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On a class of projective Ricci flat Finsler metrics

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Abstract. The projective Ricci curvature is an important projective invariant in Finsler geometry. In this paper, we study the projective Ricci curvature and characterize projective Ricci flat Randers metrics. As a natural application, we characterize projective Ricci flat Randers metrics with isotropic S-curvature. In this case, the metrics are acturally weak Einstein Finsler metrics.

1. Introduction

The Ricci curvature in Finsler geometry is a natural extension of the Ricci curvature in Riemannian geometry and plays an important role in Finsler geometry. A Finsler metric F on an n-dimensional manifold M is called a *weak Einstein metric* if it satisfies the following equation on the Ricci curvature:

$$\mathbf{Ric} = (n-1)\left(\frac{3\theta}{F} + \sigma\right) F^2, \tag{1}$$

where σ is a scalar function and $\theta = \theta_i y^i$ is a 1-form on M. F is called an *Einstein metric* if $\theta = 0$ in (1), that is,

$$\mathbf{Ric} = (n-1)\sigma F^2. \tag{2}$$

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In particular, a Finsler metric F is said to be of *Ricci constant* if F satisfies (2) with a constant σ. F is called a *Ricci flat metric* if F satisfies (2) with $\sigma = 0$, that is, $\mathbf{Ric} = 0$.

The S-curvature $\mathbf{S}$ is an important non-Riemannian quantity in Finsler geometry, which was introduced by Z. Shen when he studied volume comparison in Riemann–Finsler geometry [7]. Z. Shen proved that the S-curvature and the Ricci curvature determine the local behavior of the Busemann–Hausdorff measure of small metric balls around a point [8]. He also established a volume comparison theorem for the volume of metric balls under a lower Ricci curvature bound and a lower S-curvature bound, and generalized Bishop–Gromov volume comparison theorem in the Riemannian case [8]. Recent studies confirm the importance of S-curvature in Finsler geometry (see [3], [5] and [8]).

It is natural to consider the geometric quantities derived from Ricci curvature and S-curvature. In [9], Z. Shen considered the *projective spray* $\tilde{\mathbf{G}}$ associated with a given spray $\mathbf{G}$ on an n-dimensional manifold which is defined by $\mathbf{G}$ and its S-curvature $\mathbf{S}$ as

$$\tilde{\mathbf{G}} = \mathbf{G} + \frac{2\mathbf{S}}{n+1}\mathbf{Y},$$

where $\mathbf{Y} := y^i \frac{\partial}{\partial y^i}$ is the vertical radial field on TM. Then $\tilde{\mathbf{G}}$ is projectively invariant, and it is easy to see that the Ricci curvature $\widetilde{\mathbf{Ric}}$ of $\tilde{\mathbf{G}}$ is given by

$$\widetilde{\mathbf{Ric}} = \mathbf{Ric} + \frac{n-1}{n+1}\mathbf{S}_{|m}y^m + \frac{n-1}{(n+1)^2}\mathbf{S}^2,$$

where " | " denotes the horizontal covariant derivative with respect to Berwald connection of $\mathbf{G}$. Z. Shen also introduced the so-called *Berwald–Weyl curvature* of $\mathbf{G}$ by the Ricci scalar of $\tilde{\mathbf{G}}$, which is the Ricci curvature of $\tilde{\mathbf{G}}$ divided by $n-1$ (see Section 13.6 in [9] for more details). Recently, Z. Shen defined the concept of *projective Ricci curvature* for a Finsler metric F in Finsler geometry as

$$\mathbf{PRic} := \mathbf{Ric} + (n-1)\left\{\bar{\mathbf{S}}_{|m}y^m + \bar{\mathbf{S}}^2\right\}, \tag{3}$$

where $\bar{\mathbf{S}} := \frac{1}{n+1}\mathbf{S}$, and " | " denotes the horizontal covariant derivative with respect to the Berwald connection (or the Chern connection) of F. Actually, we can rewrite the projective Ricci curvature as

$$\mathbf{PRic} = \mathbf{Ric} + \frac{n-1}{n+1}\mathbf{S}_{|m}y^m + \frac{n-1}{(n+1)^2}\mathbf{S}^2. \tag{4}$$

It is easy to see that if two Finsler metrics are pointwise projectively related on a manifold with a fixed volume form, then their projective Ricci curvatures are equal. In other words, the projective Ricci curvature is projective invariant with respect to a fixed volume form. On the other hand, the projective Ricci curvature is actually a kind of weighted Ricci curvatures. See [6] and the definition of S-curvature in Section 2. A Finsler metric is called *projective Ricci flat* if $\mathbf{PRic} = 0$.

Randers metrics are among the simplest non-Riemannian Finsler metrics. They are defined by a Riemannian metric $\alpha = \sqrt{a_{ij}y^iy^j}$ and a 1-form $\beta = b_iy^i$ as the sum $F = \alpha + \beta$. To state our main results, let us introduce some common notations for Randers metrics. Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M. Put

$$r_{ij} := \frac{1}{2}(b_{i;j} + b_{j;i}), \quad s_{ij} := \frac{1}{2}(b_{i;j} - b_{j;i}),$$

where " ; " denotes the covariant derivative with respect to the Levi–Civita connection of α. Further, put

$$r^i{}_j := a^{im}r_{mj}, \quad s^i{}_j := a^{im}s_{mj}, \quad r_j := b^m r_{mj}, \quad s_j := b^m s_{mj},$$

$$q_{ij} := r_{im}s^m{}_j, \quad t_{ij} := s_{im}s^m{}_j, \quad q_j := b^i q_{ij} = r_m s^m{}_j, \quad t_j := b^i t_{ij} = s_m s^m{}_j,$$

where $\left(a^{ij}\right) := (a_{ij})^{-1}$ and $b^i := a^{ij}b_j$. We will denote $r_{i0} := r_{ij}y^j, s_{i0} := s_{ij}y^j$ and $r_{00} := r_{ij}y^iy^j, r_0 := r_iy^i, s_0 := s_iy^i$, etc. Let $b := \|\beta_x\|_\alpha$, $\rho := \ln\sqrt{1-b^2}$ and $\rho_i := \rho_{x^i}$.

In [1], Bao–Robles derive the formula for the Ricci curvature of a Randers metric and find two equations on α and β that characterize Einstein Randers metrics. Later, Z. Shen and G. C. Yildirim characterize weak Einstein Randers metrics and prove the following theorem: *a Randers metric $F = \alpha + \beta$ on an n-dimensional manifold M is a weak Einstein metric satisfying* (1) *if and only if α and β satisfy*

$$^{\alpha}\mathbf{Ric} = (n-1)[(\sigma-3c^2)\alpha^2+(\sigma+c^2)\beta^2+(3\theta-c_0)\beta-s_{0;0}-s_0^2]+2t_{00}+\alpha^2 t^m{}_m, \quad (5)$$

$$s^m{}_{0;m} = (n-1)\left[(\sigma+c^2)\beta + 2cs_0 + t_0 + \frac{3\theta + c_0}{2}\right], \quad (6)$$

$$r_{00} = -2s_0\beta + 2c(\alpha^2 - \beta^2), \quad (7)$$

where c is a scalar function on M and $c_0 = c_{x^i}(x)y^i$. See Theorem 7.1.1 in [4].

In this paper, we first derive a formula for the projective Ricci curvature of a Randers metric in Section 3. Based on this, we can prove the following main theorem.

Theorem 1.1. *Let $F=\alpha+\beta$ be a Randers metric on a manifold M of dimension n. Then F is a projective Ricci flat metric if and only if α and β satisfy the following equations:*

(i) ${}^{\alpha}\mathbf{Ric}=t^m_{\ m}\alpha^2+2t_{00}+(n-1)(\rho_{0;0}-\rho_0^2)$;

(ii) $s^m_{\ 0;m}=-(n-1)\rho_m s^m_{\ 0}$;

(iii) $s_0=0$ *or* $r_{00}+2\beta s_0=0$,

where ${}^{\alpha}\mathbf{Ric}$ *denotes the Ricci curvature of* α *and we have put* $\rho_0:=\rho_i y^i$.

By the definition of ρ, we have

$$\rho_{x^i}=-\frac{r_i+s_i}{1-b^2}.$$

Then

$$\rho_0=-\frac{r_0+s_0}{1-b^2} \tag{8}$$

and

$$\rho_m s^m_{\ 0}=-\frac{1}{1-b^2}(q_0+t_0). \tag{9}$$

Further,

$$\rho_{0;0}=-\frac{r_{0;0}+s_{0;0}}{1-b^2}-\frac{2(r_0+s_0)^2}{(1-b^2)^2}=-\frac{r_{0;0}+s_{0;0}}{1-b^2}-2\rho_0^2. \tag{10}$$

Hence, we can restate Theorem 1.1 as follows.

Theorem 1.2. *Let $F=\alpha+\beta$ be a Randers metric on a manifold M of dimension n. Then F is a projective Ricci flat metric if and only if α and β satisfy the following equations:*

(i) ${}^{\alpha}\mathbf{Ric}=t^m_{\ m}\alpha^2+2t_{00}-(n-1)\left[\frac{r_{0;0}+s_{0;0}}{1-b^2}+\frac{3(r_0+s_0)^2}{(1-b^2)^2}\right]$;

(ii) $s^m_{\ 0;m}=\frac{n-1}{1-b^2}(q_0+t_0)$;

(iii) $s_0=0$ *or* $r_{00}+2\beta s_0=0$.

2. Preliminaries

Let F be a Finsler metric on an n-dimensional manifold M. The geodesics of F are characterized locally by the following system of second-order ordinary differential equations

$$\left(x^i\right)''+2G^i\left(x,\dot{x}\right)=0,\quad i\in\{1,\cdots,n\},$$

where

$$G^i = \frac{1}{4} g^{il} \left\{ \left[F^2\right]_{x^k y^l} y^k - \left[F^2\right]_{x^l} \right\}, \tag{11}$$

$g_{ij}(x,y) := \frac{1}{2}\left[F^2\right]_{y^i y^j}(x,y)$ and $(g^{ij}) := (g_{ij})^{-1}$. The functions G^i are called the *geodesic coefficients* of F. Let $\sigma = \sigma(t)(a \le t \le b)$ be a geodesic on a Finsler manifold (M,F). Let $H(t,s)$ be a variation of σ such that each curve $\sigma_s(t) := H(t,s)(a \le t \le b)$ is a geodesic. Let

$$J(t) := \frac{\partial H}{\partial s}(t,0).$$

Then the vector field J is a *Jacobi field* along σ, satisfying the Jacobi equation

$$D_{\dot\sigma} D_{\dot\sigma} J(t) + \mathbf{R}_{\dot\sigma}\big(J(t)\big) = 0,$$

where $\mathbf{R}$ denotes the Riemann curvature of F. Locally, for any $x \in M$ and $y \in T_xM \backslash \{0\}$, the *Riemann curvature* $\mathbf{R}_y = R^i{}_k \frac{\partial}{\partial x^i} \otimes dx^k$ of F is given by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^m \partial y^k} y^m + 2G^m \frac{\partial^2 G^i}{\partial y^m \partial y^k} - \frac{\partial G^i}{\partial y^m}\frac{\partial G^m}{\partial y^k}. \tag{12}$$

The *Ricci curvature* is the trace of the Riemann curvature, i.e.,

$$\mathbf{Ric} = R^m{}_m. \tag{13}$$

For a Finsler metric F on M, let $(\mathbf{b}_i)_{i=1}^n$ be a basis for T_xM, and $\left(\omega^i\right)_{i=1}^n$ be the basis for T_x^*M dual to $(\mathbf{b}_i)_{i=1}^n$. Define the Busemann–Hausdorff volume form by

$$dV_{BH} := \sigma_{BH}(x) \omega^1 \wedge \cdots \wedge \omega^n,$$

where

$$\sigma_{BH}(x) := \frac{\mathrm{Vol}\big(\mathbf{B}^n(1)\big)}{\mathrm{Vol}\left\{(y^i) \in R^n | F(x, y^i \mathbf{b}_i) < 1\right\}}.$$

Here $\mathrm{Vol}\{\cdot\}$ denotes the Euclidean volume function on R^n.

If $F = \sqrt{g_{ij} y^i y^j}$ is a Riemannian metric, then

$$\sigma_{BH}(x) = \sqrt{det(g_{ij})}.$$

However, in general, for a Finsler metric F, $\sigma_{BH}(x) \neq \sqrt{det(g_{ij})}$. Define

$$\tau(x,y) := \ln\left[\frac{\sqrt{det(g_{ij}(x,y))}}{\sigma_{BH}(x)}\right].$$

Then τ is well-defined and it is called the *distortion* of F. The distortion τ characterizes the geometry of tangent space (T_xM, F_x). It is well-known that a Finsler metric F is Riemannian if and only if its distortion vanishes.

It is natural to study the rate of change of the distortion along geodesics. For a vector $y \in T_xM \setminus \{0\}$, let σ be the geodesic with $\sigma(0) = x$ and $\dot{\sigma} = y$. Put

$$\mathbf{S}(x,y) := \frac{d}{dt}\left[\tau\left(\sigma(t), \dot{\sigma}(t)\right)\right]|_{t=0}.$$

Equivalently,

$$\mathbf{S}(x,y) := \tau_{|m}(x,y)y^m, \tag{14}$$

where " | " denotes the horizontal covariant derivative with respect to the Berwald connection (or the Chern connection) of (M, F). The function $\mathbf{S}$ is called the *S-curvature* of Finsler metric F.

By the definition, the S-curvature measures the rate of change of (T_xM, F_x) in the direction $y \in T_xM$. It is easy to see that for any Berwald metric, $\mathbf{S} = 0$. In particular, $\mathbf{S} = 0$ for Riemannian metrics ([5]). Hence, S-curvature is a non-Rimannian quantity. For a Finsler metric F, the S-curvature is given by

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m}\left[\ln \sigma_{BH}\right]. \tag{15}$$

We say that F is of *isotropic S-curvature* if there exists a scalar function c on M such that $\mathbf{S} = (n+1)cF$, equivalently,

$$\frac{\tau_{|m}y^m}{F} = (n+1)c(x). \tag{16}$$

Equation (16) means that the rate of change of the tangent space (T_xM, F_x) along the direction $y \in T_xM$ at each $x \in M$ is independent of the direction y (but dependent on the point x). If c is constant, we say that F has *constant S-curvature.*

We recall that a *Randers metric* is a Finsler metric of the form $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta = b_iy^i$ is a 1-form on M. It is positive definite if and only if $b := \|\beta_x\|_\alpha < 1, x \in M$ (see, e.g., [5]).

For a Randers metric $F = \alpha + \beta$ on M, let G^i and ${}^\alpha G^i$ denote the geodesic coefficients of F and α, respectively. Then G^i and ${}^\alpha G^i$ are related by

$$G^i = {}^\alpha G^i + \alpha s^i{}_0 + \frac{1}{2F}\{-2\alpha s_0 + r_{00}\}y^i. \tag{17}$$

Furher, the Ricci curvature of $F = \alpha + \beta$ is given by

$$\mathbf{Ric} = {}^{\alpha}\mathbf{Ric} + \left(2\alpha s^m_{\ 0;m} - 2t_{00} - \alpha^2 t^m_{\ m}\right) + (n-1)\Xi, \tag{18}$$

where

$$\Xi := \frac{3}{4F^2}\left(r_{00} - 2\alpha s_0\right)^2 + \frac{1}{2F}\left[4\alpha(q_{00} - \alpha t_0) - (r_{00;0} - 2\alpha s_{0;0})\right]. \tag{19}$$

By (15) and (17), we obtain

$$\mathbf{S} = (n+1)\left[\frac{e_{00}}{2F} - (s_0 + \rho_0)\right], \tag{20}$$

where $e_{00} = r_{00} + 2\beta s_0$. For more details about Randers metrics, see [4].

3. Projective Ricci flat Randers metrics

In this section, we first derive a formula for the projective Ricci curvature of a Randers metric. Next, we characterize projective Ricci flat Randers metrics. By (4), the projective Ricci curvature is given by

$$\mathbf{PRic} = \mathbf{Ric} + \frac{n-1}{n+1}\mathbf{S}_{|m}y^m + \frac{n-1}{(n+1)^2}\mathbf{S}^2. \tag{21}$$

By (17), we have

$$G^i_m = {}^{\alpha}G^i_m + \alpha_{y^m}s^i_{\ 0} + \alpha s^i_{\ m} - \frac{F_{y^m}}{2F^2}(-2\alpha s_0 + r_{00})y^i + \frac{1}{F}(-\alpha_{y^m}s_0 - \alpha s_m + r_{m0})y^i + \frac{1}{2F}(-2\alpha s_0 + r_{00})\delta^i_{\ m}.$$

Thus

$$\mathbf{S}_{|m}y^m = y^m\frac{\partial \mathbf{S}}{\partial x^m} - G^l_m y^m \frac{\partial \mathbf{S}}{\partial y^l} = \mathbf{S}_{;m}y^m - \left[2\alpha s^m_{\ 0} + \frac{1}{F}(-2\alpha s_0 + r_{00})y^m\right]\frac{\partial \mathbf{S}}{\partial y^m}$$
$$= \mathbf{S}_{;m}y^m - 2\alpha s^m_{\ 0}\mathbf{S}_{y^m} - \frac{\mathbf{S}}{F}(-2\alpha s_0 + r_{00}). \tag{22}$$

From (20) we obtain

$$\mathbf{S}_{;m}y^m = (n+1)\left\{\frac{1}{2F}r_{00;0} + \frac{1}{F}r_{00}s_0 + \frac{1}{F}\beta s_{0;0} - \frac{1}{2F^2}e_{00}r_{00} - s_{0;0} - \rho_{0;0}\right\} =$$

$$= (n+1)\left\{\frac{1}{2F}r_{00;0} + \frac{1}{2F^2}(2\alpha s_0 - r_{00})r_{00} - \frac{1}{F}\alpha s_{0;0} - \rho_{0;0}\right\}, \tag{23}$$

$$2\alpha s^m{}_0 \mathbf{S}_{y^m} = \frac{2(n+1)}{F}\alpha q_{00} + \frac{2(n+1)}{F}\alpha s_0^2 - \frac{2(n+1)}{F}\alpha^2 t_0 - 2(n+1)\alpha(\rho_m s^m{}_0), \tag{24}$$

$$\frac{\mathbf{S}}{F}(-2\alpha s_0 + r_{00}) = \frac{n+1}{F}\left\{-\frac{2}{F}\alpha r_{00}s_0 + \frac{2}{F}\alpha^2 s_0^2 + \frac{1}{2F}r_{00}^2 + (2\alpha s_0 - r_{00})\rho_0\right\}, \tag{25}$$

where we have used $s_m s^m{}_0 = t_0$. Plugging (23), (24) and (25) into (22) yields

$$\frac{n-1}{n+1}\mathbf{S}_{|m}y^m(n-1)\left\{\frac{1}{2F}r_{00;0} + \frac{3}{F^2}\alpha r_{00}s_0 - \frac{1}{F^2}r_{00}^2 - \frac{1}{F}\alpha s_{0;0} - \rho_{0;0} - \frac{2}{F}\alpha q_{00}\right.$$
$$\left. - \frac{4}{F^2}\alpha^2 s_0^2 - \frac{2}{F^2}\alpha\beta s_0^2 + \frac{2}{F}\alpha^2 t_0 + 2\alpha(\rho_m s^m{}_0) - \frac{2}{F}\alpha s_0\rho_0 + \frac{1}{F}\rho_0 r_{00}\right\}. \tag{26}$$

Further, we have

$$\frac{n-1}{(n+1)^2}\mathbf{S}^2$$
$$= (n-1)\left\{\frac{1}{4F^2}r_{00}^2 + \frac{1}{F^2}\alpha^2 s_0^2 - \frac{1}{F^2}\alpha r_{00}s_0 + \rho_0^2 - \frac{1}{F}\rho_0 r_{00} + \frac{2}{F}\alpha\rho_0 s_0\right\}. \tag{27}$$

Substituting (18), (26) and (27) into (21), we obtain the following formula for the projective Ricci curvature of $F = \alpha + \beta$:

$$\mathbf{PRic} = {}^{\alpha}\mathbf{Ric} + 2\alpha s^m{}_{0;m} - 2t_{00} - \alpha^2 t^m{}_m$$
$$+ (n-1)\left\{-\frac{2\alpha\beta}{F^2}s_0^2 + 2\alpha(\rho_m s^m{}_0) - \rho_{0;0} - \frac{\alpha}{F^2}r_{00}s_0 + \rho_0^2\right\}. \tag{28}$$

Now we are in the position to prove Theorem 1.1.

Proof of Theorem 1.1. The proof of the sufficiency of the condition in Theorem 1.1 is immediate. To prove the necessity, let us assume that $\mathbf{PRic} = 0$, or, equivalently, $F^2\mathbf{PRic} = 0$. By (28), we obtain

$$F^{2\alpha}\mathbf{Ric} + 2F^2\alpha s^m{}_{0;m} - 2F^2 t_{00} - \alpha^2 F^2 t^m{}_m$$
$$+ (n-1)\{-2\alpha\beta s_0^2 + 2F^2\alpha(\rho_m s^m{}_0) - F^2\rho_{0;0} - \alpha r_{00}s_0 + F^2\rho_0^2\} = 0. \tag{29}$$

Equation (29) is equivalent to

$$\Xi_4\alpha^4 + \Xi_3\alpha^3 + \Xi_2\alpha^2 + \Xi_1\alpha + \Xi_0 = 0, \tag{30}$$

where

$$\Xi_4 = -t^m_{\ m}, \tag{31}$$

$$\Xi_3 = 2\left[s^m_{\ 0;m} - \beta t^m_{\ m} + (n-1)\rho_m s^m_{\ 0}\right], \tag{32}$$

$$\begin{aligned}\Xi_2 &= {}^{\alpha}\mathbf{Ric} + 4\beta s^m_{\ 0;m} - 2t_{00} - \beta^2 t^m_{\ m} + 4(n-1)\beta(\rho_m s^m_{\ 0}) \\ &\quad - (n-1)\rho_{0;0} + (n-1)\rho_0^2,\end{aligned} \tag{33}$$

$$\begin{aligned}\Xi_1 &= 2\beta\,{}^{\alpha}\mathbf{Ric} + 2\beta^2 s^m_{\ 0;m} - 4\beta t_{00} - 2(n-1)\beta s_0^2 + 2(n-1)\beta^2(\rho_m s^m_{\ 0}) \\ &\quad - 2(n-1)\beta\rho_{0;0} + 2(n-1)\beta\rho_0^2 - (n-1)r_{00}s_0,\end{aligned} \tag{34}$$

$$\Xi_0 = \left[{}^{\alpha}\mathbf{Ric} - 2t_{00} - (n-1)\rho_{0;0} + (n-1)\rho_0^2\right]\beta^2. \tag{35}$$

From (30) we obtain the following fundamental equations:

$$\Xi_4\alpha^4 + \Xi_2\alpha^2 + \Xi_0 = 0, \tag{36}$$

$$\Xi_3\alpha^2 + \Xi_1 = 0. \tag{37}$$

Rewrite (36) as

$$(\Xi_4\alpha^2 + \Xi_2)\alpha^2 + \Xi_0 = 0. \tag{38}$$

Because α^2 and β^2 are relatively prime polynomials in y, by (38) and the definition of Ξ_0 it follows that there exists a scalar function λ on M such that

$${}^{\alpha}\mathbf{Ric} - 2t_{00} - (n-1)\rho_{0;0} + (n-1)\rho_0^2 = \lambda(x)\alpha^2. \tag{39}$$

Substituting (39) into (38) yields

$$\Xi_4\alpha^2 + \Xi_2 + \lambda(x)\beta^2 = 0. \tag{40}$$

Besides, by (33), we have

$$\Xi_2 = \lambda(x)\alpha^2 + 4\beta s^m_{\ 0;m} - \beta^2 t^m_{\ m} + 4(n-1)\beta(\rho_m s^m_{\ 0}). \tag{41}$$

Rewrite (39) as

$${}^{\alpha}\mathbf{Ric} = \lambda(x)\alpha^2 + 2t_{00} + (n-1)[\rho_{0;0} - \rho_0^2]. \tag{42}$$

Substituting (41) into (40) and by (31), we get

$$(\lambda - t^m_{\ m})(\alpha^2 + \beta^2) = -4\beta[s^m_{\ 0;m} + (n-1)(\rho_m s^m_{\ 0})],$$

which implies the following:

$$\lambda = t^m_{\ m}, \tag{43}$$

$$s^m_{\ 0;m} = -(n-1)(\rho_m s^m_{\ 0}). \tag{44}$$

Further, by (42), (43) and (44) we obtain

$$\Xi_1 = 2t^m_{\ m}\alpha^2\beta - (n-1)s_0(r_{00} + 2\beta s_0), \tag{45}$$

$$\Xi_3 = -2\beta t^m_{\ m}. \tag{46}$$

Then, from (37), we get

$$s_0(r_{00} + 2\beta s_0) = 0.$$

From this we conclude that $s_0 = 0$ or $r_{00} + 2\beta s_0 = 0$. This completes the proof of Theorem 1.1. □

4. Application: projective Ricci flat Randers metrics with isotropic S-curvature

Let F be a Finsler metric on an n-dimensional manifold M. Assume that F is of isotropic S-curvature, i.e., $\mathbf{S} = (n+1)cF$. Then

$$\mathbf{S}_{|m} = (n+1)c_m F,$$

$$\mathbf{PRic} = \mathbf{Ric} + (n-1)c_0 F + (n-1)c^2 F^2,$$

where $c_m := c_{x^m}$ and $c_0 := c_m y^m$. In this case, F is a projective Ricci flat metric if and only if F is a weak Einstein metric satisfying

$$\mathbf{Ric} = (n-1)\left(\frac{3\theta}{F} + \sigma\right)F^2 \tag{47}$$

with $\theta = -c_0/3, \sigma = -c^2$.

Now, suppose that $F = \alpha + \beta$ is a Randers metric of isotropic S-curvature, $\mathbf{S} = (n+1)cF$. Then, by Lemma 3.1 in [2], α and β satisfy

$$r_{00} + 2\beta s_0 = 2c(\alpha^2 - \beta^2), \tag{48}$$

that is,

$$r_{ij} = -b_i s_j - b_j s_i + 2c(a_{ij} - b_i b_j).$$

We have

$$r_i = -b^2 s_i + 2c(1-b^2)b_i, \tag{49}$$

$$q_i = -b^2 t_i + 2c(1-b^2)s_i \tag{50}$$

and

$$q_0 + t_0 = (1-b^2)(t_0 + 2cs_0). \tag{51}$$

From Theorem 1.1 and Theorem 1.2, we obtain the following result.

Theorem 4.1. *Let $F = \alpha + \beta$ be a Randers metric on a manifold M of dimension n. Assume that F is of isotropic S-curvature, $\mathbf{S} = (n+1)cF$. Then F is a projective Ricci flat metric if and only if one of the following cases occurs:*

(i) *α and β satisfy the equations*

$$^{\alpha}\mathbf{Ric} = t^m_{\ m}\alpha^2 + 2t_{00} - (n-1)[s_{0;0} + s_0^2], \tag{52}$$

$$s^m_{\ 0;m} = (n-1)t_0, \tag{53}$$

$$r_{00} + 2\beta s_0 = 0. \tag{54}$$

In this case F is a Ricci flat metric.

(ii) *α and β satisfy the equations*

$$^{\alpha}\mathbf{Ric} = t^m_{\ m}\alpha^2 + 2t_{00} - 2(n-1)(2c^2\alpha^2 + c_0\beta), \tag{55}$$

$$s^m_{\ 0;m} = 0, \tag{56}$$

$$s_0 = 0. \tag{57}$$

Proof. *Case 1:* $r_{00} + 2\beta s_0 = 0$. Then, by (48), we know that $c = 0$ and

$$r_i = -b^2 s_i, \quad r_0 = -b^2 s_0.$$

Further, we have

$$r_{0;0} = -2(r_0 + s_0)s_0 - b^2 s_{0;0} = -2(1-b^2)s_0^2 - b^2 s_{0;0},$$

$$q_0 = -b^2 s_m s^m_{\ 0} = -b^2 t_0.$$

By Theorem 1.2, we get (52) and (53).

Case 2: $s_0 = 0$. Now, by (48), we know that $r_{00} = 2c(\alpha^2 - \beta^2)$ and

$$r_i = 2c(1-b^2)b_i, \quad r_0 = 2c(1-b^2)\beta.$$

Further, we have

$$r_{0;0} = 2(1-b^2)(c_0\beta + 2c^2\alpha^2 - 6c^2\beta^2),$$

$$q_0 = r_m s^m_{\ 0} = 2c(1-b^2)b_m s^m_{\ 0} = 2c(1-b^2)s_0 = 0, \quad t_0 = s_m s^m_{\ 0} = 0.$$

By Theorem 1.2, we get (55) and (56). □

As we mentioned in the first paragraph of this section, a Finsler metric F of isotropic S-curvature with $\mathbf{S} = (n+1)cF$ is projective Ricci flat if and only if F is a weak Einstein metric satisfying (47) with $3\theta + c_0 = 0, \sigma + c^2 = 0$. It is easy to see that Theorem 4.1 is consistent with Theorem 7.1.1 in [4], which characterizes weak Einstein Randers metrics. That is, we can also deduce Theorem 4.1 from (5), (6) and (7) with $3\theta + c_0 = 0, \sigma + c^2 = 0$.

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References

[1] D. Bao and C. Robles, On Ricci curvature and flag curvatures in Finsler geometry, In: A Sampler of Finsler Geometry, Math. Sci. Res. Inst. Publ., Vol. **50**, *Cambridge University Press*, 2004, 197–259.

[2] X. Cheng and Z. Shen, Randers metrics with special curvature properties, *Osaka J. Math.* **40** (2003), 87–101.

[3] X. Cheng and Z. Shen, A class of Finsler metrics with isotropic S-curvature, *Israel J. Math.* **169** (2009), 317–340.

[4] X. Cheng and Z. Shen, Finsler Geometry – An Approach via Randers Spaces, *Springer and Science Press*, *New York – Heidelberg – Beijing*, 2012.

[5] S. S. Chern and Z. Shen, Riemann–Finsler Geometry, Nankai Tracts in Mathematics, Vol. **6**, *World Scientific Co.*, *Singapore*, 2005.

[6] S. Ohta, Finsler interpolation inequalities, *Calc. Var. Partial Differential Equations* **36** (2009), 211–249.

[7] Z. Shen, Volume comparison and its applications in Riemann–Finsler geometry, *Advances in Math.* **128** (1997), 306–328.

[8] Z. Shen, Lectures on Finsler Geometry, *World Scientific Co.*, *Singapore*, 2001.

[9] Z. Shen, Differential Geometry of Spray and Finsler Spaces, *Kluwer Academic Publishers*, *Dordrecht – Boston – London*, 2001.

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