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On the oscillation of certain integral equations

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Abstract. The authors present conditions under which every nonoscillatory solution x of the integral equation

$$x(t) = e(t) - \int_c^t (t-s)^{\alpha-1} k(t,s) f(s, x(s)) ds, \quad c > 1, \quad 0 < \alpha \leq 1,$$

satisfies

$$|x(t)| = O(t) \quad \text{as} \quad t \to \infty, \text{ i.e., } \limsup_{t\to\infty} \frac{|x(t)|}{t} < \infty.$$

They also establish some sufficient conditions to ensure the oscillation of all solutions of this equation. The results obtained extend previous results in the literature, and the technique employed can be applied to some related integral equations that are equivalent to certain fractional differential equations.

1. Introduction

Consider the nonlinear integral equation

$$x(t) = e(t) - \int_c^t (t-s)^{\alpha-1} k(t,s) f(s, x(s)) ds, \quad c > 1, \quad 0 < \alpha \leq 1, \tag{1.1}$$

where we assume that:

(i) $e : [c, \infty) \to R$ is a continuous function;

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(ii) $k : [c, \infty) \times [c, \infty) \to R$ is continuous, and there exists a continuous function $a : [c, \infty) \to (0, \infty)$ such that

$$|k(t, s)| \leq a(t) \quad \text{for all} \quad t \geq s \geq c;$$

(iii) $f : [c, \infty) \times R \to R$ is a continuous function, and there exists a continuous function $h : [c, \infty) \to (0, \infty)$, and numbers λ and γ with $0 < \lambda < 1$ such that

$$0 \leq xf(t, x) \leq t^{\gamma-1} h(t) |x|^{\lambda+1} \quad \text{for} \quad x \neq 0 \quad \text{and} \quad t \geq c.$$

We only consider those solutions of equation (1.1) that are continuable and nontrivial in any neighborhood of ∞. Such a solution is said to be *oscillatory* if there exists $\{t_n\} \subseteq [c, \infty)$ with $t_n \to \infty$ as $n \to \infty$ such that $x(t_n) = 0$, and it is *nonoscillatory* otherwise.

Integral and fractional differential equations have gained considerably more attention in the last several years due to their applications in many areas in engineering and the sciences, such as in modeling systems and processes in physics, mechanics, chemistry, aerodynamics, and the electrodynamics of complex media. For more details, we refer the reader to the monographs of Podlubny [8] and Prudnikov *et al.* [9].

Oscillation and asymptotic behavior results for integral, as well as, integro-differential equations and fractional differential equations are not very prevalent in the literature; some results can be found in Bohner *et al.* [1] and Grace *et al.* [3], [4], [5], [6]. There do not appear to be any such results for integral equations of type (1.1). The main objective of this paper is to establish some new criteria for the oscillation and asymptotic behavior of solutions of (1.1).

2. Main results

To obtain our results in this paper, we need the following two lemmas.

Lemma 2.1 ([9])**.** *Let α, γ, and p be positive constants such that*

$$p(\alpha - 1) + 1 > 0 \quad \text{and} \quad p(\gamma - 1) + 1 > 0.$$

Then

$$\int_0^t (t - s)^{p(\alpha-1)} s^{p(\gamma-1)} ds = t^\theta B, \quad t \geq 0,$$

where $B := B[p(\gamma-1)+1, p(\alpha-1)+1]$, $B[\xi, \eta] = \int_0^1 s^{\xi-1}(1-s)^{\eta-1} ds$, $(\xi > 0,\ \eta > 0)$ *and* $\theta = p(\alpha + \gamma - 2) + 1$.

Lemma 2.2 ([7])**.** *If X and Y are nonnegative and $0 < \lambda < 1$, then*

$$X^{\lambda} - (1-\lambda)Y^{\lambda} - \lambda X Y^{\lambda-1} \le 0, \tag{2.1}$$

where equality holds if and only if $X = Y$.

In what follows, for any $t_1 \ge c$ and continuous function $w : [c,\infty) \to (0,\infty)$, we let

$$g(t; t_1, w) = e(t) + \Big((1-\lambda)\lambda^{\lambda/(1-\lambda)}\Big)\, a(t) \times \int_{t_1}^{t} (t-s)^{\alpha-1} \Big(s^{\gamma-1} w^{\lambda/(\lambda-1)}(s) h^{1/(1-\lambda)}(s)\Big)\, ds, \tag{2.2}$$

for $t \ge t_1$.

In our first theorem, we give sufficient conditions for any nonoscillatory solution x of equation (1.1) to satisfy $|x(t)| = O(t)$ as $t \to \infty$.

Theorem 2.1. *In addition to (i)–(iii), assume that the following conditions hold:*

(H$_1$) *There exists $p > 1$ such that*

$$p(\alpha-1)+1 > 0 \quad \text{and} \quad p(\gamma-1)+1 > 0;$$

(H$_2$) *Let $\theta = p(\alpha+\gamma-2)+1$, set*

$$m(t) = \begin{cases} t^{\theta/p} a(t), & \text{if } \theta \ge 0, \\ a(t), & \text{if } \theta < 0, \end{cases}$$

and assume that $m(t)$ is bounded on $[c,\infty)$;

(H$_3$) *Let $b : [c,\infty) \to (0,\infty)$ be continuous, $q = \frac{p}{p-1}$, and assume that*

$$\begin{cases} \int_c^{\infty} b^q(s)ds < \infty, & \text{if } \theta \ge 0, \\ \int_c^{\infty} s^{\theta q/p} b^q(s)ds < \infty, & \text{if } \theta < 0; \end{cases} \tag{2.3}$$

(H$_4$) $\dfrac{|g(t;c,b)|}{t}$ *is bounded on $[c,\infty)$.*

If $x(t)$ is a nonoscillatory solution of equation (1.1), then

$$\limsup_{t\to\infty} \frac{|x(t)|}{t} < \infty. \tag{2.4}$$

PROOF. Let $x(t)$ be an eventually positive solution of equation (1.1), say $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq c$. Let $F(t) = f(t, x(t))$. In view of (i)–(iii), from equation (1.1) we have

$$x(t) \leq e(t) + a(t) \int_c^t (t-s)^{\alpha-1} |f(s, x(s))| ds \leq e(t) + a(t) \int_c^{t_1} (t-s)^{\alpha-1} |F(s)|\, ds$$
$$+ a(t) \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} \left[h(s)x^{\lambda}(s) - b(s)x(s)\right] ds$$
$$+ a(t) \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} b(s)x(s)ds. \tag{2.5}$$

Letting

$$X = (h(s))^{1/\lambda} x(s) \quad \text{and} \quad Y = \left(\frac{1}{\lambda} b(s)(h^{-1/\lambda}(s))\right)^{1/(\lambda-1)}$$

in Lemma 2.2, we see that

$$h(s)x^{\lambda}(s) - b(s)x(s) \leq (1-\lambda)\lambda^{\lambda/(1-\lambda)} b^{\lambda/(\lambda-1)}(s) h^{1/(1-\lambda)}(s).$$

So from (2.5), we obtain

$$x(t) \leq e(t) + a(t) \int_c^{t_1} (t-s)^{\alpha-1} |F(s)|\, ds$$
$$+ \left((1-\lambda)\lambda^{\lambda/(1-\lambda)}\right) a(t) \int_{t_1}^t (t-s)^{\alpha-1} \left[s^{\gamma-1} b^{\lambda/(\lambda-1)}(s) h^{1/(1-\lambda)}(s)\right] ds$$
$$+ a(t) \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} b(s)x(s)ds. \tag{2.6}$$

Using the fact that $(t-s)^{\alpha-1} \leq (t_1 - s)^{\alpha-1}$, (2.6) yields

$$x(t) \leq a(t) \int_c^{t_1} (t_1 - s)^{\alpha-1} |F(s)|\, ds$$
$$+ |g(t; t_1, b)| + a(t) \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} b(s)x(s)ds. \tag{2.7}$$

Thus, in view of conditions (H_2) and (H_4),

$$\begin{aligned}\frac{x(t)}{t} := z(t) &\leq C + a(t)\int_{t_1}^{t} (t-s)^{\alpha-1} s^{\gamma-1} b(s)z(s)ds,\\ &\leq 1 + C + a(t)\int_{t_1}^{t} (t-s)^{\alpha-1} s^{\gamma-1} b(s)z(s)ds, \qquad (2.8)\end{aligned}$$

for some positive constant C. Applying Hölder's inequality and Lemma 2.1, we obtain

$$\begin{aligned}\int_{t_1}^{t}(t-s)^{\alpha-1} s^{\gamma-1}b(s)z(s)ds &\leq \left(\int_{t_1}^{t}(t-s)^{p(\alpha-1)} s^{p(\gamma-1)}ds\right)^{1/p}\left(\int_{t_1}^{t} b^q(s)z^q(s)ds\right)^{1/q}\\ &\leq \left(\int_{0}^{t}(t-s)^{p(\alpha-1)} s^{p(\gamma-1)}ds\right)^{1/p}\left(\int_{t_1}^{t} b^q(s)z^q(s)ds\right)^{1/q}\\ &\leq (Bt^{\theta})^{1/p}\left(\int_{t_1}^{t} b^q(s)z^q(s)ds\right)^{1/q},\end{aligned}$$

where $B := B[p(\gamma-1)+1, p(\alpha-1)+1]$ and $\theta = p(\alpha+\gamma-2)+1$. Hence,

$$a(t)\int_{t_1}^{t} (t-s)^{\alpha-1} s^{\gamma-1}b(s)z(s)ds \leq B^{1/p}t^{\theta/p}a(t)\left(\int_{t_1}^{t} b^q(s)z^q(s)ds\right)^{1/q}.$$

Clearly,

$$a(t)\int_{t_1}^{t}(t-s)^{\alpha-1} s^{\gamma-1}b(s)z(s)ds \leq \begin{cases} B^{1/p}B_1\left(\int_{t_1}^{t} b^q(s)z^q(s)ds\right)^{1/q}, & \text{if } \theta \geq 0,\\ B^{1/p}B_1^*\left(\int_{t_1}^{t} s^{\theta q/p}b^q(s)z^q(s)ds\right)^{1/q}, & \text{if } \theta < 0,\end{cases}$$

where B_1 and B_1^* are upper bounds for $t^{\theta/p}a(t)$ and $a(t)$, respectively. Using this inequality and the fact that

$$(x+y)^q \leq 2^{q-1}(x^q+y^q) \quad \text{for} \quad x, y \geq 0 \quad \text{and} \quad q > 1,$$

(2.8) implies

$$z^q(t) \le \begin{cases} 2^{q-1}\left((1+C)^q + (B_1 B^{1/p})^q \int\limits_{t_1}^{t} b^q(s) z^q(s) ds\right), & \text{if } \theta \ge 0, \\ 2^{q-1}\left((1+C)^q + (B_1^* B^{1/p})^q \int\limits_{t_1}^{t} s^{\theta q/p} b^q(s) z^q(s) ds\right), & \text{if } \theta < 0. \end{cases} \tag{2.9}$$

If we set $u(t) = z^q(t)$, $P_1 = 2^{q-1}(1+C)^q$, and

$$Q_1(t) = \begin{cases} 2^{q-1}(B_1 B^{1/p})^q, & \text{if } \theta \ge 0, \\ 2^{q-1}(B_1^* B^{1/p})^q, & \text{if } \theta < 0, \end{cases}$$

then $z(t) = u^{1/q}(t)$ and

$$u(t) \le \begin{cases} P_1 + Q_1 \int\limits_{t_1}^{t} b^q(s) u(s) ds, & \text{if } \theta \ge 0, \\ P_1 + Q_1 \int\limits_{t_1}^{t} s^{\theta q/p} b^q(s) u(s) ds, & \text{if } \theta < 0, \end{cases} \tag{2.10}$$

for $t \ge t_1$. By Gronwall's inequality and condition (2.3), we see that $u(t)$ is bounded, and so

$$\limsup_{t\to\infty} \frac{x(t)}{t} < \infty.$$

A similar proof holds if $x(t)$ is an eventually negative solution of the equation. This completes the proof of the theorem. □

The following example illustrates the above theorem.

Example 2.1. Consider the equation

$$x(t) = t\sin t - \int\limits_{c}^{t} (t-s)^{-\frac{1}{2p}} \left[\frac{ts^{-\frac{1}{2p}}}{t^2+1}\right] h(s) s^{-\frac{1}{2p}} |x(s)|^{\lambda-1} x(s) ds \tag{2.11}$$

with $c > 1$, $p > 1$, and $0 < \lambda < 1$. Here we have $k(t,s) = \frac{t}{t^2+1} s^{-\frac{1}{2p}}$, $a(t) = \frac{1}{t}$, $q = \frac{p}{p-1}$, $\alpha = 1 - \frac{1}{2p} = \gamma$, $p(\alpha - 1) + 1 = \frac{1}{2} = p(\gamma - 1) + 1$ and $\theta = p(1 - \frac{1}{2p} + 1 - \frac{1}{2p} - 2) + 1 = 0$. Note that $m(t) = a(t)$. We take $b(t) = h(t)$, and assume that

$$\int\limits_{c}^{\infty} h^q(s) ds < \infty \quad \text{and} \quad \limsup_{t\to\infty} \frac{1}{t^2}\left(\int\limits_{c}^{t} (t-s)^{-\frac{1}{2p}} s^{-\frac{1}{2p}} h(s) ds\right) < \infty.$$

Then all conditions of Theorem 2.1 are satisfied, so every nonoscillatory solution x of equation (2.11) satisfies

$$\limsup_{t\to\infty} \frac{x(t)}{t} < \infty.$$

Next, we establish an oscillation result for equation (1.1).

Theorem 2.2. *In addition to conditions* (i)–(iii), (H_1)–(H_2) *and* (H_4), *assume that*

$$\begin{cases} \int_c^\infty s^q b^q(s)ds < \infty, & \text{if } \theta \geq 0, \\ \int_c^\infty s^{\theta q/p} s^q b^q(s)ds < \infty, & \text{if } \theta < 0. \end{cases} \tag{2.12}$$

If

$$\liminf_{t\to\infty} g(t;c,b) = -\infty \quad \text{and} \quad \limsup_{t\to\infty} g(t;c,b) = \infty, \tag{2.13}$$

then all solutions of equation (1.1) *are oscillatory.*

Proof. Since condition (2.12) implies (H_3), the conclusion of Theorem 2.1 holds. Without any loss of generality, assume that $t_1 \geq c$ is sufficiently large so that $x(t) > 0$ for $t \geq t_1$. From equation (1.1), we have

$$\begin{aligned} x(t) &= e(t) - \int_c^{t_1} (t-s)^{\alpha-1} k(t,s) f(s,x(s))ds - \int_{t_1}^t (t-s)^{\alpha-1} k(t,s) f(s,x(s))ds \\ &\leq e(t) - \int_c^{t_1} (t-s)^{\alpha-1} k(t,s) f(s,x(s))ds + a(t) \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} h(s) x^\lambda(s)ds. \end{aligned}$$

Proceeding as in the proof of Theorem 2.1, we obtain (see (2.6))

$$\begin{aligned} x(t) &\leq e(t) + a(t) \int_c^{t_1} (t-s)^{\alpha-1} |F(s)|ds \\ &\quad + \left((1-\lambda)\lambda^{\lambda/(1-\lambda)}\right) a(t) \int_{t_1}^t (t-s)^{\alpha-1} \left[s^{\gamma-1} b^{\lambda/(\lambda-1)}(s) h^{1/(1-\lambda)}(s)\right] ds \\ &\quad + a(t) \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} b(s) s \left(\frac{x(s)}{s}\right) ds. \end{aligned} \tag{2.14}$$

Applying Hölder's inequality and Lemma 2.1 gives

$$x(t) \le g(t; t_1, b) + a(t) \int_c^{t_1} (t_1 - s)^{\alpha-1} |F(s)| ds +$$
$$+ \begin{cases} B^{1/p} B_1 \left(\int_{t_1}^t s^q b^q(s) \left(\frac{x(s)}{s} \right)^q ds \right)^{1/q}, & \text{if } \theta \ge 0, \\ B^{1/p} B_1^* \left(\int_{t_1}^t s^{\theta q/p} s^q b^q(s) \left(\frac{x(s)}{s} \right)^q ds \right)^{1/q}, & \text{if } \theta < 0. \end{cases} \tag{2.15}$$

Now, by Theorem 2.1, $(x(t)/t)$ is bounded, so from condition (2.12), the last two integrals on the right hand side of (2.15) are finite. Taking the lim inf of both sides of this inequality as $t \to \infty$, in view of condition (2.13), we obtain a contradiction to the fact that $x(t)$ is eventually positive. The proof, if x is eventually negative, is similar. □

The following corollary is an immediate consequence of the above theorem.

Corollary 2.1. *Let condition* (2.13) *in Theorem 2.2 be replaced by*

$$\lim_{t\to\infty} a(t) \int_c^t (t-s)^{\alpha-1} \left(s^{\gamma-1} b^{\lambda/(\lambda-1)}(s) h^{1/(1-\lambda)}(s) \right) ds < \infty, \tag{2.16}$$

and

$$\liminf_{t\to\infty} e(t) = -\infty \quad \text{and} \quad \limsup_{t\to\infty} e(t) = \infty.$$

Then all solutions of equation (1.1) *are oscillatory.*

We conclude this paper with the following example.

Example 2.2. Consider the equation

$$x(t) = t\cos t - \int_2^t (t-s)^{-\frac{1}{4}} \left[\frac{e^t}{st^2(e^t+1)} \right] s^{-\frac{1}{3}} x^{\frac{1}{3}}(s) ds. \tag{2.17}$$

Here we have $p = 2$, $\lambda = 1/3$, $k(t,s) = \frac{e^t}{st^2(e^t+1)}$, $a(t) = 1/t^2$, $f(t,x) = t^{-\frac{1}{3}} x^{\frac{1}{3}}$, $h(t) = 1$, $q = 2$, $\alpha = \frac{3}{4} = \gamma$ and $\theta = 0$. If we take $b(t) = t^{-2}$, then

$$\int_2^\infty s^2 b^q(s) ds = \int_2^\infty s^{-2} ds < \infty,$$

and (2.16) becomes

$$\lim_{t\to\infty}\frac{1}{t^2}\int_2^t (t-s)^{-\frac{1}{4}}s^{\frac{3}{4}}ds \le \lim_{t\to\infty}\frac{1}{t^{\frac{5}{4}}}\int_2^t (t-s)^{-\frac{1}{4}}ds < \infty.$$

We see that the hypotheses of Corollary 2.1 are satisfied, and so all solutions of equation (1.1) are oscillatory.

3. Concluding remarks

To see that the approach and results obtained here can be applied to fractional differential equations, consider the initial value problem for fractional differential equations of the form

$$\begin{cases} D_c^\alpha x = f(t,x), \\ \lim_{t\to c^+} I_c^{1-\alpha}x(t) = x_0, \end{cases} \tag{3.1}$$

where D_c^α is the differential operator of order α, $0 < \alpha \le 1$, and I_c^β is the Riemann–Liouville fractional integral opeator given by

$$I_c^\beta x(t) = \frac{1}{\Gamma(\beta)}\int_c^t (t-s)^{\beta-1}x(s)ds.$$

It is known that this fractional initial value problem is equivalent to the integral equation

$$x(t) = \frac{x_0(t-c)^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_c^t (t-s)^{\alpha-1}f(s,x(s))ds. \tag{3.2}$$

Clearly, (3.2) has the form of our equation (1.1). Similar remarks can be made for fractional initial value problems involving the Caputo derivative.

It should also be clear that it would be possible to obtain similar results for equations with $n-1 < \alpha \le n$.

Finally, it would be of interest to obtain results analogous to those in this paper even in the case where $\lambda > 1$ in condition (iii).

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