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On centralizers of an H^*-algebra

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Abstract. It is shown that the additive function T acting on a semi-simple H^*-algebra $\mathcal{A}$ with the property that $T(x^3) = T(x)x^2$ $(x \in \mathcal{A})$ is a left centralizer.

Let $\mathcal{A}$ be a *-ring. The additive function $E : \mathcal{A} \to \mathcal{A}$ is called a Jordan *-derivation if $E(x^2) = E(x)x^* + xE(x)$ holds for all $x \in \mathcal{A}$. These mappings are extensively studied due to the fact that their structure is in a close relation with the problem of representability of quadratic functionals by sesquilinear forms (e.g. [9–11]). Recently, to give a simpler and more natural proof of ŠEMRL's fundamental theorem [11], ZALAR introduced the more general concept of Jordan *-derivation pairs [12, 13]. Moreover, to prove a Jordan - von Neumann type theorem for Hilbert A-modules, he has dealt with the structure of such pairs acting on an H*-algebra [14]. These works of ZALAR inspired us to introduce an even more general concept of Jordan *-derivation pairs and in [7] we were able to show that for a large class of complex *-algebras the representability of these pairs via double centralizers still remains valid that means a significant generalization of Zalar's result. Our definition was the following:

Definition. Let $\mathcal{A}$ be a *-ring. If $E, F : \mathcal{A} \to \mathcal{A}$ are additive functions such that

$$E(x^3) = E(x)x^{*2} + xF(x)x^* + x^2E(x)$$
$$F(x^3) = F(x)x^{*2} + xE(x)x^* + x^2F(x)$$

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hold for all $x \in \mathcal{A}$, then (E, F) is called a Jordan *-derivation pair.

We also recall the definition of a left centralizer. It is an additive function $T : \mathcal{A} \to \mathcal{A}$ on the ring $\mathcal{A}$ with the property that $T(xy) = T(x)y$ for every $x, y \in \mathcal{A}$. The definition of a right centralizer should be self-explanatory.

Now, in case of a complex *-algebra the proof á la Zalar concerning the representability may go like this. One should prove that for the additive functions

$$T_1(x) = -\frac{1}{2i}(E(ix^*) - iE(x^*)), \quad S_1(x) = -\frac{1}{2i}(F(ix) + iF(x))$$

$$T_2(x) = \frac{1}{2i}(F(ix^*) - iF(x^*)), \quad S_2(x) = \frac{1}{2i}(E(ix) + iE(x))$$

$(x \in \mathcal{A})$ we have $T_1(x^3) = T_1(x)x^2$, $S_1(x^3) = x^2 S_1(x)$, $T_2(x^3) = T_2(x)x^2$, $S_2(x^3) = x^2 S_2(x)$ for every $x \in \mathcal{A}$. As a matter of fact it is not trivial at all, but after algebraic manipulations one can arrive at this. Then following Zalar's argument we should obtain that these mappings are left or right centralizers on $\mathcal{A}$. The aim of this paper is to deal with this latter question (cf. [3, Proposition 2.5]).

We begin with some basic definitons, results and notation.
Concerning the statements on H*-algebras that follow, we refer to [1]. Let $\mathcal{A}$ be a semi-simple H*-algebra and let $\{\mathcal{A}_\alpha \colon \alpha \in \Gamma\}$ denote the collection of minimal closed ideals of $\mathcal{A}$. This system is pairwise orthogonal and its orthogonal direct sum is $\mathcal{A}$, moreover, for every $\alpha \in \Gamma$ there exist a Hilbert space H_α, a real constant $c_\alpha \geq 1$ and an isometric *-isomorphism between $\mathcal{A}_\alpha$ and $HS(H_\alpha)$ $(= (HS(H_\alpha), c_\alpha\langle\, . \,, . \,\rangle))$, the H*-algebra of Hilbert-Schmidt operators. A self-adjoint idempotent element e of $\mathcal{A}$ is said to be a projection; a nonzero projection is called minimal if it cannot be represented as a sum of two mutually orthogonal nonzero projections in $\mathcal{A}$. For an element $x \in \mathcal{A}$ we have $x = 0$ if and only if $xe = 0$ for every minimal projection $e \in \mathcal{A}$.
If X is a Banach space, then let $\mathcal{L}(X)$ and $\mathcal{B}(X)$ denote the algebra of all linear and continuous linear operators on X, respectively. $\mathcal{F}(X)$ stands for the ideal of finite rank operators in $\mathcal{B}(X)$. An algebra $\mathcal{A} \subset \mathcal{B}(X)$ is said to be standard provided $\mathcal{F}(X) \subset \mathcal{A}$.

Our basic lemma is as follows

Lemma. *Let $\mathcal{A} \subset \mathcal{B}(X)$ be a standard operator algebra and $\mathcal{T} : \mathcal{A} \to \mathcal{L}(X)$ be an additive function with the property that*

$$\mathcal{T}(A^3) = \mathcal{T}(A)A^2$$

holds for every $A \in \mathcal{A}$. Then there exists an $C \in \mathcal{L}(X)$ such that

$$\mathcal{T}(A) = CA \qquad (A \in \mathcal{A}).$$

Proof. Let us first consider the restriction of $\mathcal{T}$ onto $\mathcal{F}(X)$. Suppose that $A, P \in \mathcal{F}(X)$ and P is a projection with $AP = PA = A$. Then linearizing the equation $\mathcal{T}(A^3) = \mathcal{T}(A)A^2$ by substituting $A + P$ for A, we obtain

$$\begin{aligned}(1)\quad 3\mathcal{T}(A^2 + A) &= \mathcal{T}(A^2P + APA + PA^2 + P^2A + PAP + AP^2) = \\ &= \mathcal{T}(A)AP + \mathcal{T}(A)PA + \mathcal{T}(P)A^2 + \mathcal{T}(P)PA + \mathcal{T}(P)AP + \mathcal{T}(A)P^2 = \\ &= 2\mathcal{T}(A)A + \mathcal{T}(P)A^2 + 2\mathcal{T}(P)A + \mathcal{T}(A)P.\end{aligned}$$

Replacing A by $A + P$ again, the linearization of (1) results in

$$6\mathcal{T}(A) = 4\mathcal{T}(P)A + 2\mathcal{T}(A)P. \tag{2}$$

Multiplying both sides by P on the right, it follows that $\mathcal{T}(A)P = \mathcal{T}(P)A$ and substituting it into (2), we arrive at

$$\mathcal{T}(A) = \mathcal{T}(P)A. \tag{3}$$

We next prove that $\mathcal{T}(AB) = \mathcal{T}(A)B$ for every $A, B \in \mathcal{F}(X)$. If X is finite dimensional, then it is obvious because in this case P can be chosen to be the identity operator on X. So suppose that X is of infinite dimension and let $x \in X$ be fixed. We conclude from (3) that for every $f \in X^*$ there exists a vector $x(f) \in X$ such that $\mathcal{T}(x \otimes f) = x(f) \otimes f$. Let f_1, f_2 be nonzero elements of X^*. We show that $x(f_1) = x(f_2)$. Indeed, if f_1 and f_2 are linearly independent, then the equation

$$\begin{aligned}x(f_1 + f_2) \otimes f_1 + x(f_1 + f_2) \otimes f_2 &= \mathcal{T}(x \otimes (f_1 + f_2)) = \\ = \mathcal{T}(x \otimes f_1) + \mathcal{T}(x \otimes f_2) &= x(f_1) \otimes f_1 + x(f_2) \otimes f_2,\end{aligned}$$

together with the fact that no one of the subspaces $\ker f_1$ and $\ker f_2$ is contained in the other, imply that

$$x(f_1) = x(f_1 + f_2) = x(f_2).$$

If f_1 and f_2 are linearly dependent, then choosing an $f_3 \in X^*$ such that f_1, f_3 as well as f_2, f_3 are independent, we obtain

$$x(f_1) = x(f_3) = x(f_2).$$

Let $0 \neq f, g \in X^*$ and $u \in X$ be such that $f(u) \neq 0$. Then

$$\begin{aligned}\mathcal{T}(x \otimes f \cdot u \otimes g) &= \mathcal{T}(x \otimes f(u)g) = f(u)x(f(u)g) \otimes g = \\ = f(u)x(f) \otimes g &= x(f) \otimes f \cdot u \otimes g = \mathcal{T}(x \otimes f)u \otimes g.\end{aligned}$$

Also in case $f(u)=0$, we have

$$\mathcal{T}(x\otimes f\cdot u\otimes g)=0,$$
$$\mathcal{T}(x\otimes f)u\otimes g=\mathcal{T}(P)(x\otimes f\cdot u\otimes g)=0$$

with some finite dimensional projection P. The additivity of $\mathcal{T}$ implies that

$$\mathcal{T}(AB)=\mathcal{T}(A)B \qquad A,B\in\mathcal{F}(X).$$

As an easy consequence, we immediately have the linearity of $\mathcal{T}$ on $\mathcal{F}(X)$. Now, let $f\in X^*$ and $y\in X$ be such that $f(y)=1$. Define

$$Cx=\mathcal{T}(x\otimes f)y \qquad (x\in X).$$

Then $C\in\mathcal{L}(X)$,

$$(CA)x=\mathcal{T}(Ax\otimes f)y=\mathcal{T}(A)((x\otimes f)y)=\mathcal{T}(A)x \qquad (x\in X)$$

and it implies $\mathcal{T}(A)=CA$ for every $A\in\mathcal{F}(X)$. It remains to prove that this latter equation is valid on $\mathcal{A}$ as well. To this end let $\mathcal{T}_1:\mathcal{A}\to\mathcal{L}(X)$ be defined by $\mathcal{T}_1(A)=CA$ and consider $\mathcal{T}_0=\mathcal{T}-\mathcal{T}_1$. We know that $\mathcal{T}_0(A^3)=\mathcal{T}_0(A)A^2$ $(A\in\mathcal{A})$, $\mathcal{T}_0$ is additive and it vanishes on $\mathcal{F}(X)$. Let $A\in\mathcal{A}$, suppose that $P\in\mathcal{F}(X)$ is a projection and $S=(1-P)A(1-P)$. Since $S-A\in\mathcal{F}(X)$, we have $\mathcal{T}_0(S)=\mathcal{T}_0(A)$. Moreover, from the equation

$$\mathcal{T}_0(S^3+P^3)=\mathcal{T}_0((S+P)^3)=$$
$$=\mathcal{T}_0(S+P)(S+P)^2=\mathcal{T}_0(S+P)(S^2+P^2)$$

we infer

$$0=\mathcal{T}_0(S)P+\mathcal{T}_0(P)S^2=\mathcal{T}_0(S)P+\mathcal{T}_0(P)PS^2=\mathcal{T}_0(S)P=\mathcal{T}_0(A)P.$$

Since it holds for every finite rank projection P, we have $\mathcal{T}_0(A)=0$. This completes the proof. $\square$

Now, we are in a position to prove our theorem which is a generalization of [14, Lemma 15] and [6, Theorem 2] in case of $n=2$.

Theorem. *Let $T:\mathcal{A}\to\mathcal{A}$ be an additive function on the semi-simple H^*-algebra $\mathcal{A}$ with the property that $T(x^3)=T(x)x^2$ for every $x\in\mathcal{A}$. Then T is a left centralizer. Similar statement holds for right centralizers as well.*

Proof. Let us first linearize the equation $T(x^3)=T(x)x^2$ by substituting $x+e$ for x. Then similarly to (1), we obtain

$$T(x^2e+xex+ex^2+e^2x+exe+xe^2)= \tag{4}$$
$$=T(x)xe+T(x)ex+T(e)x^2+T(e)ex+T(e)xe+T(x)e^2.$$

Let $\alpha \in \Gamma$, $x \in \mathcal{A}_\alpha$ and e be a minimal projection with $e \in \mathcal{A}_\beta$ ($\alpha \neq \beta \in \Gamma$). Then, by $T(e) = T(e)e$, the equation (4) implies $T(x)e = 0$. Thus we conclude $T(x) \in \mathcal{A}_\alpha$, and consequently we have proved that $\mathcal{A}_\alpha$ is invariant under T. Now, it follows from our Lemma that T is a left centralizer on every minimal closed ideal of $\mathcal{A}$. Consider the continuity of T. By Lemma and [5, Remark 1], it follows that T is continuous on each $\mathcal{A}_\alpha$. To apply the closed graph theorem, let (x_n) be a sequence in $\mathcal{A}$ and $y \in \mathcal{A}$ such that

$$\lim x_n = 0 \quad \text{and} \quad \lim T(x_n) = y.$$

Linearizing (4) again, replacing x by $x + e$, we have

$$\begin{aligned} &T(xe^2 + exe + xe^2 + e^2x + exe + e^2x) = \\ &= T(x)e^2 + T(e)xe + T(x)e^2 + T(e)ex + T(e)xe + T(e)ex. \end{aligned} \tag{5}$$

If $e \in \mathcal{A}$ is a minimal projection, then from (5) we infer

$$0 = \lim(T(x_n)e + T(e)x_ne + T(e)ex_n) = ye$$

and it results in $y = 0$. Consequently, T is continuous. Finally, if $x = \sum_\alpha x_a,\ \ y = \sum_\alpha y_a \in \mathcal{A}$, where $x_\alpha, y_\alpha \in \mathcal{A}_\alpha$ ($\alpha \in \Gamma$), then we conclude

$$\begin{aligned} T(xy) &= T\left(\sum_\alpha x_\alpha y_\alpha\right) = \sum_\alpha T(x_\alpha y_\alpha) = \sum_\alpha T(x_\alpha) y_\alpha = \\ &= \left(\sum_a T(x_\alpha)\right)\left(\sum_\alpha y_a\right) = T(x)y. \qquad \square \end{aligned}$$

Remarks. We first note that one can similarly prove the following statement: If $T : \mathcal{A} \to \mathcal{A}$ is an additive function having the property that $T(x^{n+1}) = T(x)x^n$ for every $x \in \mathcal{A}$ with some fixed $n \in \mathbb{N}$, then T is a left centralizer.

We also remark that our argument can be applied to gain similar results for other operator algebra like structures without an identity element. (In case of a ring with an identity after two linearizations we can achieve the desired goal.) For example, without any changes in the proof, one can show the same statement for dual B*-algebras (e.g. [8]). These B*-algebras are exactly the ones which are isometric and *-isomorphic to the c_0-direct sum of B*-algebras of compact operators on some Hilbert spaces (for another characterization, see [4, Remark 5]).

Since in the formulation of our theorem we have used only algebraic concepts, it would be desirable to study the relevant problem in a purely ring theoretical context. As for this generality we are able only to show the following statement (cf. [14, Lemma 15 and 3, Proposition 2.5]).

Proposition. *Let $\mathcal{A}$ be a 2-torsion free prime ring and $T : \mathcal{A} \to \mathcal{A}$ be an additive function. If $T(xyx) = T(x)yx$ holds for every $x, y \in \mathcal{A}$, then T is a left centralizer.*

Proof. We use a similar argument to the proof of [2, Lemma 3.1]. For the sake of a simplified writing let $T(a)$ be denoted by a'. Let $a, b, c \in \mathcal{A}$. We prove that

$$((abc)' - a'bc)x(abc - cba) = 0 \qquad (x \in \mathcal{A}).$$

To this end we compute $(abcxcba + cbaxabc)'$ in two different ways. First we have

$$(abcxcba + cbaxabc)' = (abcxcba)' + (cbaxabc)' =$$
$$= (a(bcxcb)a)' + (c(baxab)c)' = a'(bcxcb)a + c'(baxab)c.$$

On the other hand, the linearization of the equation $(xyx)' = x'yx$ implies that

$$(abcxcba + cbaxabc)' = (abc)'x(cba) + (cba)'x(abc).$$

From these equations we infer

$$((abc)' - a'bc)x(abc - cba) = (abc)'x(abc) - (abc)'x(cba) - (a'bc)x(abc)+$$
$$+(a'bc)x(cba) = (abc)'x(abc) + (cba)'x(abc) - (a'bc)x(abc)-$$
$$-c'(baxab)c = ((abc)' + (cba)' - (a'bc) - (c'ba))x(abc) = 0 \quad (x \in \mathcal{A}).$$

Now, there are two possibilities. The first is when there exist $a_0, b_0, c_0 \in \mathcal{A}$ such that $a_0b_0c_0 \neq c_0b_0a_0$. Then by [2, Lemma 1.2] we conclude that $(abc)' - a'bc = 0$ for every $a, b, c \in \mathcal{A}$. But it implies that $(ab)'xy = (abxy)' = a'bxy$ from which we obtain that $(ab)' = a'b$ holds for any $a, b \in \mathcal{A}$.

The second possibility is that $abc = cba$ for every $a, b, c \in \mathcal{A}$. But in this case we have

$$(ab - ba)x(ab - ba) = a(bxa)b - a(b(xb)a) - (b(ax)a)b + b(axb)a =$$
$$= aaxbb - aaxbb - aaxbb + aaxbb = 0,$$

consequently $ab = ba$ for any $a, b \in \mathcal{A}$, i.e. $\mathcal{A}$ is commutative. Let $0 \neq x \in \mathcal{A}$ be fixed and define a new operation on $\mathcal{A}$ in the following way:

$$a \circ b = axb \qquad (a, b \in \mathcal{A}).$$

We prove that with this operation $\mathcal{A}$ is a prime ring. Indeed, let $a, b \in \mathcal{A}$ are such that $a \circ y \circ b = 0$ for every $y \in \mathcal{A}$. Then the primeness of $\mathcal{A}$ implies that $ax = 0$ or $xb = 0$. From the commutativity we infer that $ayx = 0$ or

$xyb = 0$ holds for any $y \in \mathcal{A}$ from which we have $a = 0$ or $b = 0$. Now, it follows that

$$(a \circ a)' = (axa)' = a'xa = a' \circ a \qquad (a \in \mathcal{A})$$

and [3, Proposition 2.5] implies that

$$(axb)' = (a \circ b)' = a' \circ b = a'xb \qquad (a, b \in \mathcal{A}).$$

The proof can be completed as in the non-commutative case. □

Since the structure theorem of semi-simple H*-algebras shows that these algebras are generally not prime rings but they are semi-prime, it would be of some importance to study whether the statement of our Theorem (or more generally the one that was discussed in Remarks) remains valid in case of such rings which question is left as an open problem.

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