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Jordan triple product homomorphisms on Hermitian matrices of dimension two

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Abstract. We characterise all Jordan triple product homomorphisms, that is, mappings Φ satisfying

$$\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A)$$

on the set of all Hermitian 2×2 complex matrices.

1. Introduction

In order to understand the geometry of matrix spaces, mappings with certain properties are often studied. Among such properties is (anti-)multiplicativity. The structure of (anti-)multiplicative mappings on the algebra $\mathcal{M}_n(\mathbb{F})$ of $n \times n$ matrices over field $\mathbb{F}$ is well understood [6], but less is known about (anti-)multiplicative mappings from $\mathcal{M}_n(\mathbb{F})$ to $\mathcal{M}_m(\mathbb{F})$ for $m > n$.

In a well-known survey paper [13], Šemrl presented many facts and properties of such mappings, along with properties of preservers of Jordan and Lie product. Šemrl exposed a related problem, that is, to characterize maps that are multiplicative with respect to *Jordan triple product* (J.T.P. for short), namely maps Φ on $\mathcal{M}_n(\mathbb{F})$ satisfying

$$\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A)$$

for all $A, B \in \mathcal{M}_n(\mathbb{F})$. Such mappings were studied under additional assumption of additivity on quite general domain of certain rings [1]. In response to Šemrl,

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Kuzma characterized nondegenerate J.T.P. homomorphisms on the set $\mathcal{M}_n(\mathbb{F})$ in [8] for $n \geq 3$, in [2] Dobovišek characterized J.T.P. homomorphisms from $\mathcal{M}_n(\mathbb{F})$ to $\mathbb{F}$, and in [3] he characterized J.T.P. homomorphisms from $\mathcal{M}_2(\mathbb{F})$ to $\mathcal{M}_3(\mathbb{F})$.

In this paper, we focus on J.T.P. homomorphisms on the set of all Hermitian complex 2×2 matrices. By A^* denote the complex conjugate of the transpose of matrix A, and by $\mathcal{H}_2(\mathbb{C})$ the set of all Hermitian complex 2×2 matrices

$$\mathcal{H}_2(\mathbb{C}) = \{A \in \mathcal{M}_2(\mathbb{C}); A = A^*\}.$$

We cannot study multiplicative or anti-multiplicative maps on Hermitian matrices, since they are not closed under multiplication. But they are closed under J.T.P., so studying J.T.P. homomorphisms on Hermitian matrices makes perfect sense. Characterization of J.T.P. homomorphisms on the set of Hermitian matrices may shed a new light on the structure of Hermitian matrices, and may be useful in the areas where only Hermitian or positive (semi)definite matrices appear, such as some areas of financial mathematics.

Jordan triple product homomorphisms were already studied on the set of positive definite matrices, Gselmann [4] characterized mappings from the set of positive definite real or complex matrices to the field of real numbers. In the paper [7], similar result was proved, namely Jordan triple product homomorphisms from the set of all Hermitian $n \times n$ complex matrices to the field of complex numbers, and Jordan triple product homomorphisms from the field of complex or real numbers or the set of all nonnegative real numbers to the set of all Hermitian $n \times n$ complex matrices were characterized. Further, Hao *et al.* [5] characterized injective Jordan triple product endomorphisms on the set of complex symmetric matrices, and Molnár in [9] described continuous Jordan triple endomorphisms on the set of complex positive definite matrices of size at least 3. The special case of 2×2 positive definite complex matrices was considered separately in [10]. One may think that in this case the solution can be found straightforwardly, but this is far from being true. We generalize this result by omitting the continuity assumption and enlarging the set of matrices to all complex Hermitian matrices.

The paper is organized as follows. In Section 2, we state the characterization theorem for J.T.P. homomorphisms on $\mathcal{H}_2(\mathbb{C})$. In Section 3, we list some results on J.T.P. homomorphisms on the set $\mathcal{H}_n(\mathbb{C})$ and main results from [7], which we will find useful later on. In Sections 4–7, we treat different cases of J.T.P. homomorphisms, namely irregular, scalar, nondegenerate and degenerate cases.

The proof of characterization is very long and technical. It consists of many lemmas, whose proofs are straightforward multiplications of matrices. Thus, we leave it to the readers. If a reader wants to check the procedures, he or she can find the full length paper at arXiv.org under the identifier arXiv:1512.02849.

2. Characterization theorem

We first introduce some notation. By I we denote the identity matrix of an appropriate dimension, by $\det A$ the determinant, and by $\operatorname{rank} A$ the rank of a matrix A. By $\sigma(A)$ we denote the spectrum of a matrix A, and by $\operatorname{Syl}(A)$ the inertia of A, that is, the number of positive eigenvalues of A. The direct sum $A \oplus B$ is a block diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. The notation $A > 0$ means that a matrix $A \in \mathcal{H}_2(\mathbb{C})$ is positive definite, $A < 0$ is a negative definite matrix, and $A <> 0$ is an invertible nondefinite matrix.

We can now state our main result.

Theorem 2.1. *Let $\Phi : \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C})$. Then $\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A)$ if and only if there exists some unitary matrix $U \in \mathcal{M}_2(\mathbb{C})$ such that Φ has one of the following forms:*

(i) $\Phi(A) = U \begin{bmatrix} \varphi_1(A) & 0 \\ 0 & \varphi_2(A) \end{bmatrix} U^*$, *where $\varphi_1, \varphi_2 : \mathcal{H}_2(\mathbb{C}) \to \mathbb{R}$ are J.T.P. homomorphisms having the form $\varphi_i(A) = \psi_i(|\det A|)\eta_i(\operatorname{Syl}(A))$ for $i = 1, 2$, with $\psi_1, \psi_2 : [0, \infty) \to \mathbb{C}$ multiplicative functions, $\eta_1, \eta_2 : \{0, 1, 2\} \to \{-1, 1\}$ arbitrary mappings, and $\operatorname{Syl}(A)$ the inertia of A;*

(ii) $\Phi(A) = \pm UAU^*$;

(iii) $\Phi(A) = \pm U\bar{A}U^* = \pm UA^TU^*$;

(iv) $\Phi(A) = \begin{cases} \pm\beta(\det A) \cdot U\widetilde{\Phi}(A)U^*; & \operatorname{rank} A = 2 \\ 0; & \operatorname{rank} A \leq 1, \end{cases}$

where $\beta : \mathbb{R}^* \to \mathbb{R}^*$ is a unital multiplicative map, and $\widetilde{\Phi}$ has one of the following forms:

- $\widetilde{\Phi}(A) = A$;
- $\widetilde{\Phi}(A) = \bar{A}$;
- $\widetilde{\Phi}(A) = A^{-1}$;
- $\widetilde{\Phi}(A) = \bar{A}^{-1}$;
- $\widetilde{\Phi}(A) = \eta(A)A$;
- $\widetilde{\Phi}(A) = \eta(A)\bar{A}$;
- $\widetilde{\Phi}(A) = \eta(A)A^{-1}$;
- $\widetilde{\Phi}(A) = \eta(A)\bar{A}^{-1}$;

with

$$\eta(A) = \begin{cases} 1; & A > 0 \ \text{or} \ A <> 0 \\ -1; & A < 0\,. \end{cases}$$

It is obvious that mappings of the forms described in (i)–(iv) are J.T.P. homomorphisms on $\mathcal{H}_2(\mathbb{C})$.

3. Preliminaries

In this section, we present some properties of J.T.P. homomorphisms on the set $\mathcal{H}_n(\mathbb{C})$ we will use later on. These properties with proofs can be found in [7]. We start with a simple lemma.

Lemma 3.1 ([7] Lemma 2.1). *Let $A \in \mathcal{H}_2(\mathbb{C})$ be a Hermitian matrix. Then there exists a unitary Hermitian matrix $B \in \mathcal{H}_2(\mathbb{C})$ such that $A = B(\lambda_1 \oplus \lambda_2)B$ with $\lambda_1, \lambda_2 \in \mathbb{R}$.*

We continue with the characterization of J.T.P. homomorphisms mapping from $\mathcal{H}_2(\mathbb{C})$ to $\mathbb{C}$.

Lemma 3.2 ([7] Lemma 3.1). *Let $\Phi : \mathcal{H}_2(\mathbb{C}) \to \mathbb{C}$ be a J.T.P. homomorphism with $\Phi(I) = 1$ and $\Phi(0) = 0$. Then $\Phi(A) = 0$ for every $A \in \mathcal{H}_2(\mathbb{C})$ with* $\operatorname{rank} A < 2$.

Proposition 3.3 ([7] Theorem 3.3). *Let Φ be a mapping from $\mathcal{H}_2(\mathbb{C})$ to $\mathbb{C}$. Then Φ satisfies the identity $\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A)$ if and only if Φ has the form*

$$\Phi(A) = \Psi(|\det A|)\eta(\operatorname{Syl}(A)),$$

where $\Psi : [0, \infty) \to \mathbb{C}$ is a multiplicative function, $\eta : \{0, 1, 2\} \to \{-1, 1\}$ an arbitrary mapping, and $\operatorname{Syl}(A)$ the inertia of A.

We also need the characterization of J.T.P. homomorphisms from matrices of dimension one to $n \times n$ Hermitian matrices.

Lemma 3.4 ([7] Lemma 4.1). *Let a mapping $\Phi : \mathcal{A} \to \mathcal{H}_n(\mathbb{C})$ be a J.T.P. homomorphism, where $\mathcal{A}$ is the set $\mathbb{C}^*$, $\mathbb{R}^*$, or $\mathbb{R}^+$, such that $\Phi(\lambda)$ is invertible for every $\lambda \in \mathcal{A}$ and $\Phi(1) = I$. Then there exist a unitary matrix U and multiplicative maps $\varphi_1, \varphi_2 : \mathcal{A} \to \mathbb{R}^*$ with $\varphi_i(1) = 1$, such that*

$$\Phi(\lambda) = U(\varphi_1(\lambda) \oplus \varphi_2(\lambda))U^*, \quad \lambda \in \mathcal{A}.$$

Proposition 3.5 ([7] Theorem 4.2)**.** *Let a mapping $\Phi : \mathcal{A} \to \mathcal{H}_n(\mathbb{C})$ be a J.T.P. homomorphism, where $\mathcal{A}$ is the set $\mathbb{C}$, $\mathbb{R}$, or $\mathbb{R}^+ \cup \{0\}$. Then there exist a unitary matrix U, a diagonal matrix D with ± 1's on its diagonal and multiplicative maps $\varphi_1, \varphi_2 : \mathcal{A} \to \mathbb{R}$, such that*

$$\Phi(\lambda) = UD(\varphi_1(\lambda) \oplus \varphi_2(\lambda))U^*, \quad \lambda \in \mathcal{A}.$$

4. Irregular cases

In this section, we start with the study J.T.P. homomorphisms that map from 2×2 Hermitian matrices to 2×2 Hermitian matrices. Since $\Phi(0) = \Phi(0^3) = \Phi(0)^3$, it must be that $\sigma(\Phi(0)) \subset \{-1, 0, 1\}$. So, we consider several cases.

Case 1. If $\Phi(0)$ is invertible, then it follows from

$$\Phi(0) = \Phi(0 \cdot A \cdot 0) = \Phi(0)\Phi(A)\Phi(0)$$

that $\Phi(A) = \Phi(0)^{-1} = \Phi(0)$ for every $A \in \mathcal{H}_2(\mathbb{C})$ with $\Phi(0)$ some involution in $\mathcal{H}_2(\mathbb{C})$.

Case 2. If $\operatorname{rank} \Phi(0) = 1$, then it follows from $\Phi(0) = \Phi(0)^3$ that $\sigma(\Phi(0)) = \{0, \alpha\}$ with $\alpha \in \{-1, 1\}$. Hence, we can write

$$\Phi(0) = U \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} U^*$$

for some unitary matrix $U \in \mathcal{M}_2(\mathbb{C})$. Choose an arbitrary $A \in \mathcal{H}_2(\mathbb{C})$ and write

$$\Phi(A) = U \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix} U^*.$$

Then

$$\Phi(0) = \Phi(0)\Phi(A)\Phi(0) = U \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

Hence $a = \alpha$. On the other hand,

$$\Phi(0) = \Phi(A)\Phi(0)\Phi(A) = U \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix} U^* = U \begin{bmatrix} \alpha & b \\ \bar{b} & \alpha|b|^2 \end{bmatrix} U^*,$$

from which it follows $b = 0$. We conclude that, for every $A \in \mathcal{H}_2(\mathbb{C})$,

$$\Phi(A) = U \begin{bmatrix} \alpha & 0 \\ 0 & \varphi(A) \end{bmatrix} U^*$$

for some J.T.P. homomorphism $\varphi : \mathcal{H}_2(\mathbb{C}) \to \mathbb{R}$ with $\varphi(0) = 0$.

We split the remaining case $\Phi(0) = 0$ into several subcases, depending on the image $\Phi(I)$. Since $\Phi(I) = \Phi(I)^3$, it must be that $\sigma(\Phi(I)) \subset \{-1, 0, 1\}$.

Case 3. Let $\Phi(I) = 0$. Then $\Phi(A) = \Phi(I)\Phi(A)\Phi(I) = 0$ for every $A \in \mathcal{H}_2(\mathbb{C})$.

Case 4. If $\operatorname{rank}\Phi(I) = 1$, we write

$$\Phi(I) = U \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} U^*$$

for $\alpha \in \{-1, 1\}$ and a unitary matrix $U \in \mathcal{M}_2(\mathbb{C})$. Write

$$\Phi(A) = U \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix} U^*.$$

Then for every $A \in \mathcal{H}_2(\mathbb{C})$ we get

$$\begin{aligned} \Phi(A) = \Phi(I)\Phi(A)\Phi(I) &= U \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} \varphi(A) & 0 \\ 0 & 0 \end{bmatrix} U^*, \end{aligned}$$

for some J.T.P. homomorphism $\varphi : \mathcal{H}_2(\mathbb{C}) \to \mathbb{R}$ with $\varphi(0) = 0$ and $\varphi(I) = \alpha$.

Case 5. Let $\Phi(I)$ be invertible. From $\Phi(I) = \Phi(I)^3$ it follows that $\Phi(I)^2 = I$. Denote $P := \Phi(I)$. Then

$$\Phi(A) = \Phi(I)\Phi(A)\Phi(I) = P\Phi(A)P$$

for every $A \in \mathcal{H}_2(\mathbb{C})$, hence $\Phi(A)P = P\Phi(A)$ for every $A \in \mathcal{H}_2(\mathbb{C})$. If $P \neq \pm I$, we can write $P = U \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} U^*$ for some unitary $U \in \mathcal{M}_2(\mathbb{C})$. Since $\Phi(A)$ commutes with P, we have

$$\Phi(A) = U \begin{bmatrix} \varphi_1(A) & 0 \\ 0 & \varphi_2(A) \end{bmatrix} U^*,$$

for some J.T.P. homomorphisms $\varphi_1, \varphi_2 : \mathcal{H}_2(\mathbb{C}) \to \mathbb{R}$. If $P = -I$, define a mapping $\Phi'(A) = -\Phi(A)$. Φ' is a J.T.P. homomorphism from $\mathcal{H}_2(\mathbb{C})$ to $\mathcal{H}_2(\mathbb{C})$ with $\Phi'(0) = 0$ and $\Phi'(I) = I$. This translates directly to the last case in need of considering, Case 6.

Case 6. $\Phi : \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C})$ is a J.T.P. homomorphism with $\Phi(0) = 0$ and $\Phi(I) = I$. We refer to this case as a *regular* case.

Cases 1–6 amount to the following proposition.

Proposition 4.1. *Let $\Phi : \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C})$ be a J.T.P. homomorphism. Then Φ is regular, or $-\Phi$ is regular, or there exists a unitary matrix U, such that*

$$\Phi(A) = U \begin{bmatrix} \varphi_1(A) & 0 \\ 0 & \varphi_2(A) \end{bmatrix} U^*,$$

where $\varphi_1, \varphi_2 : \mathcal{H}_2(\mathbb{C}) \to \mathbb{R}$ are J.T.P. homomorphisms characterised in Proposition 3.3, possibly constant mappings $\varphi_i(A) = c \in \{-1, 0, 1\}$ for all $A \in \mathcal{H}_2(\mathbb{C})$.

All cases when $\Phi(I) \neq \pm I$ or $\Phi(0) \neq 0$ are covered by the form (i) of Theorem 2.1. In the case when $-\Phi$ is regular, we get the negative sign in the forms (ii)–(iv) of Theorem 2.1.

5. Nontrivial involution to a scalar

In Sections 5–7, we assume $\Phi : \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C})$ to be a regular J.T.P. homomorphism, that is, $\Phi(0) = 0$ and $\Phi(I) = I$. We now consider the image of a nontrivial involution $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Since $J^2 = I$, it is mapped to an involution. So, $\Phi(J)$ is a matrix similar to J, or a scalar matrix I or $-I$. In this section, we assume that $\Phi(J) \in \{-I, I\}$.

The proofs of the following lemmas can be found at arXiv.org under the identifier arXiv:1512.02849.

Lemma 5.1. *Let $\Phi : \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C})$ be a regular J.T.P. homomorphism and $\Phi(J) \in \{-I, I\}$. Then every nontrivial involution is mapped to $\pm I$. If matrices $A, B \in \mathcal{H}_2(\mathbb{C})$ are similar, then $\Phi(A) = \Phi(B)$.*

Lemma 5.2. *If $\Phi : \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C})$ is a regular J.T.P. homomorphism and $\Phi(J) \in \{-I, I\}$, then $\Phi(A) = 0$ for every matrix $A \in \mathcal{H}_2(\mathbb{C})$ with $\operatorname{rank} A = 1$.*

Lemma 5.3. *Let $\Phi : \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C})$ be a regular J.T.P. homomorphism with $\Phi(J) \in \{-I, I\}$. Let $A \in \mathcal{H}_2(\mathbb{C})$ be invertible. If A is positive definite or A is nondefinite, then*

$$\Phi(A) = \Phi\left(\begin{bmatrix} \det A & 0 \\ 0 & 1 \end{bmatrix}\right).$$

If A is negative definite, then

$$\Phi(A) = \Phi(-I)\Phi\left(\begin{bmatrix}\det A & 0\\ 0 & 1\end{bmatrix}\right).$$

Proposition 5.4. *Let $\Phi : \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C})$ be a regular J.T.P. homomorphism with $\Phi(J) \in \{-I, I\}$. Then there exist a unitary matrix U, unital multiplicative maps $\psi_1, \psi_2 : [0,\infty) \to [0,\infty)$ with $\psi_i(0) = 0$ for $i \in \{1,2\}$, and maps $\eta_1, \eta_2 : \{0,1,2\} \to \{-1,1\}$ which satisfy $\eta_1(2) = \eta_2(2) = 1$ and $\eta_1(1) = \eta_2(1)$, such that $\Phi(A)$ has the form*

$$\Phi(A) = U\begin{bmatrix}\psi_1(|\det A|)\eta_1(\mathrm{Syl}(A)) & 0\\ 0 & \psi_2(|\det A|)\eta_2(\mathrm{Syl}(A))\end{bmatrix}U^*,$$

for every $A \in \mathcal{H}_2(\mathbb{C})$, where $\mathrm{Syl}(A)$ *is the inertia of A.*

Proof. Consider all matrices of the form $\begin{bmatrix}x & 0\\ 0 & 1\end{bmatrix} \in \mathcal{H}_2(\mathbb{C})$. They are isomorphic to the semigroup of real numbers for multiplication, so Φ induces a J.T.P. homomorphism from $\mathbb{R}$ to $\mathcal{H}_2(\mathbb{C})$. From Proposition 3.5 we know its form, and by the previous Lemma it follows that there exist a unitary matrix U, a diagonal matrix D with ± 1's on its diagonal and multiplicative maps $\varphi_1, \varphi_2 : \mathbb{R} \to \mathbb{R}$, such that

$$\Phi(A) = UD\begin{bmatrix}\varphi_1(\det A) & 0\\ 0 & \varphi_2(\det A)\end{bmatrix}U^*,$$

for every positive definite or nondefinite matrix $A \in \mathcal{H}_2(\mathbb{C})$. This can be written in the form

$$\Phi(A) = U\begin{bmatrix}\psi_1(|\det A|)\eta_1(\mathrm{Syl}(A)) & 0\\ 0 & \psi_2(|\det A|)\eta_2(\mathrm{Syl}(A))\end{bmatrix}U^*,$$

where $\psi_1, \psi_2 : [0,\infty) \to [0,\infty)$ are multiplicative maps, and $\mathrm{Syl}(A)$ is the inertia of A. Since $\Phi(I) = I$, we obtain $\eta_1(2) = \eta_2(2) = 1$, and since Φ maps a nontrivial involution to a scalar, we obtain $\eta_1(1) = \eta_2(1)$.

We now have to prove this form also for negative definite matrices. If $\psi_1(x) = \psi_2(x)$ for every $x \geq 0$, then $\Phi(A)$ is scalar for every positive definite or nondefinite matrix $A \in \mathcal{H}_2(\mathbb{C})$. In this case, matrix U is still arbitrary. There exists a unitary matrix U and a diagonal matrix D with ± 1's on its diagonal, so that $\Phi(-I) = UDU^*$.

On the other hand, if $\psi_1(x) \neq \psi_2(x)$ for some $x \geq 0$, then $\Phi(-I)$ commutes with $\Phi\left(\begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}\right)$ by the previous Lemma, and again $\Phi(-I) = UDU^*$. Now, let $\eta_1(0)$ and $\eta_2(0)$ be defined by diagonal entries of matrix D. Every negative definite matrix $A \in \mathcal{H}_2(\mathbb{C})$ can be written in the form $A = \sqrt{-A}(-I)\sqrt{-A}$, so

$$\begin{aligned}
\Phi(A) &= \Phi(\sqrt{-A})\Phi(-I)\Phi(\sqrt{-A}) \\
&= U \begin{bmatrix} \psi_1(\sqrt{\det A}) & 0 \\ 0 & \psi_2(\sqrt{\det A}) \end{bmatrix} \begin{bmatrix} \eta_1(0) & 0 \\ 0 & \eta_2(0) \end{bmatrix} \\
&\quad \times \begin{bmatrix} \psi_1(\sqrt{\det A}) & 0 \\ 0 & \psi_2(\sqrt{\det A}) \end{bmatrix} U^* \\
&= U \begin{bmatrix} \psi_1(|\det A|)\eta_1(\mathrm{Syl}(A)) & 0 \\ 0 & \psi_2(|\det A|)\eta_2(\mathrm{Syl}(A)) \end{bmatrix} U^*,
\end{aligned}$$

which completes the proof. □

The case when a nontrivial idempotent is mapped to a scalar is covered by the form (i) of Theorem 2.1.

6. Nondegenerate case

In this section, we assume that for a regular J.T.P. homomorphism $\Phi : \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C})$ there exists $A \in \mathcal{H}_2(\mathbb{C})$ with $\operatorname{rank} A = 1$ such that $\Phi(A) \neq 0$. We refer to such regular Φ as *nondegenerate* J.T.P. homomorphism.

From Lemmas 5.1 and 5.2 it follows that nontrivial involutions cannot be mapped to scalar matrices. Thus,

$$\sigma\left(\Phi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)\right) = \{-1, 1\}.$$

Let $\Phi : \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C})$ be a nondegenerate J.T.P. homomorphism. We proceed by proving the following steps. The proofs can be found at arXiv.org under the identifier arXiv:1512.02849.

Step 1. $\operatorname{rank} \Phi(A) = 1$ for every $A \in \mathcal{H}_2(\mathbb{C})$ with $\operatorname{rank} A = 1$.

Step 2. If there exists $\lambda \in \mathbb{R}$ such that $\Phi(\lambda I)$ is not a scalar, then there exists a unitary matrix U such that

$$\Phi(A) = U \begin{bmatrix} \varphi_1(A) & 0 \\ 0 & \varphi_2(A) \end{bmatrix} U^* \quad \text{for all } A \in \mathcal{H}_2(\mathbb{C}),$$

where $\varphi_1, \varphi_2 : \mathcal{H}_2(\mathbb{C}) \to \mathbb{R}$ are distinct unital J.T.P. homomorphisms.

Step 3. There exists a multiplicative map $\Psi : \mathbb{R} \to \mathbb{R}$ with $\Psi(1) = 1$ such that $\Phi(\lambda I) = \Psi(\lambda) I$ for every $\lambda \in \mathbb{R}$.

A matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is an idempotent of rank 1, hence it is mapped to an idempotent of rank 1 by Step 1. So $\Phi\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = U \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} U^*$ for some $U \in \mathcal{H}_2(\mathbb{C})$ unitary. By taking $\Phi'(A) = U\Phi(A)U^*$, we may assume without the loss of generality that $\Phi\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. In other words, Φ preserves $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Let $\Phi : \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C})$ be a nondegenerate J.T.P. homomorphism preserving E_{11}. We proceed by proving the following steps.

Step 4. $\Phi\left(\begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix}\right) = \begin{bmatrix} \Psi(a) & * \\ * & * \end{bmatrix}$ for every $a \in \mathbb{R}$.

Step 5. $\Phi\left(\begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix}\right) = \begin{bmatrix} * & * \\ * & \Psi(c) \end{bmatrix}$ for every $c \in \mathbb{R}$.

Step 6. $\Phi\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} \Psi(a) & 0 \\ 0 & \Psi(b) \end{bmatrix}$ for every $a, b \in \mathbb{R}$.

Now, take $x, y > 0$. Then $\begin{bmatrix} x-y & \sqrt{xy} \\ \sqrt{xy} & 0 \end{bmatrix} = B \begin{bmatrix} x & 0 \\ 0 & -y \end{bmatrix} B$ for some Hermitian unitary matrix $B \in \mathcal{H}_2(\mathbb{C})$ by Lemma 3.1. Thus,

$$\Phi\left(\begin{bmatrix} x-y & \sqrt{xy} \\ \sqrt{xy} & 0 \end{bmatrix}\right) = \begin{bmatrix} \Psi(x-y) & * \\ * & 0 \end{bmatrix} = \Phi(B) \begin{bmatrix} \Psi(x) & 0 \\ 0 & \Psi(-y) \end{bmatrix} \Phi(B).$$

Hence $\Psi(x-y) = \Psi(x)+\Psi(-y)$, since trace of matrix is preserved under similarity action. Taking $y = x$, we get $\Psi(-x) = -\Psi(x)$ for $x > 0$, hence for all $x \in \mathbb{R}$. But then the equality $\Psi(x-y) = \Psi(x) + \Psi(-y)$ also holds for all $x, y \in \mathbb{R}$. Taking $z = -y$, we obtain additivity of Ψ. Since a multiplicative function $\Psi : \mathbb{R} \to \mathbb{R}$ is additive, it must be an identity by [12, Theorem 1.10].

Lemma 6.1. *Let $\Phi : \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C})$ be a nondegenerate J.T.P. homomorphism preserving E_{11}. Then $\Phi(\lambda I) = \lambda I$ for every $\lambda \in \mathbb{R}$.*

Suppose $\Phi : \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C})$ is a nondegenerate J.T.P. homomorphism preserving E_{11}. We know that $\Phi\left(\begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix}\right) = \begin{bmatrix} a & * \\ * & c \end{bmatrix}$. Define

$$\Phi\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & \gamma \\ \bar{\gamma} & 0 \end{bmatrix} \quad \text{and} \quad \Phi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & \delta \\ \bar{\delta} & 0 \end{bmatrix}.$$

Take $a > 0$. Then

$$\begin{bmatrix} \sqrt{a} & 0 \\ 0 & \frac{1}{\sqrt{a}} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{a} & 0 \\ 0 & \frac{1}{\sqrt{a}} \end{bmatrix} = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix},$$

hence $\Phi\left(\begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} a & \gamma \\ \bar{\gamma} & 0 \end{bmatrix}$. We apply Φ on both hand sides of

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

to obtain

$$\begin{aligned} \Phi\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right) \Phi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \Phi\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right) &= \begin{bmatrix} 1 & \gamma \\ \bar{\gamma} & 0 \end{bmatrix} \begin{bmatrix} 0 & \delta \\ \bar{\delta} & 0 \end{bmatrix} \begin{bmatrix} 1 & \gamma \\ \bar{\gamma} & 0 \end{bmatrix} \\ &= \begin{bmatrix} \gamma\bar{\delta} + \bar{\gamma}\delta & \gamma^2\bar{\delta} \\ \bar{\gamma}^2\delta & 0 \end{bmatrix} = \Phi\left(\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 2 & \gamma \\ \bar{\gamma} & 0 \end{bmatrix}. \end{aligned}$$

Thus, we get $\gamma^2\bar{\delta} = \gamma$. Since $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is an involution, so is $\Phi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & \delta \\ \bar{\delta} & 0 \end{bmatrix}$. This gives us $|\delta| = 1$, hence $\gamma = \delta$.

The next step is taking arbitrary $x, y, z \in \mathbb{R}$, $y, z \neq 0$, such that $\operatorname{sign} y = \operatorname{sign} z$. Note that

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix} = \begin{bmatrix} \frac{y}{z} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & x - \frac{y^2}{z} \end{bmatrix} \begin{bmatrix} \frac{y}{z} & 1 \\ 1 & 0 \end{bmatrix},$$

hence

$$\Phi\left(\begin{bmatrix} x & y \\ y & z \end{bmatrix}\right) = \begin{bmatrix} x & y\gamma \\ y\bar{\gamma} & z \end{bmatrix}. \tag{1}$$

On the other hand,

$$\Phi\left(\begin{bmatrix} x & -y \\ -y & z \end{bmatrix}\right) = \Phi\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ y & z \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} x & -y\gamma \\ -y\bar{\gamma} & z \end{bmatrix},$$

hence the equation (1) holds for all $x, y, z \in \mathbb{R}$.

Denote with Γ the unit circle of $\mathbb{C}$. From Steps 4 and 5 we know that there exists $\omega : \Gamma \to \mathbb{C}$ such that $\Phi\left(\begin{bmatrix} 0 & \beta \\ \bar{\beta} & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & \omega(\beta) \\ \overline{\omega(\beta)} & 0 \end{bmatrix}$ for every $\beta \in \Gamma$. Since $\begin{bmatrix} 0 & \omega(\beta) \\ \overline{\omega(\beta)} & 0 \end{bmatrix}$ is an involution, it must be that $|\omega(\beta)| = 1$, hence $\omega : \Gamma \to \Gamma$. Define $\rho : \Gamma \to \Gamma$ with $\rho(\beta) = \frac{\omega(\beta)}{\beta\gamma}$. Then it holds that $\Phi\left(\begin{bmatrix} 0 & \beta \\ \bar{\beta} & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & \beta\gamma\rho(\beta) \\ \overline{\beta\gamma\rho(\beta)} & 0 \end{bmatrix}$. Also, $\rho(1) = 1$.

Take $\alpha \in \Gamma$ to obtain

$$\Phi\left(\begin{bmatrix} 0 & \alpha\beta^2 \\ \overline{\alpha\beta^2} & 0 \end{bmatrix}\right) = \Phi\left(\begin{bmatrix} 0 & \beta \\ \bar{\beta} & 0 \end{bmatrix}\begin{bmatrix} 0 & \bar{\alpha} \\ \alpha & 0 \end{bmatrix}\begin{bmatrix} 0 & \beta \\ \bar{\beta} & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & \alpha\beta^2\gamma\rho(\alpha\beta^2) \\ \overline{\alpha\beta^2\gamma\rho(\alpha\beta^2)} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \beta\gamma\rho(\beta) \\ \overline{\beta\gamma\rho(\beta)} & 0 \end{bmatrix}\begin{bmatrix} 0 & \bar{\alpha}\gamma\rho(\bar{\alpha}) \\ \overline{\bar{\alpha}\gamma\rho(\bar{\alpha})} & 0 \end{bmatrix}\begin{bmatrix} 0 & \beta\gamma\rho(\beta) \\ \overline{\beta\gamma\rho(\beta)} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \alpha\beta^2\gamma\rho(\beta)^2\overline{\rho(\bar{\alpha})} \\ \overline{\alpha\beta^2\gamma\rho(\beta)^2}\rho(\bar{\alpha}) & 0 \end{bmatrix},$$

so

$$\rho(\alpha\beta^2) = \overline{\rho(\bar{\alpha})}\rho(\beta)^2. \tag{2}$$

If we insert $\alpha = 1$ and arbitrary β, we get $\rho(\beta^2) = \rho(\beta)^2$. On the other hand, if we insert $\beta = 1$ and arbitrary α, we get $\overline{\rho(\bar{\alpha})} = \rho(\alpha)$. Using these two expressions on (2), we get $\rho(\alpha\beta^2) = \rho(\alpha)\rho(\beta)^2 = \rho(\alpha)\rho(\beta^2)$. Denoting $\beta' := \beta^2$, we get

$$\rho(\alpha\beta') = \rho(\alpha)\rho(\beta') \tag{3}$$

for every $\alpha, \beta' \in \Gamma$, thus a function ρ is multiplicative.

Take arbitrary $x, z \in \mathbb{R}$ and $y \in \mathbb{C}$. Write $y = |y|e^{i\phi}$ for $\phi \in [0, 2\pi)$. Then

$$\Phi\left(\begin{bmatrix} x & y \\ \bar{y} & z \end{bmatrix}\right) = \Phi\left(\begin{bmatrix} 0 & e^{i\frac{\phi}{2}} \\ e^{-i\frac{\phi}{2}} & 0 \end{bmatrix}\begin{bmatrix} z & |y| \\ |y| & x \end{bmatrix}\begin{bmatrix} 0 & e^{i\frac{\phi}{2}} \\ e^{-i\frac{\phi}{2}} & 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0 & e^{i\frac{\phi}{2}}\gamma\rho(e^{i\frac{\phi}{2}}) \\ \overline{e^{i\frac{\phi}{2}}\gamma\rho(e^{i\frac{\phi}{2}})} & 0 \end{bmatrix} \begin{bmatrix} z & |y|\gamma \\ |y|\bar{\gamma} & x \end{bmatrix} \begin{bmatrix} 0 & e^{i\frac{\phi}{2}}\gamma\rho(e^{i\frac{\phi}{2}}) \\ \overline{e^{i\frac{\phi}{2}}\gamma\rho(e^{i\frac{\phi}{2}})} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} x & |y|e^{i\phi}\gamma\rho(e^{i\frac{\phi}{2}})\rho(e^{i\frac{\phi}{2}}) \\ \overline{|y|e^{i\phi}\gamma\rho(e^{i\frac{\phi}{2}})\rho(e^{i\frac{\phi}{2}})} & z \end{bmatrix} = \begin{bmatrix} x & y\gamma\rho(e^{i\phi}) \\ \overline{y\gamma\rho(e^{i\phi})} & z \end{bmatrix},$$

where the last equality holds due to (3).

Now, take $\beta \in \Gamma$ and calculate

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \beta \\ \bar{\beta} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1+\beta+\bar{\beta} & 1+\beta+\bar{\beta} \\ 1+\beta+\bar{\beta} & 1+\beta+\bar{\beta} \end{bmatrix}.$$

Applying Φ on both hand sides, we obtain

$$\begin{bmatrix} 1 & \gamma \\ \bar{\gamma} & 1 \end{bmatrix} \begin{bmatrix} 1 & \beta\gamma\rho(\beta) \\ \overline{\beta\gamma\rho(\beta)} & 0 \end{bmatrix} \begin{bmatrix} 1 & \gamma \\ \bar{\gamma} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+\beta\rho(\beta)+\bar{\beta}\rho(\bar{\beta}) & \gamma(1+\beta\rho(\beta)+\bar{\beta}\rho(\bar{\beta})) \\ \bar{\gamma}(1+\beta\rho(\beta)+\bar{\beta}\rho(\bar{\beta})) & 1+\beta\rho(\beta)+\bar{\beta}\rho(\bar{\beta}) \end{bmatrix} = \begin{bmatrix} 1+\beta+\bar{\beta} & \gamma(1+\beta+\bar{\beta}) \\ \bar{\gamma}(1+\beta+\bar{\beta}) & 1+\beta+\bar{\beta} \end{bmatrix},$$

which gives us $\beta+\bar{\beta} = \beta\rho(\beta)+\bar{\beta}\rho(\bar{\beta})$. Then $\operatorname{Re}\beta = \operatorname{Re}\beta\rho(\beta)$, and since $|\beta| = |\rho(\beta)| = 1$, it must be that $|\operatorname{Im}\beta| = |\operatorname{Im}\beta\rho(\beta)|$. Hence $\operatorname{Im}\beta\rho(\beta) = \pm\operatorname{Im}\beta$. Thus, we have either $\beta\rho(\beta) = \beta$ or $\beta\rho(\beta) = \bar{\beta}$. We obtain that either

$$\Phi\left(\begin{bmatrix} x & y \\ \bar{y} & z \end{bmatrix}\right) = \begin{bmatrix} x & y\gamma \\ \bar{y}\bar{\gamma} & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \bar{\gamma} \end{bmatrix} \begin{bmatrix} x & y \\ \bar{y} & z \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix}$$

or

$$\Phi\left(\begin{bmatrix} x & y \\ \bar{y} & z \end{bmatrix}\right) = \begin{bmatrix} x & \bar{y}\gamma \\ y\bar{\gamma} & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \bar{\gamma} \end{bmatrix} \begin{bmatrix} x & \bar{y} \\ y & z \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix}.$$

It is clear that these two forms of Φ cannot exist simultaneously, hence Φ always takes a single form for every matrix in $\mathcal{H}_2(\mathbb{C})$.

These findings give us the following lemma.

Lemma 6.2. *Let $\Phi : \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C})$ be a nondegenerate J.T.P. homomorphism preserving E_{11}. Then there exists a diagonal unitary matrix U such that either $\Phi(A) = UAU^*$ or $\Phi(A) = UA^TU^* = U\bar{A}U^*$.*

The main result of this section characterizes nondegenerate regular J.T.P. homomorphisms on $\mathcal{H}_2(\mathbb{C})$.

Proposition 6.3. *Let $\Phi : \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C})$. The map Φ is a nondegenerate J.T.P. homomorphism if and only if there exists a unitary matrix U such that*

$$\Phi(A) = UAU^* \quad \textit{or} \quad \Phi(A) = UA^TU^* = U\bar{A}U^*.$$

The nondegenerate case is covered by the forms (ii) and (iii) of Theorem 2.1.

7. Degenerate case

In this section, we consider regular J.T.P. homomorphisms $\Phi : \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C})$ such that $\Phi(A) = 0$ for every $A \in \mathcal{H}_2(\mathbb{C})$ with $\operatorname{rank} A \leq 1$. We refer to such regular Φ as *degenerate* J.T.P. homomorphism.

Further, we assume that

$$\sigma\left(\Phi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)\right) = \{-1, 1\}.$$

The other possibility was already considered in Section 5.

Lemma 7.1. *If there exists $\lambda \in \mathbb{R}$ such that $\Phi(\lambda I)$ is not a scalar, then there exist unitary matrix U, distinct unital multiplicative maps $\psi_1, \psi_2 : [0, \infty) \to [0, \infty)$ with $\psi_i(0) = 0$ for $i \in \{1, 2\}$, and maps $\eta_1, \eta_2 : \{0, 1, 2\} \to \{-1, 1\}$ which satisfy $\eta_1(2) = \eta_2(2) = 1$ and $\eta_1(1) \neq \eta_2(1)$, so that $\Phi(A)$ has the form*

$$\Phi(A) = U \begin{bmatrix} \psi_1(|\det A|)\eta_1(\operatorname{Syl}(A)) & 0 \\ 0 & \psi_2(|\det A|)\eta_2(\operatorname{Syl}(A)) \end{bmatrix} U^*,$$

for every $A \in \mathcal{H}_2(\mathbb{C})$, where $\operatorname{Syl}(A)$ *is the inertia of A.*

Remark 1. Notice that in this case we get a similar form as in the Proposition 5.4.

If $\Phi(\lambda I)$ is a scalar matrix for every $\lambda \in \mathbb{R}$, then there exists $\Psi : \mathbb{R} \to \mathbb{R}$ multiplicative such that $\Phi(\lambda I) = \Psi(\lambda)I$. In the remainder of this section, we assume that $\Phi(\lambda I) = \Psi(\lambda)I$. Due to regularity of Φ it holds that $\Psi(0) = 0$ and $\Psi(1) = 1$.

A set of matrices $\left\{\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} : a \in \mathbb{R}^*\right\}$ is isomorphic to the group of nonzero real numbers for multiplication, so Φ induces a J.T.P. homomorphism from $\mathbb{R}^*$ to the set of invertible matrices in $\mathcal{H}_2(\mathbb{C})$. By Lemma 3.4 it then holds that

$$\Phi\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\right) = U \begin{bmatrix} \alpha(a) & 0 \\ 0 & \beta(a) \end{bmatrix} U^*,$$

for some unitary matrix U and $\alpha, \beta : \mathbb{R}^* \to \mathbb{R}^*$ unital multiplicative maps. Without the loss of generality we may assume that

$$\Phi\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} \alpha(a) & 0 \\ 0 & \beta(a) \end{bmatrix}$$

with $\alpha(1) = \beta(1) = 1$ and $\{\alpha(-1), \beta(-1)\} = \{-1, 1\}$. We may also assume that $\alpha(-1) = -1$ and $\beta(-1) = 1$, hence

$$\Phi\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Next, we look at the image of the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and denote its upper right entry by b. It turns out that we have two cases to consider. The proofs omitted can be found at arXiv.org under the identifier arXiv:1512.02849.

Lemma 7.2. *Let $\Phi : \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C})$ be a degenerate J.T.P. homomorphism mapping scalars to scalars such that $\Phi\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Then*

$$\Phi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{cases} \pm\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; & \textit{or} \\ \begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix}; & |b| = 1. \end{cases}$$

Remark 2. In the first case, where $b = 0$, images of involutions $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ commute. In the second case, where $|b| = 1$, images do not commute. We will consider these cases in Subsections 6.1 and 6.2.

Lemma 7.3. *For an invertible matrix $A \in \mathcal{H}_2(\mathbb{C})$ define*

$$\eta(A) = \begin{cases} 1; & A > 0 \ \textit{or} \ A <> 0 \\ -1; & A < 0\,. \end{cases}$$

If $\Phi : \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C})$ is a J.T.P. homomorphism, then so is

$$\Phi'(A) = \begin{cases} \eta(A)\Phi(A); & \det A \neq 0 \\ 0; & \det A = 0\,. \end{cases}$$

7.1. Case $b = 0$. In this subsection, we have the following assumptions (C1):

- $\Phi : \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C})$ a regular J.T.P. homomorphism;

- $\Phi(A) = 0$ for every $A \in \mathcal{H}_2(\mathbb{C})$ with rank $A \leq 1$;
- $\Phi(\lambda I) = \Psi(\lambda)I$ for some $\Psi : \mathbb{R} \to \mathbb{R}$ multiplicative with $\Psi(0) = 0$;
- $\Phi\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} \alpha(a) & 0 \\ 0 & \beta(a) \end{bmatrix}$ for $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ unital multiplicative maps with $\alpha(-1) = -1$ and $\beta(-1) = 1$;
- $\Phi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

We have $\Phi\left(\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}\right) = \begin{bmatrix} \alpha(a) & 0 \\ 0 & \beta(a) \end{bmatrix}$. Let $a > 0$. It follows that

$$\Phi\left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\right) = \Psi(a)I = \Phi\left(\begin{bmatrix} \sqrt{a} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} \sqrt{a} & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} \alpha(a)^2 & 0 \\ 0 & \beta(a)^2 \end{bmatrix}.$$

Thus $\alpha(a)^2 = \beta(a)^2$, hence $\alpha(a) = \pm\beta(a)$.

It also holds that $\begin{bmatrix} \alpha(a) & 0 \\ 0 & \beta(a) \end{bmatrix} = \begin{bmatrix} \alpha(\sqrt{a})^2 & 0 \\ 0 & \beta(\sqrt{a})^2 \end{bmatrix}$, hence it follows that $\alpha(a) > 0$ and $\beta(a) > 0$, which in turn implies that $\alpha(a) = \beta(a)$ for all $a > 0$. By initial assumptions it also holds that $\alpha(-1) = -1$ and $\beta(-1) = 1$. Thus, for $a < 0$, we have $\alpha(a) = -\alpha(|a|)$ and $\beta(a) = \beta(|a|)$. Hence $\alpha(-x) = -\alpha(x)$ and $\beta(-x) = \beta(x)$ for all $x \in \mathbb{R}$ with $\alpha(x) = \beta(x) > 0$ for all $x > 0$.

We can conclude that

- $\Phi\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} -\alpha(|a|) & 0 \\ 0 & \alpha(|a|) \end{bmatrix} = \Phi\left(\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}\right)$ for $a < 0$;
- $\Phi\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} \alpha(ab) & 0 \\ 0 & \alpha(ab) \end{bmatrix}$ for $a, b > 0$;
- $\Phi\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} -\alpha(|ab|) & 0 \\ 0 & \alpha(|ab|) \end{bmatrix} = \begin{bmatrix} \alpha(ab) & 0 \\ 0 & \alpha(|ab|) \end{bmatrix}$ for $a < 0$ and $b > 0$ or $a > 0$ and $b < 0$.

From $-I$ an involution and $\Phi(\lambda I) = \Psi(\lambda)I$ it follows that $\Phi(-I) = \pm I$. If $\Phi(-I) = -I$, define $\Phi'(A) = \eta(A)\Phi(A)$, so $\Phi'(-I) = I$. Hence we can assume without the loss of generality that $\Phi(-I) = I$. Thus,

$$\Phi\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} \alpha(ab) & 0 \\ 0 & \alpha(|ab|) \end{bmatrix}$$

for every $a, b \in \mathbb{R}$, which gives us the following lemma.

Lemma 7.4. *Under assumptions* (C1) *and* $\Phi(-I) = I$, *there exists a unital odd multiplicative function* $\alpha : \mathbb{R} \to \mathbb{R}$ *with* $\alpha(-1) = -1$

$$\Phi\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} \alpha(ab) & 0 \\ 0 & \alpha(|ab|) \end{bmatrix}$$

for all $a, b \in \mathbb{R}$.

We proceed by proving the following steps under assumptions (C1) and $\Phi(-I) = I$.

Step 1. $\Phi\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix}\right) = \begin{bmatrix} \alpha(a^2 - b^2) & 0 \\ 0 & \alpha(|a^2 - b^2|) \end{bmatrix}$ for all $a, b \in \mathbb{R}$.

Step 2. $\Phi\left(\begin{bmatrix} x & y \\ y & z \end{bmatrix}\right) = \begin{bmatrix} \alpha(xz - y^2) & 0 \\ 0 & \alpha(|xz - y^2|) \end{bmatrix}$ for all $x, y, z \in \mathbb{R}$ with $xz > 0$.

Step 3. $\Phi\left(\begin{bmatrix} \pm\sqrt{1 - a^2} & a \\ a & \mp\sqrt{1 - a^2} \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ for every $a \in (0, 1)$.

Step 4. Every involution maps into a diagonal involution.

Now, take an arbitrary $A \in \mathcal{H}_2(\mathbb{C})$. We know by Lemma 3.1 that $A = BDB$ for some involution B and some diagonal matrix D. Hence

$$\Phi(A) = \pm\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \Phi(D) \left(\pm\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \Phi(D) = \begin{bmatrix} \alpha(\det A) & 0 \\ 0 & \alpha(|\det A|) \end{bmatrix}.$$

Case $b = 0$ amounts to the following proposition.

Proposition 7.5. *Let* $\Phi : \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C})$ *be a regular J.T.P. homomorphism such that*

- $\Phi(A) = 0$ *for every* $A \in \mathcal{H}_2(\mathbb{C})$ *with* $\operatorname{rank} A \leq 1$;
- Φ *maps scalars to scalars;*
- *images of* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *and* $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ *commute.*

Then there exist a unitary matrix U *and* $\alpha : \mathbb{R} \to \mathbb{R}$ *a unital multiplicative map with* $\alpha(-1) = -1$ *such that*

$$\Phi(A) = U \begin{bmatrix} \alpha(\det A) & 0 \\ 0 & \alpha(|\det A|) \end{bmatrix} U^* \quad \text{for every } A \in \mathcal{H}_2(\mathbb{C}),$$

or

$$\Phi(A) = \eta(A)U \begin{bmatrix} \alpha(\det A) & 0 \\ 0 & \alpha(|\det A|) \end{bmatrix} U^* \quad \textit{for every } A \in \mathcal{H}_2(\mathbb{C}),$$

where η is the function defined in Lemma 7.3.

This case is covered by the form (i) of Theorem 2.1.

7.2. Case $|b| = 1$. In this subsection, we consider Φ as in Lemma 7.2 such that $\Phi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix}$ for some $b \in \Gamma$.

Lemma 7.6. *Let $\Phi : \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C})$ be a degenerate J.T.P. homomorphism mapping scalars to scalars such that Φ maps $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ to a diagonal matrix for every $a \in \mathbb{R}$. Then there exists a unitary matrix U such that $\Phi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = U \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} U^*$.*

Define $\Phi' = U^*\Phi U$. Thus, $\Phi'\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Since Φ' preserves the other assumptions of the lemma, we can substitute Φ for Φ', if necessary, so we can safely assume that $\Phi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

We have $\Phi\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} \alpha(a) & 0 \\ 0 & \beta(a) \end{bmatrix}$ for $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ unital multiplicative maps with $\alpha(-1) = -1$ and $\beta(-1) = 1$. If $a \neq 0$, we write it as

$$\Phi\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\right) = \beta(a) \begin{bmatrix} \frac{\alpha(a)}{\beta(a)} & 0 \\ 0 & 1 \end{bmatrix}.$$

Define $\gamma : \mathbb{R}^* \to \mathbb{R}^*$ with $\gamma(a) = \frac{\alpha(a)}{\beta(a)}$. Then γ is a unital multiplicative map with $\gamma(-1) = -1$.

We now have the following assumptions (C2):

- $\Phi : \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C})$ a regular J.T.P. homomorphism;
- $\Phi(A) = 0$ for every $A \in \mathcal{H}_2(\mathbb{C})$ with $\operatorname{rank} A \leq 1$;
- $\Phi(\lambda I) = \Psi(\lambda)I$ for some $\Psi : \mathbb{R} \to \mathbb{R}$ multiplicative with $\Psi(0) = 0$;

- $\Phi\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\right) = \beta(a)\begin{bmatrix} \gamma(a) & 0 \\ 0 & 1 \end{bmatrix}$ for $\beta, \gamma : \mathbb{R}^* \to \mathbb{R}^*$ unital multiplicative maps with $\beta(-1) = 1$ and $\gamma(-1) = -1$;
- $\Phi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Lemma 7.7. *Under assumptions* (C2), *it holds that*

$$\Phi\left(\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}\right) = \beta(a)\begin{bmatrix} 1 & 0 \\ 0 & \gamma(a) \end{bmatrix}$$

for every $a \in \mathbb{R}^*$.

We know that

- $\Phi\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$;
- $\Phi\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$;
- $\Phi\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$;
- $\Phi\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right) = \pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

If $\Phi(-I) = I$, we multiply Φ by η from Lemma 7.3 to get $\Phi(-I) = -I$. So, we may assume without the loss of generality that $\Phi(-I) = -I$.

We proceed by proving the following steps under assumptions (C2).

Step 1. If $\Phi(-I) = -I$, then $\Phi\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \beta(ab)\begin{bmatrix} \gamma(a) & 0 \\ 0 & \gamma(b) \end{bmatrix}$ for every $a, b \in \mathbb{R}$.

Step 2. $\Phi\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right) = \pm\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Step 3. If $a, b \in \mathbb{R}$ are arbitrary, then $\Phi\left(\begin{bmatrix} a+2b & b \\ b & a \end{bmatrix}\right) = \begin{bmatrix} a'+2b' & b' \\ b' & a' \end{bmatrix}$ for some $a', b' \in \mathbb{R}$.

Lemma 7.8. *Under assumptions* (C2), *the function* $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ *satisfies functional equation*

$$\gamma\left(\frac{x - 1 + \sqrt{2x^2+2}}{x+1}\right) = \frac{\gamma(x) - 1 + \sqrt{2\gamma(x)^2+2}}{\gamma(x)+1}. \tag{f.e.}$$

Proof. Take $x, a > 0$. Then

$$\Phi\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \beta(a^2x) \begin{bmatrix} \gamma(a) & 0 \\ 0 & 1 \end{bmatrix} \left(\pm\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right) \begin{bmatrix} \gamma(x) & 0 \\ 0 & 1 \end{bmatrix} \left(\pm\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right) \begin{bmatrix} \gamma(a) & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \beta(a^2x) \begin{bmatrix} \gamma(a^2)\frac{\gamma(x)+1}{2} & \gamma(a)\frac{\gamma(x)-1}{2} \\ \gamma(a)\frac{\gamma(x)-1}{2} & \frac{\gamma(x)+1}{2} \end{bmatrix} = \Phi\left(\begin{bmatrix} a^2\frac{x+1}{2} & a\frac{x-1}{2} \\ a\frac{x-1}{2} & \frac{x+1}{2} \end{bmatrix}\right).$$

We would like the matrix $A = \begin{bmatrix} a^2\frac{x+1}{2} & a\frac{x-1}{2} \\ a\frac{x-1}{2} & \frac{x+1}{2} \end{bmatrix}$ to have the form as in Step 3, hence choose $a \in \mathbb{R}$ such that

$$a^2\frac{x+1}{2} = \frac{x+1}{2} + 2a\frac{x-1}{2}.$$

Taking for a the positive solution of this quadratic equation, we get

$$a = \frac{x-1+\sqrt{2x^2+2}}{x+1}.$$

The matrix A is therefore mapped to a matrix of the same form by Step 3, hence

$$\gamma(a^2)\frac{\gamma(x)+1}{2} = 2\gamma(a)\frac{\gamma(x)-1}{2} + \frac{\gamma(x)+1}{2}.$$

Since a is positive, it is mapped by γ to a positive solution of the new quadratic equation, thus

$$\gamma(a) = \gamma\left(\frac{x-1+\sqrt{2x^2+2}}{x+1}\right) = \frac{\gamma(x)-1+\sqrt{2\gamma(x)^2+2}}{\gamma(x)+1},$$

which concludes the proof. $\square$

Lemma 7.9. *Under assumptions* (C2), *the function* γ *has one of the following forms:*

$$\gamma(x) = x \quad \textit{for every } x \in \mathbb{R}^* \qquad \textit{or} \qquad \gamma(x) = x^{-1} \quad \textit{for every } x \in \mathbb{R}^*.$$

Proof. Let us prove the lemma for $x > 0$ first. For $x < 0$ it will then follow, since $\gamma(-x) = -\gamma(x)$.

We know that $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a multiplicative function satisfying (f.e.). By [12, Theorem 2.4] γ has the form $\gamma(x) = e^{f(\log x)}$ for every $x > 0$, where $f : \mathbb{R} \to \mathbb{R}$ is additive. From (f.e.) it follows that

$$e^{f\left(\log \frac{x-1+\sqrt{2x^2+2}}{x+1}\right)} = \frac{e^{f(\log x)} - 1 + \sqrt{2e^{2f(\log x)} + 2}}{e^{f(\log x)} + 1},$$

hence

$$f\left(\log \frac{x-1+\sqrt{2x^2+2}}{x+1}\right) = \log \frac{e^{f(\log x)} - 1 + \sqrt{2e^{2f(\log x)} + 2}}{e^{f(\log x)} + 1}.$$

Taking $x \in (1, \infty)$, we get $z = \log \frac{x-1+\sqrt{2x^2+2}}{x+1} \in (0, \log(1+\sqrt{2}))$. Substituting $y = f(\log x)$, we get

$$f(z) = \log \frac{e^{f(\log x)} - 1 + \sqrt{2e^{2f(\log x)} + 2}}{e^{f(\log x)} + 1} = \log \frac{e^y - 1 + \sqrt{2e^{2y} + 2}}{e^y + 1}.$$

Then, for $t > 0$ the following estimation manipulation

$$2t^2 + 2 \leq (\sqrt{2}t + 2 + \sqrt{2})^2, \quad \sqrt{2t^2+2} \leq \sqrt{2}t + 1 + \sqrt{2} + 1,$$
$$t - 1 + \sqrt{2t^2+2} \leq (1+\sqrt{2})(t+1), \quad \frac{t-1+\sqrt{2t^2+2}}{t+1} \leq 1 + \sqrt{2}$$

shows that $f(z) \leq \log(1+\sqrt{2})$.

Thus, additive function f is bounded on an open interval $(0, \log(1+\sqrt{2}))$, hence by [12, Theorem 1.8] it is linear. Since it has the form $f(z) = cz$ for some $c \in \mathbb{R}$, it follows that $\gamma(x) = x^c$.

We get

$$\left(\frac{x-1+\sqrt{2x^2+2}}{x+1}\right)^c = \frac{x^c - 1 + \sqrt{2x^{2c}+2}}{x^c + 1}$$

for every $x > 0$. If $c > 0$, by taking $\lim_{x\to\infty}$ we get $(1+\sqrt{2})^c = 1 + \sqrt{2}$. Thus, $c = 1$. If $c < 0$, again, by taking $\lim_{x\to\infty}$ we get $(1+\sqrt{2})^c = -1 + \sqrt{2}$, which implies $c = -1$. The last solution is $c = 0$. By taking $c = 0$, we get $\Phi\left(\begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}\right) = I$ for every $x > 0$, which is a contradiction with our assumptions, hence $c \in \{-1, 1\}$. $\square$

Next, we show that in the case $c=-1$, we can reduce the proof to the case $c=1$. We have $\gamma(x)=\frac{1}{x}$. Under assumptions (C2) and $\Phi(-I)=-I$, we have

$$\Phi\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \beta(ab)\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix} = \frac{\beta(ab)}{ab}\begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix}$$

for every $a,b \in \mathbb{R}^*$. Define

$$\Phi'(A) = \begin{cases} \Phi(A^{-1}); & \operatorname{rank} A = 2 \\ 0; & \operatorname{rank} A \leq 1. \end{cases}$$

Then, introducing new notation $\Psi'(t)=\Psi(t^{-1})$ and $\beta'(t)=\beta(t^{-1})$, we have

- $\Phi'(0)=0$;
- $\Phi'(I)=I$;
- $\Phi'(-I)=-I$;
- $\Phi'(\lambda I)=\Psi(\lambda^{-1})I=\Psi'(\lambda)I$;
- $\Phi'\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$;
- $\Phi'\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \Phi\left(\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix}\right) = \beta\left(\frac{1}{ab}\right)\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \beta'(ab)\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$;
- $\Phi'\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right) = \Phi\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right) = \pm\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$;
- $\Phi'\left(\begin{bmatrix} a+2b & b \\ b & a \end{bmatrix}\right) = \begin{bmatrix} a'+2b' & b' \\ b' & a' \end{bmatrix}$.

So, taking Φ' instead of Φ, if necessary, we may assume that $\gamma(x)=x$.

We proceed by proving the following steps under assumptions (C2), $\Phi(-I)=-I$ and $\gamma(x)=x$.

Step 4. $\Phi\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix}\right) = \beta(a^2-b^2)\begin{bmatrix} a & b \\ b & a \end{bmatrix}$.

Step 5. $\Phi\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right) = \beta(ac-b^2)\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ for every $a,b,c \in \mathbb{R}$.

Step 6. For arbitrary $x \in \Gamma$ we have $\Phi\left(\begin{bmatrix} 0 & x \\ \bar{x} & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & \lambda \\ \bar{\lambda} & 0 \end{bmatrix}$, where $\lambda \in \Gamma$.

From the previous step it follows that there exists a function on a unit circle $\lambda : \Gamma \to \Gamma$ with $\lambda(1)=1$ such that $\Phi\left(\begin{bmatrix} 0 & x \\ \bar{x} & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & \lambda(x) \\ \overline{\lambda(x)} & 0 \end{bmatrix}$.

Step 7. For arbitrary $x, y \in \Gamma$, we have $\lambda(xy) = \lambda(x)\lambda(y)$ and $\lambda(\bar{x}) = \overline{\lambda(x)}$.

Step 8. The function λ has one of the following forms:

$$\lambda(x) = x \quad \text{for every } x \in \Gamma \qquad \text{or} \qquad \lambda(x) = \bar{x} \quad \text{for every } x \in \Gamma.$$

If $\lambda(x) = \bar{x}$, define $\Phi'(A) = \Phi(\bar{A})$. Therefore, we can assume without the loss of generality that $\lambda(x) = x$.

Step 9. If $\lambda \equiv \mathrm{id}$, then $\Phi\left(\begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix}\right) = \beta(ac - b\bar{b}) \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix}$.

These findings amount to the following proposition. This case is covered by the form (iv) of Theorem 2.1.

Proposition 7.10. *Let $\Phi : \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C})$ be a regular J.T.P. homomorphism such that*

- $\Phi(A) = 0$ *for every* $A \in \mathcal{H}_2(\mathbb{C})$ *with* $\operatorname{rank} A \leq 1$*;*
- Φ *maps scalars to scalars;*
- *images of* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *and* $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ *do not commute.*

Then there exist a unitary matrix U and $\beta : \mathbb{R} \to \mathbb{R}$ a unital multiplicative map with $\beta(-1) = 1$ such that

$$\Phi(A) = \begin{cases} \beta(\det A) \cdot U\widetilde{\Phi}(A)U^*; & \operatorname{rank} A = 2 \\ 0; & \operatorname{rank} A \leq 1, \end{cases}$$

where $\widetilde{\Phi}$ has one of the following forms:

- $\widetilde{\Phi}(A) = A$;
- $\widetilde{\Phi}(A) = \bar{A}$;
- $\widetilde{\Phi}(A) = A^{-1}$;
- $\widetilde{\Phi}(A) = \bar{A}^{-1}$;
- $\widetilde{\Phi}(A) = \eta(A)A$;
- $\widetilde{\Phi}(A) = \eta(A)\bar{A}$;
- $\widetilde{\Phi}(A) = \eta(A)A^{-1}$;
- $\widetilde{\Phi}(A) = \eta(A)\bar{A}^{-1}$;

for every $A \in \mathcal{H}_2(\mathbb{C})$, where η is the function defined in Lemma 7.3.

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