Big)\Psi^{-1}\big(t^{-n}\big)\|f\|_{L^\Phi(B(x,t))}.$$

We take $r<\delta_0$, where δ_0 will be chosen small enough, and split the integration:

$$\frac{\|M_{b,\alpha}f\|_{L^\Psi(B(x,r))}}{\varphi_2(x,r)}\le C[I_{\delta_0}(x,r)+J_{\delta_0}(x,r)], \tag{5.9}$$

where

$$I_{\delta_0}(x,r):=\frac{1}{\Psi^{-1}(r^{-n})\varphi_2(x,r)}\sup_{r<t<\delta_0}\Big(1+\ln\frac{t}{r}\Big)\Psi^{-1}\big(t^{-n}\big)\,\|f\|_{L^\Phi(B(x,t))},$$

and

$$J_{\delta_0}(x,r):=\frac{1}{\Psi^{-1}(r^{-n})\varphi_2(x,r)}\sup_{\delta_0<t<\infty}\Big(1+\ln\frac{t}{r}\Big)\Psi^{-1}\big(t^{-n}\big)\|f\|_{L^\Phi(B(x,t))}.$$

We choose a fixed $\delta_0>0$ such that

$$\sup_{x\in\mathbb{R}^n}\frac{\|f\|_{L^\Phi(B(x,t))}}{\varphi_1(x,t)}<\frac{\varepsilon}{2CC_0},\quad t\le\delta_0,$$

where C and C_0 are constants from (5.9) and (5.6), which yields the estimate of the first term uniform in $r\in(0,\delta_0)$: $\sup\limits_{x\in\mathbb{R}^n} CI_{\delta_0}(x,r)<\frac{\varepsilon}{2},\ 0<r<\delta_0$.

For the second term, writing $1+\ln\frac{t}{r}\le 1+|\ln t|+\ln\frac{1}{r}$, we obtain

$$J_{\delta_0}(x,r)\le\frac{m_{\delta_0}+\widetilde{m_{\delta_0}}\,\ln\frac{1}{r}}{\Psi^{-1}(r^{-n})\varphi_2(x,r)}\|f\|_{\mathcal{M}^{\Phi,\varphi}},$$

where m_{δ_0} is the constant from (5.7) with $\delta=\delta_0$, and $\widetilde{c_{\delta_0}}$ is a similar constant with omitted logarithmic factor in the integrand. Then, by (5.2) we can choose small r such that $\sup_{x\in\mathbb{R}^n}J_{\delta_0}(x,r)<\frac{\varepsilon}{2}$, which completes the proof. □

If we take $\Phi(t) = t^p$, $\Psi(t) = t^q$ with $1 < p, q < \infty$ at Theorem 5.4, we get the following new result.

Corollary 5.5. *Let* $0 \le \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $b \in BMO(\mathbb{R}^n)$, $\varphi_1 \in \Omega_p^V$, $\varphi_2 \in \Omega_q^V$ *and*

$$\sup_{t>r} \left(1 + \ln \frac{t}{r}\right) \frac{\varphi_1(x,t)}{t^{\frac{n}{q}}} \le C_0 \frac{\varphi_2(x,r)}{r^{\frac{n}{q}}}$$

be fulfilled, where C_0 *does not depend on* $x \in \mathbb{R}^n$ *and* $r > 0$. *If*

$$\lim_{r \to 0} \frac{r^{\frac{n}{q}} \ln \frac{1}{r}}{\inf_{x \in \mathbb{R}^n} \varphi_2(x,r)} = 0$$

and

$$\sup_{t>\delta} \frac{1 + |\ln t|}{t^{\frac{n}{q}}} \sup_{x \in \mathbb{R}^n} \varphi_1(x,t) < \infty$$

for every $\delta > 0$, *then the commutator* $M_{b,\alpha}$ *is bounded from* $V\mathcal{M}^{p,\varphi_1}(\mathbb{R}^n)$ *to* $V\mathcal{M}^{q,\varphi_2}(\mathbb{R}^n)$. *In particular, this holds for the spaces* $V\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ *with* $\varphi_1(x,r) = r^{\frac{\lambda}{p}}$ *and* $\varphi_2(x,r) = r^{\frac{\lambda}{q}}$, $0 \le \lambda < n$.

Acknowledgements. The authors thank the anonymous referees for the careful reading of the paper and their very useful comments.

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(Received September 29, 2015)