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On numerical semigroups closed with respect to the action of affine maps

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Abstract. In this paper, we study numerical semigroups containing a given positive integer and closed with respect to the action of an affine map. For such semigroups we find a minimal set of generators, their embedding dimension, their genus and their Frobenius number.

1. Introduction

A numerical semigroup G is a subsemigroup of the semigroup of non-negative integers $(\mathbb{N}, +)$ containing 0 and such that $\mathbb{N}\backslash G$ is finite. A comprehensive introduction to numerical semigroups is given in [4]. Nevertheless, for the reader's convenience we recall some basic notions, which we will make use of in the current paper.

A set $S \subseteq \mathbb{N}$ generates a numerical semigroup G, namely, $G = \langle S \rangle$, if and only if

$$\gcd(S) = 1,$$

where $\gcd(S)$ is the greatest common divisor of the elements contained in S.

Any numerical semigroup G has a unique finite minimal set of generators, whose cardinality is the embedding dimension $e(G)$ of G.

The cardinality of $\mathbb{N}\backslash G$ is called the genus of G and is denoted by $g(G)$, while the integer

$$F(G) := \max\{x : x \in \mathbb{Z}\backslash G\}$$

is called the Frobenius number of G.

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If $n \in G\backslash\{0\}$, then the set

$$\mathrm{Ap}(G,n) := \{s \in G : s - n \notin G\}$$

is called the Apéry set of G with respect to n.

For any $a \in \mathbb{N}^* := \mathbb{N}\backslash\{0\}$ and $b \in \mathbb{N}$ we define the affine map

$$\begin{aligned} \vartheta_{a,b} : \mathbb{N} &\to \mathbb{N} \\ x &\mapsto ax + b. \end{aligned}$$

We give the following definition.

Definition 1.1. A subsemigroup G of $(\mathbb{N}, +)$ containing 0 is a $\vartheta_{a,b}$-semigroup if $\vartheta_{a,b}(y) \in G$ for any $y \in G\backslash\{0\}$.

The problem we deal with in the paper consists in finding the smallest $\vartheta_{a,b}$-semigroup $G_{a,b}(c)$ containing a given integer $c \in \mathbb{N}\backslash\{0,1\}$, once two positive integers a and b such that $\gcd(b,c) = 1$ are chosen. We notice that under such hypotheses $G_{a,b}(c)$ is a numerical semigroup, while the same does not hold if $\gcd(b,c) > 1$. Indeed, if $d := \gcd(b,c) > 1$, then all elements in G are divisible by d and G is not co-finite. The existence and the structure of $G_{a,b}(c)$ are dealt with in Theorem 3.1.

In the literature, some special cases of $\vartheta_{a,b}$-semigroups have been studied.

In [3], the authors studied Thabit numerical semigroups, namely, numerical semigroups defined for any $n \in \mathbb{N}^*$ as

$$T(n) := \langle \{3 \cdot 2^{n+i} - 1 : i \in \mathbb{N}\} \rangle.$$

Indeed, if we set $c := 3 \cdot 2^n - 1$, then $T(n) = G_{2,1}(c)$.

Also Mersenne numerical semigroups [1], namely, numerical semigroups defined for any $n \in \mathbb{N}^*$ as

$$M(n) := \langle \{2^{n+i} - 1 : i \in \mathbb{N}\} \rangle,$$

are $\vartheta_{2,1}$-semigroups. In this case, setting $c := 2^n - 1$, we have that $M(n) = G_{2,1}(c)$.

In [2], for a given integer $b \in \mathbb{N}\backslash\{0,1\}$ and a given positive integer n, the authors defined

$$M(b,n) := \langle \{b^{n+i} - 1 : i \in \mathbb{N}\} \rangle$$

as a submonoid of $(\mathbb{N}, +)$. If we set $c := b^n - 1$, then $M(b,n) = G_{b,b-1}(c)$. We notice that this latter is not a numerical semigroup. Indeed, we have that $\gcd(b-1,c) \neq 1$.

Synopsis of the paper. The paper is organized as follows.

- In Section 2, we introduce some notations.
- In Section 3, we present (omitting the proofs) the main results of the paper (Theorem 3.1, Theorem 3.2, Corollary 3.3 and Theorem 3.4). Some examples of $\vartheta_{a,b}$-semigroups follow.
- Section 4 and 5 contain all the necessary background and proofs supporting the results presented in Section 3. In particular, Section 5 consists of the proofs of Theorem 3.1, Theorem 3.2 and Theorem 3.4.

2. Definitions and notations

Let $\{a, b, c\} \subseteq \mathbb{N}^*$.

- If y and z are two non-negative integers such that $y < z$, then
$$\begin{aligned}[y, z[&:= \{x \in \mathbb{N} : y \leq x < z\};\\ [y, z] &:= \{x \in \mathbb{N} : y \leq x \leq z\};\\ [y, +\infty[&:= \{x \in \mathbb{N} : y \leq x\}.\end{aligned}$$
- If $k \in \mathbb{N}$, then we define
$$s_k(a) := \begin{cases} 0 & \text{if } k = 0,\\ \sum_{i=0}^{k-1} a^i & \text{otherwise,}\end{cases}$$
and, accordingly,
$$t_k(a, b, c) := a^k c + b \cdot s_k(a).$$
Moreover, we define the set
$$S(a, b, c) := \{t_k(a, b, c) : k \in \mathbb{N}\},$$
and, for any non-negative integer $\tilde{k}$,
$$S_{\tilde{k}}(a, b, c) := \{t_k(a, b, c) : k \in \mathbb{N} \text{ and } 0 \leq k \leq \tilde{k}\}.$$
- We denote by $H(a, b, c)$ the semigroup generated by $S(a, b, c)$, namely,
$$H(a, b, c) := \langle S(a, b, c)\rangle.$$
If $\gcd(b, c) = 1$, then $\gcd(c, ac + b) = 1$, too. Therefore, since $\{c, ac + b\} \subseteq S(a, b, c)$, we have that $\gcd(S(a, b, c)) = 1$, and $S(a, b, c)$ generates a numerical semigroup.

- If K is a non-empty finite subset of $\mathbb{N}$ and $\tilde{k} := \max\{k \in K\}$, then we say that a set $\{j_i\}_{i\in K}$ of non-negative integers is a-reduced if
  - $j_i \in [0, a]$ for any $i \in K\backslash\{0\}$;
  - $j_{\tilde{k}} \neq 0$;
  - if $j_k = a$ for some $k \in K\backslash\{0\}$, then $j_i = 0$ for any $i \in K\backslash\{0\}$ such that $i < k$.
- If $\{j_i\}_{i\in K_1}$ and $\{\tilde{j}_i\}_{i\in K_2}$ are two a-reduced sets of integers indexed on two subsets K_1 and K_2 of $\mathbb{N}^*$ such that

$$k_1 := \max\{k \in K_1\}, \quad k_2 := \max\{k \in K_2\},$$

  then we say that
  - $\{j_i\}_{i\in K_1} = \{\tilde{j}_i\}_{i\in K_2}$ if and only if $K_1 = K_2$ and $j_i = \tilde{j}_i$ for any $i \in K_1$;
  - $\{j_i\}_{i\in K_1} \prec \{\tilde{j}_i\}_{i\in K_2}$ if and only if $k_1 < k_2$ or $k_1 = k_2$ and $j_M < \tilde{j}_M$, where $M := \max\{k \in K_1 : j_k \neq \tilde{j}_k\}$.

3. Main results and examples

In this and the following sections $\{a, b, c\}$ is a subset of $\mathbb{N}^*$, where $c \geq 2$ and $\gcd(b, c) = 1$. The following holds.

Theorem 3.1. *We have that* $G_{a,b}(c) = H(a, b, c)$.

In the following theorem a minimal set of generators for $G_{a,b}(c)$ is provided.

Theorem 3.2. *Let* $\tilde{k} := \min\{k \in \mathbb{N} : s_k(a) > c - 1\}$*. Then* $S_{\tilde{k}-1}(a, b, c)$ *is a minimal set of generators for* $G_{a,b}(c)$*.*

As an immediate consequence of Theorem 3.2, we obtain the embedding dimension of $G_{a,b}(c)$, since for each $\tilde{k} \in \mathbb{N}$ we have that $|S_{\tilde{k}}(a, b, c)| = \tilde{k} + 1$.

Corollary 3.3. *Let* $\tilde{k} := \min\{k \in \mathbb{N} : s_k(a) > c - 1\}$*. Then* $e(G_{a,b}(c)) = \tilde{k}$*.*

In the following theorem, we determine the Frobenius number $F(G_{a,b}(c))$ and the genus $g(G_{a,b}(c))$ of $G_{a,b}(c)$.

Theorem 3.4. *For any* $l \in [1, c-1]$ *there exists and is unique an* a*-reduced set of integers* $\{j_i^{(l)}\}_{i=1}^{k_l}$*, for some positive integer* k_l*, such that*

$$l = \sum_{i=1}^{k_l} j_i^{(l)} \cdot s_i(a).$$

Moreover, if we define

$$x_l := \begin{cases} 0 & \text{if } l = 0, \\ \sum_{i=1}^{k_l} j_i^{(l)} \cdot t_i(a,b,c) & \text{if } l \in [1, c-1], \end{cases}$$

then the following hold:

(1) $x_l = \min\{x \in G_{a,b}(c) : x \equiv bl \pmod{c}\}$;

(2) $\mathrm{Ap}(G_{a,b}(c), c) = \{x_l : l \in [0, c-1]\}$;

(3) $F(G_{a,b}(c)) = x_{c-1} - c$;

(4) $g(G_{a,b}(c)) = \frac{1}{c} \cdot \sum_{l=1}^{c-1} x_l - \frac{c-1}{2}$.

As a by-product of Theorem 3.4, we get the following membership criterion: if $n \in \mathbb{N}$ and $n \equiv x_l \pmod{c}$ for some $l \in [0, c-1]$, then $n \in G_{a,b}(c)$ if and only if $n \geq x_l$.

Example 3.5. In this example, we study the semigroup $G_{3,1}(3)$.

Adopting the notations introduced above, we have that

$$a = 3, \quad b = 1, \quad c = 3.$$

Moreover,

$$x_0 = 0, \quad x_1 = 1 \cdot t_1(3,1,3) = 1 \cdot (3 \cdot 3 + 1) = 10, \quad x_2 = 2 \cdot t_1(3,1,3) = 20.$$

Therefore,

$$F(G_{3,1}(3)) = 17, \quad g(G_{3,1}(3)) = 9,$$

according to Theorem 3.4.

Since

$$\min\{k \in \mathbb{N} : s_k(3) > 2\} = 2,$$

we have that

$$G_{3,1}(3) = \langle S_1(3,1,3) \rangle,$$

according to Theorem 3.2.

The non-negative integers smaller than 21, belonging to $G_{3,1}(3)$, are listed in the following table (the numbers in bold are the elements of $S_1(3,1,3)$):

0	**3**	6	9	12	15	18
			10	13	16	19
						20

Example 3.6. In this example, we study the semigroup $G_{3,1}(5)$.

We have that

$$a = 3, \quad b = 1, \quad c = 5.$$

Moreover,

$$x_0 = 0, \quad x_1 = 1 \cdot t_1(3, 1, 5) = 1 \cdot (3 \cdot 5 + 1) = 16, \quad x_2 = 2 \cdot t_1(3, 1, 5) = 32,$$

$$x_3 = 3 \cdot t_1(3, 1, 5) = 48, \quad x_4 = 1 \cdot t_2(3, 1, 5) = 3^2 \cdot 5 + 4 = 49.$$

Therefore,

$$F(G_{3,1}(5)) = 44, \quad g(G_{3,1}(5)) = 27,$$

according to Theorem 3.4.

Since

$$\min\{k \in \mathbb{N} : s_k(3) > 4\} = 3,$$

we have that

$$G_{3,1}(5) = \langle S_2(3, 1, 5) \rangle,$$

according to Theorem 3.2.

The non-negative integers smaller than 50, belonging to $G_{3,1}(5)$, are listed in the following table (the numbers in bold are the elements of $S_2(3, 1, 5)$):

0	**5**	10	15	20	25	30	35	40	45
			16	21	26	31	36	41	46
						32	37	42	47
									48
									49

Example 3.7. In this example, we study the semigroup $G_{2,3}(4)$.

We have that

$$a = 2, \quad b = 3, \quad c = 4.$$

Moreover,

$$x_0 = 0, \quad x_1 = 1 \cdot t_1(2, 3, 4) = 1 \cdot (2 \cdot 4 + 3) = 11,$$

$$x_2 = 2 \cdot t_1(2, 3, 4) = 22, \quad x_3 = 1 \cdot t_2(2, 3, 4) = 1 \cdot (4 \cdot 4 + 3 \cdot 3) = 25.$$

Therefore,

$$F(G_{2,3}(4)) = 21, \quad g(G_{2,3}(4)) = 13,$$

according to Theorem 3.4.

Since

$$\min\{k \in \mathbb{N} : s_k(2) > 3\} = 3,$$

we have that

$$G_{2,3}(4) = \langle S_2(2,3,4) \rangle,$$

according to Theorem 3.2.

The non-negative integers smaller than 28, belonging to $G_{2,3}(4)$, are listed in the following table (the numbers in bold are the elements of $S_2(2,3,4)$):

0	**4**	8	12	16	20	24
						25
					22	26
		11	15	19	23	27

4. Background

In this section, we prove some technical lemmas which we will repeatedly use in Section 5.

Lemma 4.1. *We have that $S(a,b,c) \subseteq G_{a,b}(c)$.*

Proof. We prove by induction on $k \in \mathbb{N}$ that any $t_k(a,b,c)$ belongs to $G_{a,b}(c)$. If $k = 0$, then $t_0(a,b,c) = c \in G_{a,b}(c)$.

Suppose now that $t_k(a,b,c) \in G_{a,b}(c)$ for some non-negative integer k. Then

$$t_{k+1}(a,b,c) = a \cdot t_k(a,b,c) + b = \vartheta_{a,b}(t_k(a,b,c)) \in G_{a,b}(c). \qquad \square$$

Lemma 4.2. *$H(a,b,c)$ is a subsemigroup of $(\mathbb{N},+)$ closed with respect to the action of the map $\vartheta_{a,b}$.*

Proof. By definition, $H(a,b,c)$ is a subsemigroup of $(\mathbb{N},+)$. We prove that $H(a,b,c)$ is closed with respect to the action of the map $\vartheta_{a,b}$.

Consider an element

$$y = \sum_{k \in K} j_k \cdot t_k(a,b,c) \in H(a,b,c),$$

where K is a non-empty finite subset of $\mathbb{N}$ and $\{j_k\}_{k \in K}$ is a set of positive integers. Let $\tilde{k}$ be a chosen element of K. Then

$$ay + b = \sum_{k \in K} a j_k \cdot t_k(a,b,c) + b =$$

$$= \sum_{\substack{k\in K\\ k\neq\tilde{k}}} aj_k \cdot t_k(a,b,c) + a(j_{\tilde{k}}-1)\cdot t_{\tilde{k}}(a,b,c) + a\cdot t_{\tilde{k}}(a,b,c) + b$$

$$= \sum_{\substack{k\in K\\ k\neq\tilde{k}}} aj_k \cdot t_k(a,b,c) + a(j_{\tilde{k}}-1)\cdot t_{\tilde{k}}(a,b,c) + t_{\tilde{k}+1}(a,b,c).$$

Since this latter is a linear combination of elements in $S(a,b,c)$ with coefficients in $\mathbb{N}$, we conclude that $ay+b\in H(a,b,c)$. □

Lemma 4.3. *For any $k\in\mathbb{N}$ we have that the set*

$$I_k(a,b,c) := \{a^k c + bi : i\in[0,s_k(a)]\}$$

is contained in $H(a,b,c)$.

Proof. We prove the claim by induction on k. Proving the base step is trivial, since $c\in H(a,b,c)$ and

$$I_0(a,b,c) = \{c\}.$$

Suppose now that $I_k(a,b,c)\subseteq H(a,b,c)$ for some $k\in\mathbb{N}$. For any $r\in[0,s_k(a)[$ and any $j\in[0,a]$ we have that

$$a^{k+1}c + b(ar+j)\in H(a,b,c).$$

In fact,

$$a^{k+1}c + b(ar+j) = (a-j)\cdot(a^k c + br) + j\cdot(a^k c + b(r+1)),$$

where

$$\{a^k c + br, a^k c + b(r+1)\}\subseteq I_k(a,b,c).$$

Therefore,

$$\{a^{k+1}c + bi : i\in[0,a\cdot s_k(a)]\}\subseteq H(a,b,c).$$

Finally,

$$t_{k+1}(a,b,c) = \vartheta_{a,b}(t_k(a,b,c))\in H(a,b,c).$$

Hence, $I_{k+1}(a,b,c)\subseteq H(a,b,c)$ and the inductive step is proved. □

Lemma 4.4. *Let k be a non-negative integer such that $s_k(a)\geq c-1$. Then*

$$[t_k(a,b,c),+\infty[\subseteq H(a,b,c).$$

Proof. Let $y \in [t_k(a,b,c), +\infty[$. Then

$$y \equiv r \pmod{c}$$

for some $r \in [0, c-1]$.

Since $\gcd(b,c) = 1$, there exists an integer $i \in [0, c-1] \subseteq [0, s_k(a)]$ such that

$$a^k c + bi \equiv r \pmod{c}.$$

Therefore,

$$y - a^k c - bi \equiv 0 \pmod{c},$$

namely,

$$y = a^k c + bi + cq$$

for some non-negative integer q. Since

$$\{a^k c + bi, c\} \subseteq H(a,b,c),$$

according to Lemma 4.3, we conclude that $y \in H(a,b,c)$, and the result follows. $\square$

Lemma 4.5. *If k is a positive integer such that $s_k(a) \leq c-1$, then*

$$t_k(a,b,c) \notin \langle S_{k-1}(a,b,c) \rangle.$$

Proof. Suppose by contradiction that

$$\sum_{i \in K} j_i \cdot t_i(a,b,c) = t_k(a,b,c)$$

for some positive integers $\{j_i\}_{i \in K}$ indexed on a non-empty set $K \subseteq [0, k-1]$. Before proceeding, we define $K^* := K \backslash \{0\}$.

We distinguish three different cases.

Case 1. $\sum_{i \in K} j_i \cdot a^i \leq a^k$. We distinguish four subcases.

- *Subcase 1.* $a = 1$. Then

$$\sum_{i \in K} j_i 1^i \leq 1^k.$$

  This latter is possible only if $|K| = 1$ and the only integer j_i is equal to 1. Therefore, $K = \{r\}$ for some $r \in [0, k-1]$ and $j_r = 1$. Hence,

$$\sum_{i \in K} j_i \cdot t_i(a,b,c) = c + b \cdot s_r(1),$$

  while

$$t_k(a,b,c) = c + b \cdot s_k(1).$$

  Since $s_r(1) < s_k(1)$, we conclude that $\sum_{i \in K} j_i \cdot t_i(a,b,c) \neq t_k(a,b,c)$.

- *Subcase 2.* $a > 1$ and $\sum_{i \in K^*} j_i = 0$. Then $K = \{0\}$ and

$$\sum_{i \in K} j_i \cdot t_i(a, b, c) = j_0 \cdot c \equiv 0 \pmod{c}.$$

Since

$$t_k(a, b, c) \equiv b \cdot s_k(a) \pmod{c}$$

and

$$b \cdot s_k(a) \not\equiv 0 \pmod{c}$$

because $\gcd(b, c) = 1$ and $1 \leq s_k(a) \leq c - 1$, we conclude that $\sum_{i \in K} j_i \cdot t_i(a, b, c) \neq t_k(a, b, c)$.

- *Subcase 3.* $a > 1$ and $\sum_{i \in K^*} j_i = 1$. Then either $K = \{r\}$ or $K = \{0, r\}$ for some positive integer r. In both cases, $j_r = 1$.

If $K = \{r\}$, then

$$\sum_{i \in K} j_i \cdot t_i(a, b, c) = a^r c + b \cdot s_r(a) < t_k(a, b, c).$$

If $K = \{0, r\}$, then

$$\sum_{i \in K} j_i \cdot t_i(a, b, c) = j_0 \cdot c + a^r c + b \cdot s_r(a).$$

We notice that

$$t_k(a, b, c) \equiv b \cdot s_k(a) \pmod{c}, \qquad \sum_{i \in K} j_i \cdot t_i(a, b, c) \equiv b \cdot s_r(a) \pmod{c}.$$

Since $\gcd(b, c) = 1$ and

$$0 < s_k(a) - s_r(a) < c - 1,$$

we conclude that

$$t_k(a, b, c) \not\equiv \sum_{i \in K} j_i \cdot t_i(a, b, c) \pmod{c},$$

and, consequently, $t_k(a, b, c) \neq \sum_{i \in K} j_i \cdot t_i(a, b, c)$.

- *Subcase 4.* $a > 1$ and $\sum_{i\in K^*} j_i \geq 2$. Then

$$\begin{aligned}
&\sum_{i\in K} j_i \cdot t_i(a,b,c)\\
&= \sum_{i\in K} j_i \cdot a^i c + b \cdot \sum_{i\in K^*} j_i \cdot s_i(a) \leq a^k \cdot c + b \cdot \left(\sum_{i\in K^*} j_i \cdot \frac{a^i - 1}{a-1}\right)\\
&= a^k \cdot c + \frac{b}{a-1} \cdot \left(\sum_{i\in K^*} j_i \cdot a^i - \sum_{i\in K^*} j_i\right) \leq a^k \cdot c + \frac{b}{a-1} \cdot \left(a^k - \sum_{i\in K^*} j_i\right)\\
&\leq a^k \cdot c + \frac{b}{a-1} \cdot \left(a^k - 2\right) < a^k \cdot c + b \cdot \frac{a^k - 1}{a-1} = t_k(a,b,c).
\end{aligned}$$

Case 2. $\sum_{i\in K} j_i \cdot a^i > a^k$ and $\sum_{i\in K} j_i \cdot s_i(a) > s_k(a)$. Then

$$\sum_{i\in K} j_i \cdot t_i(a,b,c) > t_k(a,b,c),$$

in contradiction with the initial assumption.

Case 3. $\sum_{i\in K} j_i \cdot a^i > a^k$ and $\sum_{i\in K} j_i \cdot s_i(a) \leq s_k(a)$. Since

$$\left(\sum_{i\in K} j_i a^i - a^k\right) \cdot c = b \cdot \left(s_k(a) - \sum_{i\in K} j_i s_i(a)\right)$$

and

$$\left(\sum_{i\in K} j_i a^i - a^k\right) \cdot c > 0,$$

we get that

$$0 < s_k(a) - \sum_{i\in K} j_i s_i(a) \leq c - 1.$$

This latter fact implies that

$$b \cdot \left(s_k(a) - \sum_{i\in K} j_i s_i(a)\right) \not\equiv 0 \pmod{c},$$

in contradiction with the fact that

$$\left(\sum_{i\in K} j_i a^i - a^k\right) \cdot c \equiv 0 \pmod{c}.$$

Hence, also in this case $\sum_{i\in K} j_i \cdot t_i(a,b,c) \neq t_k(a,b,c)$. □

Lemma 4.6. *Let k be a positive integer. If x is a positive integer such that*

$$s_k(a) \leq x < s_{k+1}(a),$$

then there exists and is unique an a-reduced set of integers $\{j_i\}_{i=1}^{k}$ such that

$$x = \sum_{i=1}^{k} j_i \cdot s_i(a).$$

Proof. We prove the claim by induction on $k \in \mathbb{N}^*$. If $k = 1$, then $s_1(a) \leq x < s_2(a) = 1 + a$. Therefore, $x = j_1 \cdot s_1(a)$, where $j_1 = x$.

Suppose that $k > 1$ and $s_k(a) \leq x < s_{k+1}(a)$. Then there exist and are unique two non-negative integers q and r such that

$$\begin{cases} x = qs_k(a) + r \\ 0 \leq r < s_k(a) \end{cases}$$

and $q \leq a$.

If $r = 0$, then $q \geq 1$. The result follows, setting $j_k := q$ and $j_i := 0$ for any $i < k$.

If $r > 0$, then $1 \leq q < a$ and $s_{\tilde{k}}(a) \leq r < s_{\tilde{k}+1}(a)$ for some positive integer $\tilde{k}$. By inductive hypothesis, we have that

$$r = \sum_{i=1}^{\tilde{k}} j_i \cdot s_i(a)$$

for some a-reduced set of integers $\{j_i\}_{i=1}^{\tilde{k}}$. Therefore, the result follows setting $j_k := q$ and $j_i := 0$ for any $i \in [\tilde{k} + 1, k - 1]$. $\square$

Lemma 4.7. *If*

$$x = \sum_{i \in K} j_i \cdot t_i(a, b, c),$$

where $\{j_i\}_{i \in K}$ is a set of positive integers indexed on a finite subset K of $\mathbb{N}$, then

$$x = \sum_{i \in \tilde{K}} \tilde{j}_i \cdot t_i(a, b, c)$$

for some a-reduced set of integers $\{\tilde{j}_i\}_{i \in \tilde{K}}$ indexed on a finite subset $\tilde{K}$ of $\mathbb{N}$.

PROOF. We notice that

$$a \cdot t_{i_2}(a,b,c) + t_{i_1}(a,b,c) = t_{i_2+1}(a,b,c) + a \cdot t_{i_1-1}(a,b,c)$$

for any choice of positive integers i_1 and i_2 such that $i_1 \leq i_2$.

We define $l := 0$, $K(l) := K$ and $\tilde{j}_i := j_i$ for any $i \in K(l)$. Then we enter the following iterative procedure.

(1) If $\{\tilde{j}_i\}_{i\in K(l)}$ is a-reduced, then we break the procedure, else we define

$$M(l) := \max\{i \in K(l) : \tilde{j}_i \geq a\}, \quad m(l) := \min\{i \in K(l)\backslash\{0\} : \tilde{j}_i \neq 0\}.$$

(2) We set

$$\tilde{j}_{m(l)-1} := \tilde{j}_{m(l)-1} + a, \quad \tilde{j}_{m(l)} := \tilde{j}_{m(l)} - 1,$$
$$\tilde{j}_{M(l)} := \tilde{j}_{M(l)} - a, \quad \tilde{j}_{M(l)+1} := \tilde{j}_{M(l)+1} + 1.$$

(3) We set

$$K(l+1) := K(l) \cup \{m(l)-1, M(l)+1\}, \quad l := l+1,$$

and go to step (1).

We notice that for each l we have that

$$\sum_{i\in K(l+1)} j_i = \sum_{i\in K(l)} j_i,$$

and at least one of the following holds:

$$m(l+1) = m(l) - 1 \quad \text{or} \quad \sum_{i\in K(l+1)\backslash\{0\}} j_i < \sum_{i\in K(l)\backslash\{0\}} j_i.$$

In particular, when $m(l) = 1$ for some integer l, we have that

$$\sum_{i\in K(l+1)\backslash\{0\}} j_i < \sum_{i\in K(l)\backslash\{0\}} j_i.$$

Therefore, after some iterations the procedure breaks.

Hence,

$$x = \sum_{i\in\tilde{K}} \tilde{j}_i \cdot t_i(a,b,c),$$

where $\tilde{K} := K(l)$. $\square$

Example 4.8. Suppose that

$$a = 2, \quad b = 3, \quad c = 4.$$

Let $K = \{1, 2, 4\}$ and

$$j_1 = 2, \quad j_2 = 4, \quad j_4 = 3.$$

We have that

$$t_0(2,3,4) = 4, \quad t_1(2,3,4) = 11, \quad t_2(2,3,4) = 25,$$

$$t_3(2,3,4) = 53, \quad t_4(2,3,4) = 109, \quad t_5(2,3,4) = 221.$$

Let

$$x = \sum_{i \in K} j_i \cdot t_i(2,3,4) = 449.$$

We use the iterative procedure described in the proof of Lemma 4.7 with the aim to write

$$x = \sum_{i \in \tilde{K}} \tilde{j}_i \cdot t_i(2,3,4)$$

for some a-reduced set of integers $\{\tilde{j}_i\}_{i \in \tilde{K}}$.

We set $l := 0$, $K(0) := K$ and $\tilde{j}_i = j_i$ for any $i \in K(0)$. Since $\{\tilde{j}_i\}_{i \in K(0)}$ is not a-reduced, we define

$$M(0) := 4, \quad m(0) := 1.$$

Then we set

$$\tilde{j}_0 := 2, \quad \tilde{j}_1 := 1, \quad \tilde{j}_2 := 4, \quad \tilde{j}_4 := 1, \quad \tilde{j}_5 := 1,$$

and

$$K(1) := \{0, 1, 2, 4, 5\}, \quad l := 1.$$

Since $\{\tilde{j}_i\}_{i \in K(1)}$ is not a-reduced, we define

$$M(1) := 2, \quad m(1) := 1.$$

Then we set

$$\tilde{j}_0 := 4, \quad \tilde{j}_1 := 0, \quad \tilde{j}_2 := 2, \quad \tilde{j}_3 := 1, \quad \tilde{j}_4 := 1, \quad \tilde{j}_5 := 1,$$

and

$$K(2) := \{0, 1, 2, 3, 4, 5\}, \quad l := 2.$$

We notice that $\{\tilde{j}_i\}_{i \in K(2)}$ is a-reduced and define $\tilde{K} := K(2)$.

We have that

$$x = \sum_{i \in \tilde{K}} \tilde{j}_i \cdot t_i(2,3,4).$$

Lemma 4.9. *Let k be a positive integer. If $\{j_i\}_{i=1}^k$ is an a-reduced set of integers, then*

$$\sum_{i=1}^{k} j_i \cdot t_i(a,b,c) < t_{k+1}(a,b,c).$$

Proof. We prove the claim by induction on k. If $k = 1$, then

$$\sum_{i=1}^{1} j_i \cdot t_i(a,b,c) \leq a \cdot t_1(a,b,c) = t_2(a,b,c) - b < t_2(a,b,c).$$

Let $k > 1$. We distinguish two cases.

- If $j_k = a$, then

$$\sum_{i=1}^{k} j_i \cdot t_i(a,b,c) = a \cdot t_k(a,b,c) < t_{k+1}(a,b,c).$$

- If $j_k \leq a - 1$, then, by inductive hypothesis, we have that

$$\begin{aligned}\sum_{i=1}^{k} j_i \cdot t_i(a,b,c) &= \sum_{i=1}^{k-1} j_i \cdot t_i(a,b,c) + j_k \cdot t_k(a,b,c) \\ &< t_k(a,b,c) + (a-1) \cdot t_k(a,b,c) < t_{k+1}(a,b,c). \qquad \square\end{aligned}$$

Lemma 4.10. *Suppose that*

- *k_1 and k_2 are two positive integers such that $k_1 \leq k_2$;*
- *$\{j_i\}_{i=1}^{k_1}$ and $\{\tilde{j}_i\}_{i=1}^{k_2}$ are two different a-reduced sets of integers;*
- *x and y are two integers such that*

$$x = \sum_{i=1}^{k_1} j_i \cdot t_i(a,b,c), \quad y = \sum_{i=1}^{k_2} \tilde{j}_i \cdot t_i(a,b,c).$$

The following hold.

- *If $\{j_i\}_{i=1}^{k_1} \prec \{\tilde{j}_i\}_{i=1}^{k_2}$, then $x < y$.*
- *If $\{\tilde{j}_i\}_{i=1}^{k_2} \prec \{j_i\}_{i=1}^{k_1}$, then $y < x$.*

Proof. Without loss of generality we suppose that $\{j_i\}_{i=1}^{k_1} \prec \{\tilde{j}_i\}_{i=1}^{k_2}$. If $k_1 < k_2$, then

$$x = \sum_{i=1}^{k_1} j_i \cdot t_i(a,b,c) < t_{k_1+1}(a,b,c) \leq t_{k_2}(a,b,c) \leq y,$$

according to Lemma 4.9.

If $k_1 = k_2$, then we define $M := \max\{i \in [1, k_2] : j_i \neq \tilde{j}_i\}$. Since $\{j_i\}_{i=1}^{k_1} \prec \{\tilde{j}_i\}_{i=1}^{k_2}$, we have that $j_M < \tilde{j}_M$. Therefore,

$$\begin{aligned}
x &= \sum_{i=1}^{k_1} j_i \cdot t_i(a,b,c) = \sum_{i=1}^{k_2} j_i \cdot t_i(a,b,c) \\
&= \sum_{i=1}^{M-1} j_i \cdot t_i(a,b,c) + j_M \cdot t_M(a,b,c) + \sum_{i=M+1}^{k_2} j_i \cdot t_i(a,b,c) \\
&< t_M(a,b,c) + j_M \cdot t_M(a,b,c) + \sum_{i=M+1}^{k_2} \tilde{j}_i \cdot t_i(a,b,c) \\
&\leq \tilde{j}_M \cdot t_M(a,b,c) + \sum_{i=M+1}^{k_2} \tilde{j}_i \cdot t_i(a,b,c) \leq \sum_{i=1}^{k_2} \tilde{j}_i \cdot t_i(a,b,c) = y. \qquad \square
\end{aligned}$$

In analogy with Lemma 4.9 and Lemma 4.10, we state two more lemmas, whose proofs (which are omitted) follow the same lines as the proofs of the two lemmas above.

Lemma 4.11. *If $\{j_i\}_{i=1}^{k}$ is an a-reduced set of integers for some positive integer k, then*

$$\sum_{i=1}^{k} j_i \cdot s_i(a) < s_{k+1}(a).$$

Lemma 4.12. *Suppose that*

- *k_1 and k_2 are two positive integers such that $k_1 \leq k_2$;*
- *$\{j_i\}_{i=1}^{k_1}$ and $\{\tilde{j}_i\}_{i=1}^{k_2}$ are two different a-reduced sets of integers;*
- *x and y are two integers such that*

$$x = \sum_{i=1}^{k_1} j_i \cdot s_i(a), \quad y = \sum_{i=1}^{k_2} \tilde{j}_i \cdot s_i(a).$$

The following hold.

- *If $\{j_i\}_{i=1}^{k_1} \prec \{\tilde{j}_i\}_{i=1}^{k_2}$, then $x < y$.*
- *If $\{\tilde{j}_i\}_{i=1}^{k_2} \prec \{j_i\}_{i=1}^{k_1}$, then $y < x$.*

5. Proofs

5.1. Proof of Theorem 3.1. First, we notice that $H(a,b,c) \subseteq G_{a,b}(c)$, according to Lemma 4.1.

According to Lemma 4.2, $H(a,b,c)$ is a subsemigroup of $(\mathbb{N},+)$ closed with respect to the action of the map $\vartheta_{a,b}$.

Moreover, $\mathbb{N} \backslash H(a,b,c)$ is finite. In fact, according to Lemma 4.4, we have that

$$[t_k(a,b,c), +\infty[\subseteq H(a,b,c)$$

for any positive integer k such that $s_k(a) \geq c-1$.

Hence, $H(a,b,c)$ is a numerical semigroup and $G_{a,b}(c) = H(a,b,c)$.

5.2. Proof of Theorem 3.2. According to Lemma 4.5, we have that

$$t_k(a,b,c) \notin \langle S_{k-1}(a,b,c) \rangle$$

for any positive integer $k < \tilde{k}$. In fact, for any such k we have that $s_k(a) \leq c-1$. Nevertheless,

$$t_{\tilde{k}}(a,b,c) \in \langle S_{\tilde{k}-1}(a,b,c) \rangle.$$

The latter assertion holds, since

$$\begin{cases} t_{\tilde{k}}(a,b,c) - a^{\tilde{k}}c = b(qc+r) \\ 0 \leq r < c \end{cases}$$

for some non-negative integers q and r. Since

$$r \leq c-1 < s_{\tilde{k}}(a),$$

we have that

$$a^{\tilde{k}}c + br \in G_{a,b}(c),$$

according to Lemma 4.3. Therefore,

$$t_{\tilde{k}}(a,b,c) \in \langle S_{\tilde{k}-1}(a,b,c) \rangle,$$

namely, $S_{\tilde{k}-1}(a,b,c)$ is a minimal set of generators for $G_{a,b}(c)$.

5.3. Proof of Theorem 3.4. We prove separately the 4 assertions.

(1) Let $l \in [1, c-1]$. According to Lemma 4.6, there exists and is unique an a-reduced set $\{j_i^{(l)}\}_{i=1}^{k_l}$ such that

$$l = \sum_{i=1}^{k_l} j_i^{(l)} \cdot s_i(a).$$

Moreover,

$$x_l = \sum_{i=1}^{k_l} j_i^{(l)} \cdot t_i(a,b,c) = \sum_{i=1}^{k_l} j_i^{(l)} \cdot a^i c + b \cdot \sum_{i=1}^{k_l} j_i^{(l)} \cdot s_i(a) \equiv bl \pmod{c}.$$

Let $x \in G_{a,b}(c)$. Since

$$\{bl : l \in [0, c-1]\}$$

is a set of representatives of the residue classes in $\mathbb{Z}/c\mathbb{Z}$, we can say that

$$x \equiv bl \pmod{c}$$

for some $l \in [0, c-1]$.

If $x \not\equiv 0 \pmod{c}$, then

$$x = \sum_{i \in \tilde{K}} \tilde{j}_i \cdot t_i(a,b,c)$$

for some a-reduced set of integers $\{\tilde{j}_i\}_{i \in \tilde{K}}$, according to Lemma 4.7.

We distinguish two cases.

- If $0 \in \tilde{K}$ and $\tilde{j}_0 \neq 0$, then
$$x = \tilde{j}_0 \cdot c + \sum_{i \in \tilde{K} \setminus \{0\}} \tilde{j}_i \cdot t_i(a,b,c) > \sum_{i \in \tilde{K} \setminus \{0\}} \tilde{j}_i \cdot t_i(a,b,c) \equiv bl \pmod{c}.$$
- If $0 \notin \tilde{K}$ or $\tilde{j}_0 = 0$, then $\{j_i^{(l)}\}_{i=1}^{k_l} \preceq \{\tilde{j}_i\}_{i \in \tilde{K}}$ and $x_l \leq x$, according to Lemma 4.10.
Indeed, suppose by contradiction that $\{\tilde{j}_i\}_{i \in \tilde{K}} \prec \{j_i^{(l)}\}_{i=1}^{k_l}$. We notice that
$$x \equiv b \cdot \sum_{i \in \tilde{K}} \tilde{j}_i \cdot s_i(a) \equiv bl \pmod{c},$$

namely,

$$\sum_{i\in\tilde{K}} \tilde{j}_i \cdot s_i(a) \equiv l \pmod{c}.$$

This latter is absurd, because

$$1 \leq \sum_{i\in\tilde{K}} \tilde{j}_i \cdot s_i(a) < \sum_{i=1}^{k_l} j_i^{(l)} \cdot s_i(a) = l,$$

according to Lemma 4.12.

(2) This assertion follows from (1).

(3) Since $x_i < x_{c-1}$ for any $i \in [0, c-1[$, we have that $F(G_{a,b}(c)) = x_{c-1} - c$.

(4) This assertion follows from Selmer's formulas.

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