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Fitting heights of solvable groups with no nontrivial prime power character degrees

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Abstract. We construct solvable groups where the only degree of an irreducible character that is a prime power is 1 and that have arbitrarily large Fitting heights. We will show that we can construct such groups that also have a Sylow tower. We also will show that we can construct such groups using only three primes.

1. Introduction

Throughout this paper, all groups are finite, and if G is a group, then we write $\mathrm{Irr}(G)$ for the irreducible characters of G, and $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ are the character degrees of G.

In the paper [1], we said that a nonabelian group G is a *composite degree group* (CDG for short) if 1 is the only prime power that lies in $\mathrm{cd}(G)$. Solvable groups satisfying this condition had earlier been studied in the paper [4]. Examples of solvable CDGs can be found in [4, Example 3.4] and in [1, Section 4]. We mentioned in [1, Section 4] that we did not know of any examples of solvable CDGs that had Fitting height larger than 3. We now remedy this by presenting CDGs with arbitrarily large Fitting heights.

Theorem 1.1. *Let $l > 1$ be an integer. Then there exists a solvable CDG G such that the Fitting height of G is l.*

We will show that the groups in Theorem 1.1 can be chosen to have a Sylow tower. It is not difficult to see that if G is a CDG, then $|G|$ must be divisible by at

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least three primes. We will show that there exist solvable CDGs with arbitrarily large derived length whose orders are divisible by only three primes.

2. Modules and extra-special groups

In this section, we construct extra-special groups whose quotients are modules for other groups.

We begin by reviewing a construction that can be found in [3] among other places. Let k be a field, and let V be a finite dimensional vector space for k. Write $\hat{V}$ for the dual vector space for V. That is, $\hat{V}$ is the set of k-linear transformations from V to k. Let $\{e_1, \dots, e_n\}$ be a basis for V, and define $\lambda_i : V \to k$ by $\lambda_i(e_j) = \delta_{ij}$ where δ_{ij} is the Kronecker delta and then extending linearly. It is not difficult to see that $\{\lambda_1, \dots, \lambda_n\}$ forms a basis for $\hat{V}$. We now define the group $E(V)$ as follows: let $E(V) = \{(v, \alpha, z) \mid v \in V, \alpha \in \hat{V}, z \in k\}$, and we define multiplication in $E(V)$ by

$$(v_1, \alpha_1, z_1)(v_2, \alpha_2, z_2) = (v_1 + v_2, \alpha_1 + \alpha_2, z_1 + z_2 + \alpha_2(v_1)).$$

It can be checked that $E(V)$ is a group. One can show that $\{(v, 0, z) \mid v \in V, z \in k\}$ and $\{(0, \alpha, z) \mid \alpha \in \hat{V}, z \in k\}$ are normal abelian subgroups whose product is $E(V)$, and whose intersection is $\{(0, 0, z) \mid z \in k\}$. Also, one can show that the commutators $[(e_i, 0, 0), (0, \lambda_j, 0)] = (0, 0, \lambda_j(e_i)) = (0, 0, \delta_{ij})$. It follows that $E(V)' = Z(E(V)) = \{(0, 0, z) \mid z \in k\}$. In the case where k has order p for some prime p, it now follows that $E(V)$ is an extra-special group of order p^{2n+1}.

Suppose that G is a group and k is a field of prime order, and suppose that V is a finite dimensional $k[G]$-module. It is not difficult to see that $\hat{V}$ will also be a $k[G]$-module where $\alpha \cdot g$ for $\alpha \in \hat{V}$ and $g \in G$ is defined by $\alpha \cdot g(v \cdot g) = \alpha(v)$. One can now see that G acts on $E(V)$ by $(v, \alpha, z) \cdot g = (v \cdot g, \alpha \cdot g, z)$. One can observe that this action is an action by automorphisms that centralizes $Z(E(V))$.

Suppose that $x \in k$ is a nonzero element of the field k. We define a map σ_x on $E(V)$ by $(v, \alpha, z)\sigma_x = (xv, x\alpha, x^2 z)$. It is not difficult to see that σ_x will be an automorphism of $E(V)$, and that the order of σ_x equals the multiplicative order of x in k. In addition, the action of $\langle \sigma_x \rangle$ on $E(V)/E(V)'$ is Frobenius, and if the order of x is odd, then in fact the action of $\langle \sigma_x \rangle$ on $E(V)$ is Frobenius. Finally, it is not difficult to see that σ_x commutes with the action of G since $(xv) \cdot g = x(v \cdot g)$ and $(x\alpha) \cdot g = x(\alpha \cdot g)$ for $v \in V$, $\alpha \in \hat{V}$, and $g \in G$.

Thus, we have proved the following:

Lemma 2.1. *If G is a group, k is a field of prime order, and V is a finite dimensional $k[G]$-module, then G acts on $E(V)$ via automorphism such that $Z(E(V))$ is centralized. Furthermore, if m divides $|k|-1$, then $E(V)$ has an automorphism of order m that commutes with the action of G and its action on $E(V)/E(V)'$ is Frobenius, and if m is odd, this automorphism can be taken so that its action on $E(V)$ is Frobenius.*

Before leaving this section, we consider how the action of G on V determines the action of G on $\hat{V}$ when the action is coprime.

Lemma 2.2. *Suppose that p is a prime, G is a p'-group, and k is the field of order p. If V is a finite dimensional $k[G]$-module, so that $C_V(G)=0$, then $C_{\hat{V}}(G)=0$.*

Proof. We begin by noting that G acts coprimely on V, so we can apply Fitting's theorem to see that $V=C_V(G)\oplus[V,G]$ where $[V,G]=\{v-v^g \mid v\in V, g\in G\}$. The assumption that $C_V(G)=0$ implies that $V=[V,G]$. Suppose that $\varphi\in C_{\hat{V}}(G)$. We then have $\varphi(v)=\varphi^g(v^g)=\varphi(v^g)$ for all $v\in V$ and $g\in G$. This implies that $\varphi(v-v^g)=\varphi(v)-\varphi(v^g)=0$ for all $v\in V$ and $g\in G$. Since $V=[V,G]$, this implies that $\varphi(V)=0$, and so, $\varphi=0$. We conclude that $C_{\hat{V}}(G)=0$. □

3. Construction

In this section, we present our construction. We begin with a simple lemma suggested by the referee.

Lemma 3.1. *Let $G=EH$, where E is normal in G and $C_H(E)=1$. Assume that E is a p-group for some prime p and that $O_p(H)=1$. Then $C_G(E)\leq E$.*

Proof. Let $C=C_G(E)$, and let $M=H\cap EC$. Since $C_H(E)=1$, we see that $M\cap C=1$. Because E is normal in G, we see that C and hence EC are normal in G. This implies that M is normal in H and applying Dedekind's lemma, we obtain $EC=EM$.

Now, $C\leq MC\leq EC$, so $MC=C(MC\cap E)$, and thus $|MC:C|=|MC\cap E: MC\cap E\cap C|$. Since $MC\cap E\leq E$ and E is a p-group, it follows that $|MC:C|$ is a power of p. Also, $|MC:C|=|M:M\cap C|=|M|$, so M is a p-group. Now, $M\leq O_p(H)=1$, so $M=1$. Then $EC=EM=E$, and we conclude that $C\leq E$, as desired. □

The following theorem encodes our key construction.

Theorem 3.2. *Let H be a CDG with a unique minimal normal subgroup N, and assume that N is a q-group for some prime q. Let p be a prime different from q so that $p-1$ is divisible by an odd prime r that is different from q. Then there exists an extra-special p-group E so that if $G = E \rtimes (H \times Z_r)$, then G is a CDG, $F(G) = E$, and $Z(E)$ is the unique minimal normal subgroup of G. In particular, if H is solvable, then the Fitting height of G is one more than the Fitting height of H.*

Proof. Let V be an H-module of characteristic p such that $C_V(N) = 0$. Note that $|V|$ and $|N|$ are coprime. Let $E = E(V)$. As we saw in the previous section, E is an extra-special p-group, and we define the action of H on E as in that section. Since r divides $p-1$, we see that k contains an element x whose multiplicative order is r. Thus, $\langle \sigma_x \rangle \cong Z_r$ where σ_x is defined as in the previous section, and using that section we can define an action of Z_r on E. Since r is odd, we see that the action of Z_r on E is Frobenius. We also saw that the action of H and σ_x on E commute; so in fact, we have an action of $H \times Z_r$ on E, and we take $G = E \rtimes (H \times Z_r)$ under this action.

We first prove that G is a CDG. Since H is a CDG, it follows that $G/E \cong H \times Z_r$ is a CDG. Thus, it suffices to show that the characters in $\mathrm{Irr}(G)$ that do not have E in their kernel do not have prime power degree. Since EZ_r is a Frobenius group, it follows that r divides the degree of every irreducible character of EZ_r whose kernel does not contain E, and this implies that r divides the degree of every irreducible character of G whose kernel does not contain E. Since E is an extra-special p-group, p will divide the degree of every irreducible character of G whose kernel does not contain E'. Thus, we need only consider those irreducible characters of G whose kernels do not contain E but do contain E'. Let χ be such a character of G, and let λ be an irreducible constituent of χ_E, and since E' is contained in the kernel of χ, we conclude that λ is linear.

Since $C_V(N) = 0$, we may use Lemma 2.2 to see that $C_{\hat{V}}(N) = 0$. Now, E/E' is the direct sum of two N-modules whose centralizers of N are trivial, so $C_{E/E'}(N) = 1$. It follows that $N_\lambda < N$, and applying Clifford's theorem, we have that $|N : N_\lambda|$ divides the degree of every irreducible constituent of λ^N, and hence, q divides $\chi(1)$. This proves that G is a CDG.

Observe that $H \times Z_r$ acts faithfully on E since otherwise the kernel of the action would contain N, and this leads to a contradiction since $C_V(N) = 0$. Also, $O_p(H \times Z_r) = 1$, so it follows by Lemma 3.1 that E contains $C_G(E)$. From this, we see that $F(G)$ is a p-group, and thus $F(G) = E$ since $O_p(H \times Z) = 1$. Also, since E contains $C_G(E)$, it is clear that E contains all minimal normal subgroups of G. □

In Theorem 3.2, the hypothesis that H has a unique minimal normal subgroup is stronger than we really need. One could weaken this hypothesis to require that $F(H)$ be a q-group. In the proof, we then choose V to be a module for H with the property that no irreducible $F(H)$-submodule of V is centralized by $F(H)$.

We also note in the proof of Theorem 3.1 that if V is chosen to be an irreducible, faithful module for H, then necessarily we have $C_V(N) = 0$ since $C_V(N)$ will be a proper H-submodule of V.

We now find CDGs with arbitrarily large Fitting heights by inductively applying Theorem 3.2. In particular, we are ready to prove Theorem 1.1. This next result includes Theorem 1.1.

Theorem 3.3. *There exists an infinite family of solvable CDGs $G_1, G_2, \ldots$ so that G_i has Fitting height $i+1$ and has a unique minimal normal subgroup. Furthermore, there exists an infinite family of solvable CDGs $G_1, G_2, \ldots$ so that each G_i satisfies the above conclusions and has a Sylow tower.*

PROOF. We prove the first conclusion by working via induction on i. We start by finding a solvable CDG with Fitting height 2. We could choose one of the examples in [1, Section 4], however, we can find an easy example. Let E be an extra-special group of order 7^3 and exponent 7. It is not difficult to see that E has an automorphism α of order 2 that inverts all the elements of $E/Z(E)$ and centralizes $Z(E)$. Using Lemma 2.1, we see that E has an automorphism β of order 3 whose action on E is Frobenius. Also, it is easy to see that α and β commute. We take $G_1 = E \rtimes \langle \alpha\beta \rangle$. It is easy to see that $\mathrm{cd}(G_1) = \{1, 6, 21\}$, so G_1 is a CDG, and $Z(E)$ is the unique minimal normal subgroup of G_1. Notice that G_1 has Fitting height 2 and $Z(E)$ is a 7-group. Also, G_1 will have a Sylow tower. This proves the base case. We now prove the inductive step. At the i-th step, we have the solvable CDG G_i which has Fitting height $i+1$ and a unique minimal normal subgroup. Since G_i is solvable, we know that this minimal normal subgroup will be a q_i-subgroup for some prime q_i. We can then find primes p_i and r_i that are different from q_i so that r_i is odd and r_i divides $p_i - 1$. We apply Theorem 3.2 using G_i, p_i, q_i, and r_i to obtain the CDG G_{i+1} with Fitting height $i+2$ and having a unique minimal normal subgroup. This proves the first conclusion.

To prove the second conclusion, we assume at each step that we choose the primes p_i and r_i so that they do not divide $|G_i|$. Now, G_{i+1} will have a normal Sylow p_i-subgroup P_i. Also, we see that $G_{i+1}/P_i \cong G_i \times Z_{r_i}$. In addition, Z_{r_i}

will be the normal Sylow r_i-subgroup of $G_i \times Z_{r_i}$ and $G_i \times Z_{r_i}/Z_{r_i} \cong G_i$ which inductively has a Sylow tower. Therefore, G_{i+1} has a Sylow tower. □

Next, we give an easy proof that every CDG has order divisible by three distinct primes.

Lemma 3.4. *If G is a CDG, then $|G|$ is divisible by at least three distinct primes.*

Proof. If G is not solvable, then this is immediate by Burnside's $p^a q^b$-theorem. Thus, we may assume G is solvable, and we let M be maximal so that G/M is nonabelian. We know by [2, Lemma 12.3] that either G/M is a p-group for some prime p or G/M is a Frobenius group. However, if G/M were a p-group, then it would have a prime power character degree other than 1. Thus, G/M is a Frobenius group with Frobenius complement N/M. We know that $|G : N|$ and $|N : M|$ are relatively prime and $|G : N|$ is a character degree, so $|G : N|$ is divisible by at least two distinct primes, and so, $|G : M|$ is divisible by three distinct primes. □

We now show that we can find a CDG whose order is divisible by only three distinct primes and has arbitrarily large Fitting height.

Theorem 3.5. *There exist three distinct primes p_1, p_2, and r, and an infinite family of solvable CDG's $G_1, G_2, \ldots$, so that G_i has Fitting height $i+1$ and is a $\{p_1, p_2, r\}$-group.*

Proof. Let r be an odd prime, let p_1 and p_2 be distinct primes so that r divides both $p_1 - 1$ and $p_2 - 1$. Many such triples of primes exist. One possibility is $(p_1, p_2, r) = (7, 13, 3)$. Let k be the field of order p_2, and let V be an irreducible $k[Z_{p_1}]$ module. By Lemma 2.1, we know that $Z_{p_1} \times Z_r$ acts via automorphisms on $E(V)$ so that Z_{p_1} centralizes $Z(E(V))$, and the action of Z_r on $E(V)$ is Frobenius. Take $G_1 = E(V) \rtimes (Z_{p_1} \times Z_r)$. It is not difficult to see that $\mathrm{cd}(G_1) = \{1, rp_1, r(p_2)^n\}$ where n is the dimension of V, G_1 is a $\{p_1, p_2, r\}$-group, and G_1 has a unique minimal normal subgroup that happens to be a p_2-group. This is the base case for induction. Continuing inductively, we will have G_i is a $\{p_1, p_2, r\}$-group with Fitting height $i+1$, and with a unique minimal normal subgroup that will be a p_1-group when i is even, and a p_2-group when i is odd. We will apply Theorem 3.2 with $p = p_2$ and $q = p_1$ when i is even, and $p = p_1$ and $q = p_2$ when i is odd, and $r = r$ for all i to obtain G_{i+1}. We see that G_{i+1} is also a $\{p_1, p_2, r\}$-group, it has Fitting height $i+2$ and a unique minimal normal subgroup that will be a p_1-subgroup when $i+1$ is even (i.e, i is odd), and a p_2-subgroup when $i+1$ is odd (i.e., i is even). This yields the desired result. □

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