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On class-preserving Coleman automorphisms of semidirect products of finite nilpotent groups by finite groups

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Abstract. Let G be a semidirect product of a finite nilpotent group by a finite group. It is shown that under some conditions class-preserving Coleman automorphisms of 2-power order of G are inner. Interest in such automorphisms arose from the study of the normalizer problem for integral group rings.

1. Introduction

Let G be a finite group. Then, we have the following characteristic subgroups of the automorphism group of G:

$\mathrm{Aut}_{\mathrm{c}}(G)$ denotes the class-preserving automorphism group of G, in which every automorphism sends $g \in G$ to some conjugate of g;

$\mathrm{Aut}_{\mathrm{Col}}(G)$ denotes the Coleman automorphism group of G, in which the restriction of every automorphism to each Sylow subgroup of G equals the restriction of some inner automorphism of G;

$\mathrm{Aut}_{\mathbb{Z}}(G)$ denotes the group of automorphisms of G each of which induces an inner automorphism of $\mathbb{Z}G$, the integral group ring of G over $\mathbb{Z}$.

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Set

$$\mathrm{Out_c}(G) = \mathrm{Aut_c}(G)/\mathrm{Inn}(G); \quad \mathrm{Out_{Col}}(G) = \mathrm{Aut_{Col}}(G)/\mathrm{Inn}(G);$$

$$\mathrm{Out}_{\mathbb{Z}}(G) = \mathrm{Aut}_{\mathbb{Z}}(G)/\mathrm{Inn}(G).$$

Interest in class-preserving Coleman automorphisms arises from the fact that these two kinds of automorphisms play an important role in the study of the normalizer problem of integral group rings. We denote by $U(\mathbb{Z}G)$ the group of units of $\mathbb{Z}G$, by $\mathrm{Z}(U(\mathbb{Z}G))$ the center of $U(\mathbb{Z}G)$, and by $\mathrm{N}_{U(\mathbb{Z}G)}(G)$ the normalizer of G in $U(\mathbb{Z}G)$. Obviously, G and $\mathrm{Z}(U(\mathbb{Z}G))$ are subgroups of $\mathrm{N}_{U(\mathbb{Z}G)}(G)$. Then a question arising naturally is: does $\mathrm{N}_{U(\mathbb{Z}G)}(G) = G \cdot \mathrm{Z}(U(\mathbb{Z}G))$ hold for any finite group G? Historically, this question is referred to as the normalizer problem (see [1, problem 43]). If the question has a positive answer for G, then we say that the normalizer property holds for G. It is known that the normalizer property holds for G if and only if $\mathrm{Out}_{\mathbb{Z}}(G) = 1$. It is also known by Coleman's lemma [2] that $\mathrm{Out}_{\mathbb{Z}}(G) \leq \mathrm{Out_c}(\mathrm{G}) \cap \mathrm{Out_{Col}}(G)$. In addition, a result due to Krempa states that $\mathrm{Out}_{\mathbb{Z}}(G)$ is an elementary abelian 2-group. Thus, if we can show that $\mathrm{Out_c}(G) \cap \mathrm{Out_{Col}}(G)$ is an odd group under some conditions, then $\mathrm{Out}_{\mathbb{Z}}(G) = 1$, i.e., the normalizer property holds for G. Recently, many positive results on the normalizer problem have been presented by many authors. For instance, Li, Sehgal and Parmenter [3] proved that the normalizer property holds for finite Blackburn groups. Petit Lobão and Sehgal [4] showed that the normalizer property holds for the wreath product $G = NwrS_m$ of finite nilpotent group N by symmetric group S_m. In addition, other confirmative results on this problem can also be found in [5]–[19].

Glauberman [20] was the first to study the groups of those automorphisms of G that fix every element of a Sylow 2-subgroup and their connection with Schreier's conjecture. The aim of this paper is to investigate class-preserving Coleman automorphisms of the semidirect products $G = N \rtimes H$, where N is a finite nilpotent group of even order and H is a finite group. Marciniak and Roggenkamp [18] constructed a finite metabelian group $G = (\mathrm{C}_2^4 \times \mathrm{C}_3) \rtimes \mathrm{C}_2^3$ for which $\mathrm{Out_c}(G) \cap \mathrm{Out_{Col}}(G)$ is of even order. This example also illustrates that if $G = N \rtimes H$, where N is a finite nilpotent group of even order and H is a finite group, then in general it is not the case that $\mathrm{Out}_c(G) \cap \mathrm{Out_{Col}}(G)$ is of odd order. However, in this paper we shall prove the following main result.

Theorem 1.1. *Let $G = N \rtimes H$ be the semidirect product of N by H, where N is a finite nilpotent group of even order and H is a finite group. Assume that H*

acts faithfully on the center of each Sylow subgroup of N. Then every class-preserving Coleman automorphism of 2-power order of G is inner, i.e., $\mathrm{Out_c}(G) \cap \mathrm{Out_{Col}}(G)$ *is of odd order.*

As direct consequences of Theorem 1.1, we have the following result.

Corollary 1.2. *Let $G = NwrH$ be the wreath product of N by H, where N is a finite nilpotent group of even order and H is a finite group. Then every class-preserving Coleman automorphism of 2-power order of G is inner, i.e.,* $\mathrm{Out_c}(G) \cap \mathrm{Out_{Col}}(G)$ *is of odd order. In particular, the normalizer property holds for G.*

We fix some notation used in this paper. Let σ be an automorphism of a finite group G, and H be a subgroup of G. Denote by $\sigma|_H$ the restriction of σ to H. Let N be a normal subgroup of G. If σ fixes N, i.e., $N^\sigma = N$, then σ induces an automorphism of G/N, which is denoted by $\sigma|_{G/N}$. Let $x \in G$ be a fixed element. Denote by $\mathrm{conj}(x)$ the automorphism of G induced by x via conjugation, i.e., $g^{\mathrm{conj}(x)} = g^x$ for any $g \in G$. Denote by $\pi(G)$ the set of all primes dividing the order of G. For any $p \in \pi(G)$, we use $O_p(G)$ to denote the largest normal p-subgroup of G, and $O_{p'}(G)$ to denote the largest normal p'-subgroup of G, respectively. Denote by $\mathrm{Syl}_p(G)$ the set of all Sylow p-subgroups of G. Other notation used will be mostly standard, see [1], [22].

2. Preliminaries

In this section, we present some results which will be used in the proof of the main theorem.

Lemma 2.1. *Let $G = N \rtimes H$ be the semidirect product of a finite nilpotent group N by a finite group H. Assume that P is an arbitrary Sylow subgroup of N, and H acts faithfully on $Z(P)$. Then $C_G(P) \leq N$. In particular, $C_G(N) \leq N$.*

Proof. Let $g \in C_G(P)$. Note that $G = N \rtimes H$, so we may set $g = xh$ with $x \in N$ and $h \in H$. Since N is a finite nilpotent group, thus $N = \times_{p \in \pi(N)} P$, and $Z(P) \neq 1$, where $P \in Syl_p(N)$. For any $y \in Z(P)$, on the one hand, we have $y^g = y$. On the other hand, we have $y^g = y^{xh} = y^h$. Consequently, we obtain $y^h = y$. Note that H acts faithfully on $Z(P)$, which implies that $h = 1$, and thus $g = x \in N$, i.e., $C_G(P) \leq N$. In particular, $C_G(N) \leq C_G(P) \leq N$, we are done. $\square$

Recall that a finite group G is said to be p-constrained if $C_{\bar{G}}(O_p(\bar{G})) \leq O_p(\bar{G})$, where $\bar{G} = G/O_{p'}(G)$.

Lemma 2.2 ([21], Corollary 2.4)**.** *Let G be a finite group such that G is a p-constrained group and $O_{p'}(G) = 1$. Assume that P is a Sylow p-subgroup of G, and σ is an automorphism of G such that $\sigma|_P = id|_P$. Then $\sigma = \mathrm{conj}(x)$ for some $x \in Z(P)$.*

Lemma 2.3. *Let G be a finite group, H be a subgroup of G, and let σ be an automorphism of G of p-power order, where p is a prime. If there is $x \in G$ such that $\sigma|_H = \mathrm{conj}(x)|_H$, then there exists some $\gamma \in \mathrm{Inn}(G)$ such that $\sigma\gamma|_H = id|_H$, and $\sigma\gamma$ is still of p-power order.*

Proof. Set $o(\sigma) = p^i$, where $i \in \mathbb{N}$. Write $\beta := \mathrm{conj}(x)$. Then $\sigma|_H = \beta|_H$, i.e., $\sigma\beta^{-1}|_H = id|_H$. Let $n \in \mathbb{N}$ such that $(\sigma\beta^{-1})^n$ be the p-part of $\sigma\beta^{-1}$ with $(n,p) = 1$. Then there exists $s, t \in \mathbb{Z}$ such that $sn + tp^i = 1$. Obviously, $(\sigma\beta^{-1})^{sn}$ is of p-power order and $(\sigma\beta^{-1})^{sn}|_H = id|_H$. Note that $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$, so there exists some $\gamma \in \mathrm{Inn}(G)$ such that $(\sigma\beta^{-1})^{sn} = \sigma^{sn}\gamma = \sigma^{1-tp^i}\gamma = \sigma\gamma$. Hence, γ is the desired inner automorphism. $\square$

Lemma 2.4. *Let G be a finite group, and let N be a subgroup of G. Let σ be an automorphism of G of p-power order with p a prime. Suppose that σ fixes N, and $\sigma|_N = \mathrm{conj}(x)|_N$ for some $x \in G$. Then there exists a p-element $y \in G$ such that $\sigma|_N = \mathrm{conj}(y)|_N$.*

Proof. Let $o(\sigma) = p^i$, $o(x) = p^j t$, where $i, j, t \in \mathbb{N}$ and $(p,t) = 1$. Set $k = \max\{i, j\}$. Since $(p^k, t) = 1$, it follows that there exists $u, v \in \mathbb{Z}$ such that $up^k + vt = 1$. Write $y = x^{vt}$. Then it is obvious that y is a p-element. For any $z \in N$, since $z = z^{\sigma^{up^k}} = z^{x^{up^k}}$, it follows that $z^\sigma = z^x = z^{x^{up^k+vt}} = (z^{x^{up^k}})^{x^{vt}} = z^{x^{vt}} = z^y$, namely, $\sigma|_N = \mathrm{conj}(y)|_N$. $\square$

Lemma 2.5. *Let G be a finite group, and let N be a normal subgroup of G. Let σ be an automorphism of G of p-power order with p a prime. If σ induces an inner automorphism of G/N, i.e., $\sigma|_{G/N} = \mathrm{conj}(x)|_{G/N}$ for some $x \in G$, then there exists a p-element $y \in G$ such that $\sigma|_{G/N} = \mathrm{conj}(y)|_{G/N}$.*

Proof. The proof is similar to that of Lemma 2.4, so we omit it. $\square$

Lemma 2.6 ([22], Lemma 3.2.8)**.** *Let N be a normal subgroup of G with factor group $\bar{G} = G/N$, and let P be a p-subgroup of G. Assume that $(|N|, p) = 1$. Then $C_{\bar{G}}(\bar{P}) = \overline{C_G(P)}$.*

Lemma 2.7 ([13], Lemma 2)**.** *Let p be a prime, and σ an automorphism of G of p-power order. Assume further that there is $N \trianglelefteq G$ such that σ fixes all elements of N, and that σ induces the identity on G/N. Then σ induces the identity on $G/O_p(Z(N))$. If σ fixes in addition a Sylow p-subgroup of G element-wise, then σ is an inner automorphism.*

Let N^m be the direct product of m copies of a finite group N. We say a subgroup H of N^m is extensive in N^m if the intersection of H with $(1, \cdots, 1, \underbrace{N}_{i-th}, 1, \cdots, 1)$ is nontrivial for any $i \in \{1, 2, \cdots, m\}$.

Lemma 2.8 ([16], Lemma 2.2)**.** *Suppose that N^m is the direct product of m copies of a finite nilpotent group N. Then the center of any Sylow p-subgroup of N^m is extensive in N^m for any $p \in \pi(N)$.*

Lemma 2.9 ([7], [8])**.** *If Sylow 2-subgroups of G are cyclic, dihedral or generalized quaternion, then $\mathrm{Out_c}(G) \cap \mathrm{Out_{Col}}(G)$ is of odd order.*

3. Proof of the main theorem

In this section, we shall present a proof of the main theorem. For readers' convenience, we shall rewrite it here as

Theorem 3.1. *Let $G = N \rtimes H$ be the semidirect product of N by H, where N is a finite nilpotent group of even order and H is a finite group. Assume that H acts faithfully on the center of each Sylow subgroup of N. Then every class-preserving Coleman automorphism of 2-power order of G is inner, i.e., $\mathrm{Out_c}(G) \cap \mathrm{Out_{Col}}(G)$ is of odd order.*

Proof. Let $\sigma \in \mathrm{Aut_c}(G) \cap \mathrm{Aut_{Col}}(G)$ be of 2-power order. We have to show that $\sigma \in \mathrm{Inn}(G)$. The proof of Theorem 3.1 splits into two cases according to the order of N:

Case 1. $O_2(N) \neq 1$, but $O_{2'}(N) = 1$, i.e., N is a finite 2-group.

In this case, N is a normal 2-subgroup of G. Since $O_{2'}(G)$ is a normal $2'$-subgroup of G, it follows that $[N, O_{2'}(G)] \leq N \cap O_{2'}(G) = 1$. Thus, $O_{2'}(G) \leq C_G(N)$. By Lemma 2.1, $C_G(N) \leq N$, so $O_{2'}(G) \leq N$, which forces $O_{2'}(G) = 1$. Since $O_2(G)$ is the largest normal 2-subgroup of G, we have $N \leq O_2(G)$. It follows that $C_G(O_2(G)) \leq C_G(N)$. Again by Lemma 2.1, $C_G(N) \leq N$, so $C_G(O_2(G))$ is a normal 2-subgroup of G. Consequently, we obtain that G is a 2-constrained group with $O_{2'}(G) = 1$. Let P be a Sylow 2-subgroup of G. Then, by the definition of

Coleman automorphisms, there exists some $x \in G$ such that $\sigma|_P = \mathrm{conj}(x)|_P$, or, equivalently, $\sigma\mathrm{conj}(x^{-1})|_P = id|_P$. Now, by Lemma 2.2, we obtain that $\sigma\mathrm{conj}(x^{-1}) \in \mathrm{Inn}(G)$, implying further that $\sigma \in \mathrm{Inn}(G)$, we are done.

Case 2. $O_2(N) \neq 1$ and $O_{2'}(N) \neq 1$.

Let H_2 be a Sylow 2-subgroup of H, and let N_2 be the Sylow 2-subgroup of N, respectively. Write $P := N_2 \rtimes H_2$. Then P is a Sylow 2-subgroup of G.

Claim 1. We may assume that $\sigma|_P = id|_P$, and σ is of 2-power order.

Since $\sigma \in \mathrm{Aut}_{\mathrm{Col}}(G)$, there exists some $g \in G$ such that $\sigma|_P = \mathrm{conj}(g)|_P$. By Lemma 2.3, there exists some $\gamma \in \mathrm{Inn}(G)$ such that $\sigma\gamma|_P = id|_P$, and $\sigma\gamma$ is still of 2-power order. Without loss of generality, substituting σ with $\sigma\gamma$, we may assume that $\sigma|_P = id|_P$, and σ is of 2-power order, as claimed.

Claim 2. $\sigma|_N = id|_N$.

Let $\pi(N) = \{p_1, p_2, \ldots, p_r\}$, where $r \geq 2$. By hypothesis, N is nilpotent, so we may set $N = P_1 \times \cdots \times P_r$, where P_i is the Sylow p_i-subgroup of N for each $i \in \{1, 2, \ldots, r\}$. Since $\sigma \in \mathrm{Aut_c}(G)$, for each P_i and each $x_i \in P_i$, there exist $n_i \in N$ and $h_i \in H$ such that

$$x_i^\sigma = n_i^{-1} h_i^{-1} x_i h_i n_i. \tag{3.1}$$

Note that $N = P_1 \times \cdots \times P_r$, so we can decompose each n_i as $n_i = n_{i1} n_{i2} \cdots n_{ii} \cdots n_{ir}$ with $n_{ik} \in P_k$, where $k = 1, 2, \ldots, r$. Since $P_i \trianglelefteq G$ and $h_i^{-1} x_i h_i \in P_i$, (3.1) may be written as

$$x_i^\sigma = n_{ii}^{-1} h_i^{-1} x_i h_i n_{ii}. \tag{3.2}$$

For any $z_i \in Z(P_i)$, by (3.2), we have

$$z_i^\sigma = h_i^{-1} z_i h_i. \tag{3.3}$$

Now, take any two distinct Sylow subgroups P_i and P_j of N. Then, on the one hand, by (3.3), for any $z_i \in Z(P_i)$ and any $z_j \in Z(P_j)$ we have

$$(z_i z_j)^\sigma = z_i^\sigma z_j^\sigma = h_i^{-1} z_i h_i h_j^{-1} z_j h_j. \tag{3.4}$$

On the other hand, since $\sigma \in \mathrm{Aut_c}(G)$, there exist $n \in N$ and $h \in H$ such that

$$(z_i z_j)^\sigma = n^{-1} h^{-1} z_i z_j h n = h^{-1} z_i h h^{-1} z_j h. \tag{3.5}$$

Consequently, (3.4) and (3.5) yield that

$$(h_i^{-1} z_i h_i)(h_j^{-1} z_j h_j) = (h^{-1} z_i h)(h^{-1} z_j h). \tag{3.6}$$

But note that $h_i^{-1}z_ih_i, h^{-1}z_ih \in P_i$ and $h_j^{-1}z_jh_j, h^{-1}z_jh \in P_j$, so (3.6) implies that $h_i^{-1}z_ih_i = h^{-1}z_ih$ and $h_j^{-1}z_jh_j = h^{-1}z_jh$, or, equivalently,

$$(h_ih^{-1})^{-1}z_i(h_ih^{-1}) = z_i, \tag{3.7}$$

and

$$(h_jh^{-1})^{-1}z_j(h_jh^{-1}) = z_j. \tag{3.8}$$

Since H acts faithfully on $Z(P_i)$ and $Z(P_j)$, the equations (3.7) and (3.8) imply that $h_i = h = h_j$. As P_i and P_j are arbitrary, we actually proved that $h = h_1 = h_2 = \cdots = h_r$. Thus, we may rewrite (3.2) as

$$x_i^\sigma = n_{ii}^{-1}h^{-1}x_ihn_{ii}. \tag{3.9}$$

Write $n := n_{11}n_{22}\cdots n_{rr}$. Then, by (3.9), for any $x = x_1x_2\cdots x_r \in N$ with $x_i \in P_i$,

$$x^\sigma = \prod_{i=1}^{r} x_i^\sigma = n^{-1}h^{-1}xhn. \tag{3.10}$$

To complete the proof of Claim 2, we consider the action of σ on $Z(N_2)$, where N_2 is the Sylow 2-subgroup of N. Take any $x \in Z(N_2)$. Then, on the one hand, by Claim 1, we have

$$x^\sigma = x. \tag{3.11}$$

On the other hand, by (3.10), we have

$$x^\sigma = n^{-1}h^{-1}xhn = h^{-1}xh. \tag{3.12}$$

Consequently, (3.11) and (3.12) yield that $h^{-1}xh = x$, from which one gets that $h = 1$ since H acts faithfully on $Z(N_2)$. Thus, by (3.10), we have

$$\sigma|_N = \mathrm{conj}(n)|_N. \tag{3.13}$$

Without loss of generality, by Lemma 2.4, we may assume that n is a 2-element in N. We now consider the action of σ on N_2, the Sylow 2-subgroup of N. On the one hand, by Claim 1, we have

$$\sigma|_{N_2} = id|_{N_2}. \tag{3.14}$$

On the other hand, by (3.13), we have

$$\sigma|_{N_2} = \mathrm{conj}(n)|_{N_2}. \tag{3.15}$$

Consequently, (3.14) and (3.15) yield that $n \in Z(N_2)$, and hence $n \in Z(N)$. It follows from (3.13) that $\sigma|_N = id|_N$, as claimed.

Claim 3. $\sigma|_{G/N} = id|_{G/N}$.

Note that $G/O_{2'}(N) \cong N_2 \rtimes H$ is the semidirect product of N_2 by H, so, by Case 1, $\mathrm{Out_c}(G/O_{2'}(N)) \cap \mathrm{Out_{Col}}(G/O_{2'}(N))$ is of odd order. Note further that $\sigma \in \mathrm{Aut_c}(G) \cap \mathrm{Aut_{Col}}(G)$ implies $\sigma|_{G/O_{2'}(N)} \in \mathrm{Aut_c}(G/O_{2'}(N)) \cap \mathrm{Aut_{Col}}(G/O_{2'}(N))$. Consequently, we have $\sigma|_{G/O_{2'}(N)} \in \mathrm{Inn}(G/O_{2'}(N))$. Thus, there exists some $x \in G$ such that

$$\sigma|_{G/O_{2'}(N)} = \mathrm{conj}(x)|_{G/O_{2'}(N)}. \tag{3.16}$$

Without loss of generality, by Lemma 2.5, we may assume that x is a 2-element in G, and hence x belongs to some Sylow 2-subgroup of G. By Sylow's theorem, there exists some $g \in G$ such that $x \in P^g = N_2 \rtimes H_2^g$, where P is the fixed Sylow 2-subgroup of G as above. Set $x = ab^g$ with $a \in N_2$ and $b \in H_2$. Then we may write (3.16) as

$$\sigma|_{G/O_{2'}(N)} = \mathrm{conj}(ab^g)|_{G/O_{2'}(N)}. \tag{3.17}$$

Thus, on the one hand, by (3.17), we have

$$\sigma|_{N/O_{2'}(N)} = \mathrm{conj}(ab^g)|_{N/O_{2'}(N)}. \tag{3.18}$$

On the other hand, since by Claim 2 $\sigma|_N = id|_N$, it follows that

$$\sigma|_{N/O_{2'}(N)} = id|_{N/O_{2'}(N)}. \tag{3.19}$$

Consequently, for any $y \in Z(N_2)$, where N_2 is the Sylow 2-subgroup of N, (3.18) and (3.19) yield that $(\bar{y}^{\bar{g}^{-1}})^{\bar{b}} = \bar{y}^{\bar{g}^{-1}}$. By Lemma 2.6, we obtain that $\bar{b} = 1$. Thus, (3.17) turns into $\sigma|_{G/O_{2'}(N)} = \mathrm{conj}(a)|_{G/O_{2'}(N)}$, which implies that $\sigma|_{G/N} = \mathrm{conj}(a)|_{G/N} = id|_{G/N}$, as claimed.

Claim 4. $\sigma \in \mathrm{Inn}(G)$.

By Lemma 2.7, Claims 1 and 3 yield that $\sigma \in \mathrm{Inn}(G)$. This completes the proof of Theorem 3.1. □

As immediate consequences of Theorem 3.1, we have the following results:

Corollary 3.2. *Let $G = NwrH$ be the wreath product of N by H, where N is a finite nilpotent group of even order and H is a finite group. Then every class-preserving Coleman automorphism of 2-power order of G is inner, i.e., $\mathrm{Out_c}(G) \cap \mathrm{Out_{Col}}(G)$ is of odd order. In particular, the normalizer property holds for G.*

Proof. Let $|H| = m$. Then $G = NwrH = N^m \rtimes H$, where N^m is the direct product of m copies of N. Let $p \in \pi(N)$, and let $P \in \mathrm{Syl}_p(N^m)$. We will show that H acts faithfully on $Z(P)$. Since N is a nilpotent group, N^m is also a nilpotent group. Obviously, H acts on $Z(P)$, $C_H(Z(P))$ is the kernel of the action of H on $Z(P)$. Let $h \in C_H(Z(P))$, we have $y^h = y$ for any $y \in Z(P)$. By Lemma 2.8, $Z(P)$ is extensive in N^m, so $h = 1$ showing that H acts faithfully on $Z(P)$. Thus, the assertion follows from Theorem 3.1. $\square$

Corollary 3.3. *Let $G = N \rtimes H$ be the semidirect product of a finite nilpotent group N by a finite group H whose Sylow 2-subgroups are either cyclic, dihedral or generalized quaternion. Assume that H acts faithfully on the center of each Sylow subgroup of N. Then $\mathrm{Out_c}(G) \cap \mathrm{Out_{Col}}(G)$ is of odd order.*

Proof. If N is of odd order, then the Sylow 2-subgroups of G are necessarily either cyclic, dihedral or generalized quaternion. Thus, the assertion follows from Lemma 2.9. If N is of even order, the assertion follows from Theorem 3.1. $\square$

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