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# On inequalities for alternating trigonometric sums

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**Abstract.** We present various inequalities for alternating trigonometric sums. Among others, we prove that the double-inequality

$$\frac{1-\sqrt{2}}{3} \le \sum_{k=1}^{n} (-1)^{k-1} \, \frac{\sin^2((2k-1)x)}{2k-1} \le 1$$

is valid for all natural numbers $n$ and real numbers $x$. Both bounds are sharp.

## 1. Introduction and statement of results

Two classical results on trigonometric polynomials state that for all $n \in \mathbf{N}$ and $x \in (0, \pi)$,

$$0 < \sum_{k=1}^{n} \frac{\sin(kx)}{k} \tag{1.1}$$

and

$$-1 < \sum_{k=1}^{n} \frac{\cos(kx)}{k}. \tag{1.2}$$

Both bounds are sharp. The validity of (1.1) was conjectured by Fejér in 1910, and the first proofs were published by Jackson [9] and Gronwall [8] in 1911 and 1912, respectively. The counterpart (1.2) was established by Young [14] in 1913.

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The inequalities of Fejér–Jackson–Gronwall and Young have stimulated the research of many mathematicians, who discovered remarkable new proofs, generalizations, refinements and numerous variants of (1.1) and (1.2). Moreover, it was shown that inequalities for trigonometric polynomials have interesting applications in various fields, like, for example, the theory of univalent functions and Fourier analysis.

Here, we demonstrate that the inequalities (1.1) and (1.2) have an application in the theory of absolutely monotonic functions. An infinitely differentiable function defined on an interval $I$ is said to be absolutely monotonic if the function and all its derivatives are nonnegative on $I$. A detailed study of absolutely monotonic functions can be found in Widder [13, Chapter IV].

For $x \in (-1,1)$ and $\theta \in (0,\pi)$, we define the two functions

$$f_\theta(x) = \frac{1}{1-x} \arctan \frac{x\sin(\theta)}{1-x\cos(\theta)}$$

and

$$g_\theta(x) = \frac{1}{1-x}\left(x - \frac{1}{2}\log\big(1-2x\cos(\theta)+x^2\big)\right).$$

**Proposition.** *For all $\theta \in (0,\pi)$, the functions $f_\theta(x)$ and $g_\theta(x)$ are absolutely monotonic on $(0,1)$.*

Proof. Using the identities (see Gould [7, (2.23), (2.24)])

$$\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n} x^n = \arctan \frac{x\sin(\theta)}{1-x\cos(\theta)}$$

and

$$\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n} x^n = -\frac{1}{2}\log\big(1-2x\cos(\theta)+x^2\big),$$

which are valid for $|x|<1$, we arrive at the series representations

$$f_\theta(x) = \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{\sin(k\theta)}{k} x^n$$

and

$$g_\theta(x) = \sum_{n=1}^{\infty}\Big(1+\sum_{k=1}^{n}\frac{\cos(k\theta)}{k}\Big)x^n.$$

It follows that if $\theta \in (0,\pi)$, then

$$\frac{d^n}{dx^n} f_\theta(x) > 0 \quad \text{and} \quad \frac{d^n}{dx^n} g_\theta(x) > 0 \quad (0 \le n \in \mathbf{Z}; 0<x<1).$$

This means that $f_\theta$ and $g_\theta$ are absolutely monotonic on $(0,1)$. $\square$

We remark that ASKEY and GASPER [4] proved that the function $x \mapsto (1/x)f_\theta(x)$ is absolutely monotonic on $(0,1)$ if $\theta \in (0,\pi)$.

The inequalities (1.1) and (1.2) can be refined if we assume that $n \geq 2$. We have

$$x^2\left(\cot\frac{x}{2} - \frac{\pi - x}{2}\right) < \sum_{k=1}^{n} \frac{\sin(kx)}{k} \quad (2 \leq n \in \mathbf{N}; 0 < x < \pi) \tag{1.3}$$

and

$$-\frac{5}{6} \leq \sum_{k=1}^{n} \frac{\cos(kx)}{k} \quad (2 \leq n \in \mathbf{N}; 0 < x < \pi). \tag{1.4}$$

The constant $-5/6$ is best possible. Proofs for (1.3) and (1.4) can be found in ALZER and KOUMANDOS [1], and in BROWN and KOUMANDOS [6], respectively. We note that inequality (1.4) plays an important role in the proof of Theorem 1 below.

Among the many variants and analogues of (1.1) and (1.2) given in the literature, we mention three inequalities which have motivated our work. They are all concerned with sums having only terms involving odd multiples of $x$. The first is an elegant counterpart of (1.1) due to KOSCHMIEDER [10]:

$$0 < \sum_{k=1}^{n} \frac{\sin((2k-1)x)}{2k-1} \quad (n \in \mathbf{N}; 0 < x < \pi).$$

This is a special case of a more general result. If $(a_k)_{k\geq 1}$ is a decreasing sequence of positive numbers, then

$$0 < \sum_{k=1}^{n} a_k \sin((2k-1)x) \quad (n \in \mathbf{N}; 0 < x < \pi).$$

Proofs for the next inequalities are given in ALZER and KOUMANDOS [2]:

$$-\frac{1}{18}\sqrt{3} \leq \sum_{k=1}^{n} (-1)^{k-1} \frac{\sin((2k-1)x)}{k} \quad (n \in \mathbf{N}; 0 < x < \pi) \tag{1.5}$$

and

$$-\mathrm{Si}(\pi) < \sum_{k=1}^{n} (-1)^{k-1} \frac{\cos((2k-1)x)}{k} < \mathrm{Si}(\pi) \quad (n \in \mathbf{N}; x \in \mathbf{R}). \tag{1.6}$$

Here, $\mathrm{Si}(x) = \int_0^x \big(\sin(t)/t\big)\,dt$ denotes the sine integral. The constant bounds in (1.5) and (1.6) are sharp. For more information on inequalities connected with trigonometric sums, we refer to Askey [3], Askey and Gasper [5], Milovanović, Mitrinović and Rassias [12, Chapter 4], and the references given therein.

In this paper, we present new inequalities for certain alternating trigonometric sums. Our first theorem offers a relative of (1.1).

**Theorem 1.** *For all natural numbers $n$ and real numbers $x$, we have*

$$-\frac{1}{8} \le \sum_{k=1}^{n} (-1)^{k-1} \frac{\sin^2(kx)}{k}. \tag{1.7}$$

*The lower bound is sharp.*

Using the summation by parts method, we obtain the following extension of (1.7).

**Corollary 1.** *Let $(a_k)_{k\ge 1}$ be a sequence of nonnegative real numbers such that $(ka_k)_{k\ge 1}$ is decreasing. Then, for $n \in \mathbf{N}$ and $x \in \mathbf{R}$,*

$$-\frac{a_1}{8} \le \sum_{k=1}^{n} (-1)^{k-1} a_k \sin^2(kx).$$

*In particular, for any $\alpha \in [0,1)$,*

$$-\frac{1}{8(1-\alpha)} \le \sum_{k=1}^{n} (-1)^{k-1} \frac{\sin^2(kx)}{k-\alpha}.$$

The next theorem provides a generalization of (1.7) involving two parameters.

**Theorem 2.** *Let $\lambda$ and $\mu$ be real numbers with $\lambda \ge \mu \ge 0$. For all natural numbers $n$ and real numbers $x$, we have*

$$\frac{5\mu-\lambda}{8} \le \sum_{k=1}^{n} (-1)^{k-1} \frac{\lambda \sin^2(kx) + \mu \cos^2(kx)}{k}. \tag{1.8}$$

*The lower bound is sharp.*

*Remark 1.* We denote the sum given in (1.8) by $D_n(\lambda,\mu;x)$. If $\lambda > \mu$, then

$$\lim_{n\to\infty} D_{2n}(\lambda,\mu;\pi/2) = \mu \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} + (\lambda-\mu) \sum_{k=1}^{\infty} \frac{1}{2k-1} = \infty.$$

This implies that there are no constant upper bounds for the sums in (1.8) and (1.7).

An application of Theorem 2 leads to a class of nonnegative trigonometric sums.

**Theorem 3.** *Let $c$ be a real number. The inequality*

$$0 \le \sum_{k=1}^{n} (-1)^{k-1} \frac{\sin^2(kx) + c\,\cos^2(kx)}{k} \tag{1.9}$$

*holds for all natural numbers $n$ and real numbers $x$ if and only if $c \in [1/5, 1]$.*

The following result is a striking companion of (1.7).

**Theorem 4.** *For all natural numbers $n$ and real numbers $x$, we have*

$$\frac{1-\sqrt{2}}{3} \le \sum_{k=1}^{n} (-1)^{k-1} \frac{\sin^2((2k-1)x)}{2k-1} \le 1. \tag{1.10}$$

*Both bounds are sharp.*

Again, the summation by parts method yields the next corollary.

**Corollary 2.** *Let $(b_k)_{k\ge1}$ be a sequence of nonnegative real numbers such that $((2k-1)b_k)_{k\ge1}$ is decreasing. Then, for $n \in \mathbf{N}$ and $x \in \mathbf{R}$,*

$$\frac{1-\sqrt{2}}{3}\, b_1 \le \sum_{k=1}^{n} (-1)^{k-1}\, b_k \sin^2((2k-1)x) \le b_1.$$

*In particular, for any $\beta \in [1, 2)$,*

$$\frac{1-\sqrt{2}}{3(2-\beta)} \le \sum_{k=1}^{n} (-1)^{k-1} \frac{\sin^2((2k-1)x)}{2k-\beta} \le \frac{1}{2-\beta}.$$

Finally, we offer an analogue of (1.8), an extension of (1.10).

**Theorem 5.** *Let $\lambda$ and $\mu$ be real numbers with $\lambda \ge \mu \ge 0$. For all natural numbers $n$ and real numbers $x$, we have*

$$\begin{aligned} &\frac{(1+\sqrt{2})\mu + (1-\sqrt{2})\lambda}{3} \\ &\quad \le \sum_{k=1}^{n} (-1)^{k-1} \frac{\lambda \sin^2((2k-1)x) + \mu \cos^2((2k-1)x)}{2k-1} \le \lambda. \end{aligned} \tag{1.11}$$

*Both bounds are sharp.*

In the next section, we collect some lemmas which we need to establish Theorem 4. The proofs of our theorems are given in Section 3.

## 2. Lemmas

Throughout this paper, we maintain the following notations:

$$\begin{aligned} C_n(x) &= \sum_{k=1}^{n} (-1)^{k-1} \frac{1-\cos((2k-1)x)}{2k-1} \\ &= \int_0^x \sum_{k=1}^{n} (-1)^{k-1} \sin((2k-1)t) dt = (-1)^{n-1} \int_0^x \frac{\sin(2nt)}{2\cos(t)}\, dt, \end{aligned}$$

$$F_m(x) = \int_0^x \frac{\sin(mt)}{2\sin(t)}\, dt = C_n(x+\pi/2) - C_n(\pi/2) \quad (m=2n),$$

and

$$x_k = \frac{k\pi}{m}.$$

**Lemma 1.** *If $m \geq 6$, then*

$$F_m(x_1) - F_m(x_2) < 0.263.$$

Proof. We obtain

$$F_m(x_1) - F_m(x_2) = \frac{1}{2m} \int_{\pi}^{2\pi} \frac{-\sin(t)}{t} \frac{t}{\sin(t/m)}\, dt.$$

Since $t \mapsto t/\sin(t)$ is increasing on $(0,\pi]$, we obtain for $t \in [\pi, 2\pi]$:

$$\frac{t}{\sin(t/m)} \leq \frac{\pi}{3\sin(\pi/3)} m \leq 1.21m.$$

Thus,

$$F_m(x_1) - F_m(x_2) \leq -\frac{1.21}{2} \int_{\pi}^{2\pi} \frac{\sin(t)}{t}\, dt = \frac{1.21}{2}\big(\mathrm{Si}(\pi) - \mathrm{Si}(2\pi)\big) = 0.2624\ldots. \qquad \square$$

**Lemma 2.** *If $m = 2n \geq 2$, then*

$$F_m(\pi/2) - F_m(x_1) = C_n(\pi/2 - x_1).$$

Proof. We have

$$\begin{aligned} &2(-1)^{n-1}\big(F_m(\pi/2) - F_m(x_1) - C_n(\pi/2 - x_1)\big) \\ &\quad = 2\,(-1)^{n-1}\big(C_n(\pi) - C_n(\pi/2 + x_1) - C_n(\pi/2 - x_1)\big) \end{aligned}$$

$$= \left( \int_0^{\pi} - \int_0^{\pi/2+x_1} - \int_0^{\pi/2-x_1} \right) \frac{\sin(2nt)}{\cos(t)}\, dt$$

$$= \left( \int_{\pi/2-x_1}^{\pi} - \int_0^{\pi/2+x_1} \right) \frac{\sin(2nt)}{\cos(t)}\, dt = 0. \qquad \square$$

**Lemma 3.** *Let $m \geq 6$ be an even integer. If $r = [(m-2)/4]$ and $s = [m/4]$, then*

$$F_m(x_1) > F_m(x_3) > \cdots > F_m(x_{2r+1}) \tag{2.1}$$

*and*

$$0 = F_m(x_0) < F_m(x_2) < F_m(x_4) < \cdots < F_m(x_{2s}). \tag{2.2}$$

Proof. As shown in Kwong [11], integration by parts yields, for $1 \leq k \leq r$,

$$F_m(x_{2k+1}) - F_m(x_{2k-1}) = \int_{x_{2k-1}}^{x_{2k+1}} \frac{\sin(mt)}{2\sin(t)}\, dt$$

$$= -\int_{(2k-1)\pi/m}^{(2k+1)\pi/m} \frac{(1+\cos(mt))\cos(t)}{2m\sin^2(t)}\, dt < 0$$

and, for $0 \leq k \leq s-1$,

$$F_m(x_{2k+2}) - F_m(x_{2k}) = \int_{2k\pi/m}^{(2k+2)\pi/m} \frac{(1-\cos(mt))\cos(t)}{2m\sin^2(t)}\, dt > 0.$$

This leads to (2.1) and (2.2). $\square$

**Lemma 4.** *Let $m \geq 6$ be an even integer. Then, for $x \in [0, \pi/2]$,*

$$0 \leq F_m(x) \leq F_m(x_1). \tag{2.3}$$

Proof. Since $F_m$ is increasing on $[x_{2k}, x_{2k+1}]$ ($k = 0, ..., [(m-2)/4]$), and decreasing on $[x_{2k+1}, x_{2k+2}]$ ($k = 0, ..., [(m-4)/4]$), we conclude that $F_m$ attains a local maximum at $x_{2k+1}$, and a local minimum at $x_{2k}$. Let $s = [m/4]$. If $m/2$ is odd, then $x_{2s+1} = \pi/2$, and if $m/2$ is even, then $x_{2s} = \pi/2$. This means that $\pi/2$ is either a local maximum or a local minimum of $F_m$. Using (2.1) and (2.2) gives

$$F_m(0) = 0 \leq F_m(\pi/2) \leq F_m(x_1).$$

This implies that (2.3) is valid for all $x \in [0, \pi/2]$. $\square$

*Remark 2.* A proof of the right-hand side of (2.3) is also given in Alzer and Koumandos [1].

**Lemma 5.** *If $n \geq 3$ and $x \in [0, \pi]$, then*

$$C_n(\pi/2 - x_1) \leq C_n(x).$$

Proof. Let $m = 2n$. We show that

$$F_m(x - \pi/2) \geq F_m(-x_1). \tag{2.4}$$

*Case 1.* $0 \leq x \leq \pi/2$. Then, $0 \leq \pi/2 - x \leq \pi/2$. Applying $F_m(-x) = -F_m(x)$ and Lemma 4 gives

$$F_m(x - \pi/2) = -F_m(\pi/2 - x) \geq -F_m(x_1) = F_m(-x_1).$$

*Case 2.* $\pi/2 \leq x \leq \pi$. We have $0 \leq x - \pi/2 \leq \pi/2$, so that Lemma 4 yields

$$F_m(x - \pi/2) \geq 0 \geq -F_m(x_1) = F_m(-x_1).$$

From (2.4) we conclude that

$$C_n(x) = F_m(x - \pi/2) + C_n(\pi/2) \geq F_m(-x_1) + C_n(\pi/2) = C_n(\pi/2 - x_1). \quad \square$$

**Lemma 6.** *Let $n \geq 1$ be an odd integer. Then, for $x \in [0, \pi]$,*

$$C_n(x) \leq 2.$$

Proof. We have

$$C_1(x) = 1 - \cos(x) \leq 2.$$

Let $n \geq 3$ be odd and $m = 2n$. We consider two cases.

*Case 1.* $0 \leq x \leq \pi/2$. Then, $0 \leq \pi/2 - x \leq \pi/2$. Applying $F_m(-x) = -F_m(x)$, Lemma 4 and

$$C_n(\pi/2) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} \leq C_3(\pi/2) = \frac{13}{15} \tag{2.5}$$

gives

$$C_n(x) = C_n(\pi/2) - F_m(\pi/2 - x) \leq C_n(\pi/2) \leq \frac{13}{15}.$$

*Case 2.* $\pi/2 \le x \le \pi$. From Lemma 4 we obtain

$$C_n(x) = C_n(\pi/2) + F_m(x - \pi/2) \le C_n(\pi/2) + F_m(x_1). \tag{2.6}$$

If $s \in (0, \pi]$, then $x \mapsto x \sin(s/x)$ is increasing on $[1, \infty)$, so that for $m \ge 6$,

$$F_m(x_1) = \frac{1}{2} \int_0^\pi \frac{\sin(s)}{m \sin(s/m)}\, ds \le \frac{1}{2} \int_0^\pi \frac{\sin(s)}{6 \sin(s/6)}\, ds = \frac{14}{15}. \tag{2.7}$$

Combining (2.5), (2.6) and (2.7) gives

$$C_n(x) \le \frac{27}{15}.$$

This completes the proof of Lemma 6. □

## 3. Proofs of theorems

Proof of Theorem 1. Let $A_n(x)$ be the sum in (1.7). Since

$$A_n(x + \pi) = A_n(x) \quad \text{and} \quad A_n(\pi/2 - x) = A_n(\pi/2 + x),$$

it suffices to prove (1.7) for $x \in [0, \pi/2]$. We have

$$A_1(x) = \sin^2(x), \;\; A_2(x) + \frac{1}{8} = \frac{1}{8}\big(4\cos^2(x) - 3\big)^2, \;\; A_3(x) = A_2(x) + \frac{1}{3}\sin^2(3x).$$

It follows that (1.7) holds for $n = 1, 2, 3$. Let $n \ge 4$. Applying

$$\sin^2(y) = \frac{1}{2} - \frac{1}{2}\cos(2y) \tag{3.1}$$

gives

$$-2A_n(x) = G_n + H_n(2x),$$

where

$$G_n = \sum_{k=1}^{n} \frac{(-1)^k}{k} \quad \text{and} \quad H_n(x) = \sum_{k=1}^{n} (-1)^{k-1} \frac{\cos(kx)}{k}.$$

We set $x = (\pi - z)/2$. Then, $0 \le z \le \pi$. Using

$$G_n \le G_4 = -\frac{7}{12}$$

and (1.4) yields

$$-2A_n(x) \le -\frac{7}{12} + H_n(2x) = -\frac{7}{12} - \sum_{k=1}^{n} \frac{\cos(kz)}{k} \le -\frac{7}{12} + \frac{5}{6} = \frac{1}{4}.$$

This leads to (1.7). Moreover, since $A_2(\pi/6) = -1/8$, we conclude that $-1/8$ is the best possible constant lower bound. □

Proof of Theorem 2. As in Section 1, we denote the sum in (1.8) by $D_n(\lambda,\mu;x)$. Then,

$$D_n(\lambda,\mu;x) = -\mu G_n + (\lambda-\mu)A_n(x).$$

We obtain

$$D_1(\lambda,\mu;x) = \mu + (\lambda-\mu)\sin^2(x) \geq \mu \geq \frac{5\mu-\lambda}{8}.$$

Let $n \geq 2$. We apply (1.7) and

$$\frac{1}{2} = -G_2 \leq -G_n.$$

Thus,

$$D_n(\lambda,\mu;x) \geq \frac{1}{2}\mu - \frac{1}{8}(\lambda-\mu) = \frac{5\mu-\lambda}{8} = D_2(\lambda,\mu;\pi/6).$$

This completes the proof of Theorem 2. □

Proof of Theorem 3. Let $c \in [1/5, 1]$. Applying Theorem 2, we obtain

$$D_n(1,c;x) \geq \frac{5c-1}{8} \geq 0.$$

This settles (1.9). We assume that (1.9) is valid for all $n$ and $x$. Then,

$$D_2(1,c;\pi/6) = \frac{5}{8}\Big(c - \frac{1}{5}\Big) \geq 0.$$

Thus, $c \geq 1/5$. We have

$$D_n(1,c;\pi/2) = \frac{1+c}{2}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k} + \frac{1-c}{2}\sum_{k=1}^{n}\frac{1}{k} \geq 0.$$

If $c > 1$, then

$$\lim_{n\to\infty} D_n(1,c;\pi/2) = -\infty.$$

A contradiction. Hence, $c \leq 1$. □

Proof of Theorem 4. Let $S_n(x)$ be the sine sum in (1.10). Since

$$S_n(x+\pi) = S_n(x) \quad \text{and} \quad S_n(\pi/2 - x) = S_n(\pi/2 + x),$$

it remains to prove (1.10) for $x \in [0, \pi/2]$. Using (3.1), we obtain the representation

$$S_n(x) = \frac{1}{2}C_n(y) \quad \text{with} \quad y = 2x \in [0,\pi].$$

Thus, we have to show that

$$-0.276.... = \frac{2}{3}(1-\sqrt{2}) \leq C_n(y) \leq 2. \tag{3.2}$$

Applying Lemma 6 and

$$C_{2N}(y) \leq C_{2N+1}(y) \quad (N \geq 0)$$

reveals that the right-hand side of (3.2) is valid for all $n \geq 1$.

We find that

$$C_1(y) = 1 - \cos(y) \geq 0 > \frac{2}{3}(1-\sqrt{2})$$

and

$$C_2(y) = \frac{2}{3}(1-\sqrt{2}) + \frac{1}{3}\big(\sqrt{2}+\cos(y)\big)\big(\sqrt{2}-2\cos(y)\big)^2 \geq \frac{2}{3}(1-\sqrt{2}).$$

Let $n \geq 3$. Applying Lemmas 5, 2, 4 and 1 leads to

$$-C_n(y) \leq -C_n(\pi/2 - x_1) = F_m(x_1) - F_m(\pi/2) < F_m(x_1) - F_m(x_2) < 0.263.$$

Hence,

$$C_n(y) \geq -0.263 > \frac{2}{3}(1-\sqrt{2}).$$

This completes the proof of the left-hand side of (3.2). Since

$$S_1(\pi/2) = 1 \quad \text{and} \quad S_2(\pi/8) = \frac{1-\sqrt{2}}{3},$$

we conclude that the bounds given in (1.10) are sharp. □

Proof of Theorem 5. Let $R_n(\lambda,\mu;x)$ denote the sum given in (1.11). Then we have

$$R_n(\lambda,\mu;x) = (\lambda-\mu)S_n(x) + \mu\sum_{k=1}^{n}\frac{(-1)^{k-1}}{2k-1}.$$

Applying (1.10) and

$$\frac{2}{3} \leq \sum_{k=1}^{n}\frac{(-1)^{k-1}}{2k-1} \leq 1$$

leads to

$$\begin{aligned} R_2(\lambda,\mu;\pi/8) &= \frac{(1+\sqrt{2})\mu + (1-\sqrt{2})\lambda}{3} = \frac{1-\sqrt{2}}{3}(\lambda-\mu) + \frac{2}{3}\mu \leq R_n(\lambda,\mu;x) \\ &\leq (\lambda-\mu)+\mu = \lambda = R_1(\lambda,\mu;\pi/2). \end{aligned}$$

This settles (1.11) and reveals that the given bounds are the best possible. □

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