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On distance functions induced by Finsler metrics

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Dedicated to Professor Lajos Tamássy on the occasion of his 94th birthday

Abstract. In this paper, we find the necessary and sufficient condition under which a distance function is induced by a Finsler metric. Then, we study some analytical properties of distance functions induced by Finsler metrics. Projectively flat Finsler metrics on a convex domain in $\mathbb{R}^n$ are regular solutions to Hilbert's Fourth Problem. We find necessary and sufficient condition for a Finsler metric to be projectively flat through its induced distance function.

1. Introduction

One of the important problems in Finsler geometry is to find interesting properties of the distance function induced by a Finsler metric. Let ϱ^F be the distance function induced by a Finsler metric F on a manifold M. Then (M, ϱ^F) is a metric space if F is absolutely homogeneous, and it is a quasi-metric space if F is homogeneous. Quasi-metric spaces often occur in the investigations of metrizability of topological spaces [9], [11], [12]. According to the Busemann–Mayer relation, there is a one-one relation between Finsler spaces (M, F) and (quasi-) metric spaces (M, ϱ^F).

In [14], Tamássy shows that an arbitrary distance function ϱ is not necessarily induced by a Finsler metric F, and then it needs not to be equal to ϱ^F. This means that the family of quasi-metric spaces $\{(M, \varrho)\}$ is wider than $\{(M, F)\}$. Then he gives a necessary and sufficient condition for a quasi-metric ϱ to be equal

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to ϱ^F, i.e., a (quasi-)metric space (M, ϱ) to be equivalent (with respect to the distance) with a Finsler space (M, F).

In [5], Burago–Burago–Ivanov study the mentioned problem in the case that one consider a Finsler metric as a length structure. Loosely speaking, their analytical approach concludes that "a metric structure coincides with a length structure (for example, Finsler metric) if one can travel between any two points in the metric space in a way that two sides of triangle inequality get arbitrary close enough. For more details, see [5, Theorem 2.4.16].

In this paper, we provide a simple criterion to derive significant properties of the quasi-metric induced by a Finsler metric (Theorem 3.2). It is based on a simple fact about geodesics: "a distance function increases at its highest possible rate, along a geodesic". First, we prove the mentioned fact (Proposition 3.1), and then provide our criterion by means of so-called "parallelism property", introduced in [14, Propositions 3.1 and 3.2]. This not only gives a better understanding on the relation between Finsler spaces and quasi-metric spaces, but also obtains a suitable geometrical point of view about the geodesics that have been stated before. Indeed, it provides a new presentation for a geodesic without any ODE (though still footmark of derivation is obvious). Also, we find a kind of duality results that can be applied if one needs to find Finsler metric associated with a distance function (see Subsection 3.1).

A Finsler metric is said to be locally projectively flat if at any point there is a local coordinate system in which the geodesics are straight lines as point sets. The origin of the problem of projectively flat Finsler metrics is formulated in Hilbert's Fourth Problem that asked to determine the metrics on an open subset in R^n, whose geodesics are straight lines [6]. Let (M, F) be an n-dimensional Finsler manifold, and ϱ^F be the distance induced by F. Suppose that $p, q \in M$ such that q has a length minimizing geodesic to p. In Section 4, we show that the rank of the matrix $\left[\varrho^F(y,x)\right]_{y^j x^i}$ is equal to $n-1$ (Lemma 4.1). Using this fact, we find a necessary and sufficient condition for the quasi-metric induced by a Finsler metric under which the Finsler metric is projectively flat (Theorem 4.2).

2. Preliminaries

Let $\varrho : A \times A \longrightarrow \mathbb{R}$ be a real function, where A is a nonempty set. Then ϱ is called quasi-metric if it satisfies the following conditions:

(i) $\varrho(a, b) \geq 0, \quad \forall a, b \in A$;

(ii) $\varrho(a, b) = 0$ if and only if $a = b$;

(iii) $\varrho(a,b)+\varrho(b,c)\geq\varrho(a,c), \qquad \forall a,b,c\in A.$

See [7], [11], [12] and [14].

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x\in M$, by $TM=\cup_{x\in M}T_xM$ the tangent bundle of M, and by $TM_0=TM\backslash\{0\}$ the slit tangent bundle. A function $F:TM\longrightarrow[0,+\infty)$ is called a Finsler metric if it satisfies the following conditions: (i) F is C^∞ on TM_0, and is continuous on TM; (ii) $F(x,\lambda y)=\lambda F(x,y)$, $\forall\lambda>0$; (iii) g_{ij} is a positive definite matrix, where

$$g_{ij}=\frac{1}{2}\frac{\partial^2F(x_0,y)^2}{\partial y^i\partial y^j},$$

and $y=y^i\frac{\partial}{\partial x^i}$ is an arbitrary chart containing x_0.

In [14], Tamássy introduces the notion of "parallelism property" for an arbitrary curve. He proves the following result which simplifies parallelism property, and we explicitly state it here for our further considerations.

Proposition 2.1. *Let $g:[a,b]\longrightarrow M$ be a C^∞ curve, and ϱ a quasi-metric on a manifold M. Then g satisfies parallelism property if and only if*

$$\frac{\partial I}{\partial t}(a,\tau)=\frac{\partial I}{\partial t}(s,\tau)=\frac{\partial I^+}{\partial t}(\tau,\tau)\quad(\forall\ a\leq s<\tau<b),$$

where

$$I(s,t):=\varrho\big(g(s),g(t)\big),\qquad \frac{\partial I^+}{\partial t}(\tau,\tau):=\lim_{t\to\tau^+}\frac{\partial I}{\partial t}(\tau,t).$$

Proof. See the definition of parallelism property in [14, page 489]. □

In [14], it is shown that a quasi-metric induced by a Finsler metric has the following properties:

(a) $\varrho(p_0,q)$ is continuous at $p_0=q$;

(b) $\varrho(p_0,q)$ is C^∞ on a slit open domain containing p_0;

(c) for every smooth curve c, the following limit exists:

$$F_\varrho\big(p_0,\dot c(0)\big)=\lim_{t\to0^+}\frac{d\varrho\big(p_0,c(t)\big)}{dt}, \tag{1}$$

where $c(0)=p_0$. By the nature of directional derivative, it is easy to see that the above definition is meaningful. Also $F_\varrho(p_0,y)$ is continuous at $y=0$, and $F_\varrho(p_0,y)$ is C^∞ when $y\neq0$.

(d) The function F_ϱ defined by (1) is a Finsler metric.

If ϱ does not satisfy one of the above conditions, then it is not obtained from a Finsler metric. In the rest of this article, we suppose that every quasi-metric satisfies the above properties.

3. Finslerian geodesics

Here, we state some results which are our special favourites because they construct our viewpoint to the problem of the equality of Finsler metrics and quasi-metrics. First, we give some new definitions.

Definition 3.1. Let $A \subseteq \mathbb{R}^n$ be a nonempty set. For an arbitrary positive real number c, let us define

$$K_c^M(A) := \Big\{\Lambda \in (\mathbb{R}^n)^* \mid \sup_{a\in A} \Lambda(a) \le c\Big\}, \quad I_c^M(A) := \Big\{\Lambda \in (\mathbb{R}^n)^* \mid \sup_{a\in A} \Lambda(a) = c\Big\}.$$

Since $\mathbb{R}^n = (\mathbb{R}^n)^*$, it follows that

$$K_c^M(A) = \Big\{x \in \mathbb{R}^n \mid \sup_{a\in A}(x\cdot a) \le c\Big\}, \quad I_c^M(A) = \Big\{x \in \mathbb{R}^n \mid \sup_{a\in A}(x\cdot a) = c\Big\},$$

where "$\cdot$" denotes the ordinary inner product on $\mathbb{R}^n$.

A supporting hyperplane of a set A in Euclidean space $\mathbb{R}^n$ is a hyperplane that has the following two properties: (i) A is entirely contained in one of the two closed half-spaces bounded by the hyperplane; (ii) A has at least one boundary-point on the hyperplane.

Let us put

$$\mathbb{R}^n_0 := \mathbb{R}^n - \{0\}, \quad (\mathbb{R}^n)^*_0 := (\mathbb{R}^n)^* - \{0\}, \quad (\mathbb{R}^n)^{**}_0 := (\mathbb{R}^n)^{**} - \{0\}.$$

Suppose that $\mathrm{co}(A)$ denotes the convex hull of the set of A. Then we have the following.

Lemma 3.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth map with $\nabla f \neq 0$ on $\mathbb{R}^n_0$, and $A = f^{-1}(b) \subseteq \mathbb{R}^n_0$ be a nonempty compact set. Suppose that $\mathrm{co}(A)$ is a strictly convex set containing 0 and $\partial\big(\mathrm{co}(A)\big) = A$. Then, for any $c > 0$, the following hold:*

(i) *For any functional $\Lambda \in (\mathbb{R}^n)^*_0$, there is a unique $x_0 \in \mathrm{co}(A)$ such that $\Lambda(x_0) = \sup_{a\in \mathrm{co}(A)} \Lambda(a) > 0$ and $x_0 \in A$.*

(ii) *For each $y \in A$, there is a unique supporting hyperplane of $\mathrm{co}(A)$ such as L which crosses y, and $\nabla f(y)$ is a normal vector of L.*

(iii) *If $\Lambda_1, \Lambda_2 \in (\mathbb{R}^n)^*_0$ are not collinear, then they take their maximum on $\mathrm{co}(A)$ at different points.*

(iv) *Let $\Lambda = \sum_{i=1}^r \lambda_i\Lambda_i$ be a convex combination of mutually different elements of $I_c^M\big(\mathrm{co}(A)\big)$ such that $\{\lambda_i\}_{1\le i\le r} \neq \{0, 1\}$. Then the following holds:*

$$\sup_{a\in \mathrm{co}(A)} \Lambda(a) < c.$$

(v) *$K_c^M\big(\mathrm{co}(A)\big)$ is a nonempty compact strictly convex set, and the following holds:*

$$\partial K_c^M\big(\mathrm{co}(A)\big) = I_c^M\big(\mathrm{co}(A)\big).$$

(vi) *Let $\Lambda \in (\mathbb{R}^n)^*$ be the unique linear functional, which satisfies the following:*

$$\ker(\Lambda) = \nabla f(y)^{\perp}, \quad \Lambda(y) = c,$$

for an $y \in A$. Then

$$\sup_{a\in \mathrm{co}(A)} \Lambda(a) = c.$$

(vii) *Let $a_1, a_2 \in \mathbb{R}_0^n = (\mathbb{R}^n)_0^{**}$ be not collinear. Then they take their maximum on $K_c^M\big(\mathrm{co}(A)\big)$ at different points of $I_c^M\big(\mathrm{co}(A)\big)$.*

(viii) *The following hold:*

$$K_c^M\big(\mathrm{co}(A)\big) = K_c^M\big(A\big), \quad I_c^M\big(\mathrm{co}(A)\big) = I_c^M\big(A\big).$$

Also, the following map is a bijection from $I_c^M\big(A\big)$ to A:

$$\begin{cases} \max : I_c^M\big(A\big) \longrightarrow A, \\ \Lambda \longmapsto z, \end{cases}$$

where Λ takes its maximum over $\mathrm{co}(A)$ at z, i.e., $\Lambda(z) = c$.

Proof. (i) Let Λ denote the set of all non-zero elements of $(\mathbb{R}^n)^*$. Suppose that $z \in \mathbb{R}^n$ is its dual (i.e., $\Lambda(a) = z \cdot a, \forall a \in \mathbb{R}^n$). Since $\partial\big(\mathrm{co}(A)\big) = A$, it follows that $\mathrm{co}(A) - A = \mathrm{int}\big(\mathrm{co}(A)\big)$ contains 0. This means that there is a positively scaled multiple of z in $\mathrm{co}(A)$ such that

$$\sup_{a\in \mathrm{co}(A)} \Lambda(a) \geq \Lambda(\lambda z) > 0, \quad \text{(for an } \lambda > 0).$$

Compactness of A implies that $\mathrm{co}(A)$ is a compact set. This fact guarantees the existence of x_0 with the following property

$$\Lambda(x_0) = \sup_{a\in \mathrm{co}(A)} \Lambda(a). \tag{2}$$

Now, we are going to prove the uniqueness of x_0. On the contrary, assume that x_1 is another point in $\mathrm{co}(A)$ that satisfies (2). By assumption, $\mathrm{co}(A)$ is a strictly convex set, and then there are elements in $\mathrm{int}\big(\mathrm{co}(A)\big)$ where Λ reaches its maximum value on $\mathrm{co}(A)$. Let x_2 be such an element. Obviously, for an $\epsilon > 0$,

$(1+\epsilon)x_2$ remains in $\mathrm{co}(A)$, which is not possible. This contradiction implies the uniqueness of x_0.

(ii) The supporting hyperplane theorem states that if S is a convex set in the topological vector space $X = \mathbb{R}^n$, and x_0 is a point on the boundary of S, then there exists a supporting hyperplane containing x_0 [4]. Since $\mathrm{co}(A)$ is a convex set, the mentioned theorem implies the existence of a supporting hyperplane of $\mathrm{co}(A)$ such as L which crosses y.

Let L' be an arbitrary supporting hyperplane of $\mathrm{co}(A)$ that crosses y. If V' is a normal vector of L', then we have

$$L' = \left\{ z \in \mathbb{R}^n \mid V' \cdot z = \alpha \right\}$$

for a real number α. Since L' is a supporting hyperplane of $\mathrm{co}(A)$, then, without loss of generality (i.e., replacing V' with $-V'$ if necessary) one can assume that the following holds:

$$V' \cdot z \leq \alpha, \quad \forall z \in \mathrm{co}(A).$$

Let $\gamma : (c, d) \to A$ be an arbitrary smooth curve on A such that $\gamma(t_0) = y$. Then, we get

$$V' \cdot \gamma(t) \leq \alpha \quad \text{and} \quad V' \cdot \gamma(t_0) = \alpha.$$

Let us put $\psi(t) := V' \cdot \gamma(t)$. Then $\psi'(t_0) = 0$. In this case, we have

$$V' \cdot \dot{\gamma}(t_0) = 0,$$

where $\dot{\gamma}(t_0)$ is an arbitrary vector in $T_y A$. By [3, Theorem 5.5], which is known as preimage theorem, A is a regular submanifold of $\mathbb{R}^n$ of dimension $n-1$. This means that $T_y A$ is a subspace of $T_y \mathbb{R}^n = \mathbb{R}^n$. It follows that

$$L' = y + V'^{\perp} = T_y A.$$

Since $A = f^{-1}(\text{constant})$ implies that $\nabla f(y)$ has the same property as V', it follows that

$$L' = y + \nabla f(y)^{\perp} = T_y A.$$

This shows that $\nabla f(y)$ is a normal vector of every supporting hyperplane of $\mathrm{co}(A)$ at y, and then $T_y A$ is the only possible supporting hyperplane of $\mathrm{co}(A)$ at y.

(iii) Let $\Lambda_1, \Lambda_2 \in (\mathbb{R}^n)^*_0$ be not collinear. We are going to show that they take their maximum on $\mathrm{co}(A)$ at different points. On the contrary, suppose that there is a $z \in \mathrm{co}(A)$ where both Λ_1 and Λ_2 take their maximum values on $\mathrm{co}(A)$. Let z_1 and z_2 be duals of Λ_1 and Λ_2, respectively. It is easy to see that both of z_1 and

z_2 cannot be collinear with $\nabla f(z)$. Without loss of generality, suppose that z_1 is not collinear with $\nabla f(z)$. Part (ii) shows that $z+\ker(\Lambda_1)$ is not a supporting hyperplane of $\mathrm{co}(A)$. Thus, $\mathrm{co}(A)$ has members in both sides of $z+\ker(\Lambda_1)$. This implies that there is an element in $\mathrm{co}(A)$ at which Λ_1 has greater value than $\Lambda_1(z)$. This is a contradiction. Then, we get the proof.

(iv) Let $\Lambda = \sum_{i=1}^{r} \lambda_i \Lambda_i$ be a convex combination of mutually different elements of $I_c^M\big(\mathrm{co}(A)\big)$ such that $\{\lambda_i\}_{1\leq i\leq r} \neq \{0,1\}$. We are going to show that $\sup_{a\in \mathrm{co}(A)} \Lambda(a) < c$ holds. On the contrary, suppose that Λ satisfies

$$\sup_{a\in \mathrm{co}(A)} \Lambda(a) \geq c.$$

Then, for a unique $z \in A$, we get

$$\sum_{i=1}^{r} \lambda_i \Lambda_i(z) = \Lambda(z) \geq c,$$

where $\Lambda_i \in I_c^M\big(\mathrm{co}(A)\big)$, $(1 \leq i \leq r)$. Then $\Lambda_i(z) \leq c$. This implies $\Lambda_i(z) = c$. However, Λ_i, $1 \leq i \leq r$ are different functionals in $I_c^M\big(\mathrm{co}(A)\big)$, not collinear and necessarily non-zero. These functionals take their maximums on $\mathrm{co}(A)$ at the same point, which is not possible.

(v) By the Cauchy–Schwartz inequality, the functional norms and Euclidean norms coincide on $\mathbb{R}^n$. More precisely, $\|\Lambda\| = \|z\|$, where z is the dual of Λ. Let $x \in \mathrm{co}(A)$ and $\Lambda_1,\ \Lambda_2 \in (\mathbb{R}^n)^*$. Then, we have

$$|\Lambda_1(x) - \Lambda_2(x)| \leq \|\Lambda_1 - \Lambda_2\| \times M,$$

where $M := \sup_{a\in \mathrm{co}(A)} \|a\|$. Without loss of generality, one can suppose that

$$\Lambda_1(z_1) = \sup_{a\in \mathrm{co}(A)} \Lambda_1(a) \geq \sup_{a\in \mathrm{co}(A)} \Lambda_2(a) = \Lambda_2(z_2), \quad z_1, z_2 \in \mathrm{co}(A).$$

It follows that

$$\begin{aligned}\big|\sup_{a\in \mathrm{co}(A)} \Lambda_1(a) - \sup_{a\in \mathrm{co}(A)} \Lambda_2(a)\big| &\leq \Lambda_1(z_1) - \Lambda_2(z_2) \leq \Lambda_1(z_1) - \Lambda_2(z_1)\\ &\leq |\Lambda_1(z_1) - \Lambda_2(z_1| \leq \|\Lambda_1 - \Lambda_2\| \times M.\end{aligned}$$

It shows that

$$\begin{cases}\zeta : (\mathbb{R}^n)^* \longrightarrow \mathbb{R}\\ \Lambda \longmapsto \sup_{a\in \mathrm{co}(A)} \Lambda(a)\end{cases}$$

is a continuous map. Now, consider $\Lambda \in K_c^M\big(\mathrm{co}(A)\big)$ such that $\Lambda \notin I_c^M\big(\mathrm{co}(A)\big)$. Then

$$\sup_{a\in \mathrm{co}(A)} \Lambda(a) < c.$$

Thus, for some $\sup_{a\in\mathrm{co}(A)} \Lambda(a) < \epsilon_0 < c$, we have $\Lambda \in \zeta^{-1}(0,\epsilon_0)$. This implies that Λ is an interior point of $K_c^M\big(\mathrm{co}(A)\big)$. Therefore,

$$\partial K_c^M\big(\mathrm{co}(A)\big) \subseteq I_c^M\big(\mathrm{co}(A)\big). \tag{3}$$

Now, suppose that Λ is an arbitrary element of $I_c^M\big(\mathrm{co}(A)\big)$. Every neighbourhood of Λ contains $(1+\epsilon)\Lambda$, for an $\epsilon > 0$. Obviously, the following holds:

$$\sup_{a\in\mathrm{co}(A)} (1+\epsilon)\Lambda(a) = (1+\epsilon)c. \tag{4}$$

By (4), we get $(1+\epsilon)\Lambda \notin K_c^M\big(\mathrm{co}(A)\big)$. Consequently, there is not any neighbourhood of Λ which lies in $K_c^M\big(\mathrm{co}(A)\big)$. Then

$$I_c^M\big(\mathrm{co}(A)\big) \subseteq \partial K_c^M\big(\mathrm{co}(A)\big). \tag{5}$$

By (3) and (5), we get $\partial K_c^M\big(\mathrm{co}(A)\big) = I_c^M\big(\mathrm{co}(A)\big)$.

$K_c^M\big(\mathrm{co}(A)\big) = \zeta^{-1}[0,c]$ is a closed set. Now, we are going to show that it is a bounded set. On the contrary, suppose that $K_c^M\big(\mathrm{co}(A)\big)$ is unbounded. By assumption, there is an $\epsilon_0 > 0$ neighbourhood of 0 such as $N_{\epsilon_0}(0)$ lies in $\mathrm{co}(A)$. Let $\Lambda \in K_c^M\big(\mathrm{co}(A)\big)$ satisfy following:

$$\|\Lambda\| > \frac{2c}{\epsilon_0}.$$

Obviously, there is a scaled version of the dual of Λ, namely z, such that $\|z\| = \epsilon_0/2$. Clearly, $z \in \mathrm{co}(A)$. Then we get

$$|\Lambda(z)| = \|\Lambda\|\ \|z\| > c,$$

which is not possible. Thus, $K_c^M\big(\mathrm{co}(A)\big)$ is a bounded set. Finally, we are going to prove that $K_c^M\big(\mathrm{co}(A)\big)$ is a strictly convex set. Let $\Lambda_1, \Lambda_2 \in K_c^M\big(\mathrm{co}(A)\big)$. If $\Lambda_1, \Lambda_2 \in I_c^M\big(\mathrm{co}(A)\big)$, then, by part (iv), we get the proof. Suppose that at least one of them is not a member of $I_c^M\big(\mathrm{co}(A)\big)$. By the similar method used in the proof of (iv), one can show that the following inequality holds:

$$\sup_{a\in\mathrm{co}(A)} \Big(\lambda\Lambda_1 + (1-\lambda)\Lambda_2\Big)(a) \le \max\Big\{ \sup_{a\in\mathrm{co}(A)} \Lambda_1(a), \sup_{a\in\mathrm{co}(A)} \Lambda_2(a)\Big\},$$

where $0 < \lambda < 1$. This means that

$$\sup_{a \in \mathrm{co}(A)} \Big(\lambda \Lambda_1 + (1-\lambda)\Lambda_2\Big)(a) < c,$$

and thus

$$\lambda \Lambda_1 + (1-\lambda)\Lambda_2 \in \mathrm{int}\Big(K_c^M\big(co(A)\big)\Big).$$

(vi) Let L be the unique supporting hyperplane of $\mathrm{co}(A)$ which crosses y, and $\nabla f(y)$ be its normal vector as proved in (ii). We want to prove that the following holds:

$$\sup_{a \in \mathrm{co}(A)} \Lambda(a) = c.$$

On the contrary, suppose that $\sup_{a \in \mathrm{co}(A)} \Lambda(a) \neq c$. Then we have

$$\sup_{a \in \mathrm{co}(A)} \Lambda(a) > c.$$

This implies that for some $z \in \mathrm{co}(A)$, $\Lambda(z) > c$ hold. Since $\Lambda(0) = 0$, $\Lambda(y) = c$ and $\Lambda(z) > c$, it follows that $y + L$ is not an affine supporting hyperplane. But this is a contradiction.

(vii) With the same argument used in the proof of (i), one can show that $a \in \mathbb{R}^n_0 = (\mathbb{R}^n)^{**}_0$ takes its maximum over $K_c^M\big(\mathrm{co}(A)\big)$ at a unique point in $I_c^M\big(\mathrm{co}(A)\big)$. First, we prove that if $y_0 \in A$ takes its maximum over $K_c^M\big(\mathrm{co}(A)\big)$ at Λ_0, then Λ_0 takes its maximum over $\mathrm{co}(A)$ at y_0. On the contrary, suppose that Λ_0 takes its maximum at y_1. Let $\overline{\Lambda}$ denote a functional defined in (vi) considering y_0, then $\overline{\Lambda}(y_0) = c$ and $\overline{\Lambda} \in K_c^M\big(\mathrm{co}(A)\big)$. Since y_0 takes it maximum over $K_c^M\big(\mathrm{co}(A)\big)$ at Λ_0, it follows that

$$\Lambda_0(y_0) \geq \overline{\Lambda}(y_0) = c.$$

Since $y_1 \neq y_0$, by (i) we get that $\Lambda_0(y_0) < \Lambda_0(y_1)$. This contradicts with $\Lambda_0 \in K_c^M\big(\mathrm{co}(A)\big)$. Thus, our claim holds. Now, let us back to (vii), and suppose that a_1 and a_2 are two non-zero elements of $\mathbb{R}^n$ which are not collinear. Note that every non-zero functional, with every positive rescaled multiple of itself, over $K_c^M\big(\mathrm{co}(A)\big)$ reaches its maximum at the same point. Then, without loss of generality, one can assume that $a_1, a_2 \in A$. We want to show that a_1 and a_2 reach their maximum at different points of $K_c^M\big(\mathrm{co}(A)\big)$. On the contrary, suppose that a_1 and a_2 take their maximum at $\Lambda \in I_c^M\big(\mathrm{co}(A)\big)$. On the other hand, Λ takes its maximum over $\mathrm{co}(A)$ at a_1 and a_2. This contradicts with (i), and we get the result.

(viii) $K_c^M\big(\mathrm{co}(A)\big) = K_c^M\big(A\big)$, and $I_c^M\big(\mathrm{co}(A)\big) = I_c^M\big(A\big)$ are simple consequences of (i), which means that every functional takes its maximum at boundary.

(i) shows that max is a well-defined function. (iii) proves that it is an injective one. By (vi), max is an onto function. This completes the proof. □

Now, we are going to consider some optimizing properties of geodesics with respect to their tangent spaces.

Proposition 3.1. *Let (M, F) be a Finsler manifold. Suppose that $\epsilon_0 > 0$, and the curve $c : [a, b+\epsilon_0] \longrightarrow M$ is the length minimizing geodesic from $p = c(a)$ to $c(b + \epsilon_0)$ and $q = c(b)$. Let V_q be tangent to c with the same direction at q, i.e., $V_q = \lambda\dot{c}$ for an $\lambda > 0$. Then, the following holds:*

$$V_q.\varrho^F(p, x) \geq V_q.\varrho^F(p', x), \quad \forall\, p' \in M, \tag{6}$$

where ϱ^F is the distance function induced by F.

Proof. In the case of $\dim(M) = 1$, it is easy to see that (6) holds. Then, suppose that $\dim(M) \geq 2$.

Without loss of generality, one can assume that $\lambda = 1$, which means $V_q = \dot{c}$. Then,

$$\left.\frac{d\varrho^F\big(p, c(t)\big)}{dt}\right|_{t=b} = V_q.\varrho^F(p, x), \qquad \left.\frac{d\varrho^F\big(p', c(t)\big)}{dt}\right|_{t=b} = V_q.\varrho^F(p', x),$$

and for an arbitrary $0 < \epsilon < \epsilon_0$, we have

$$\varrho^F\big(p, c(b+\epsilon)\big) = \varrho^F(p, q) + \varrho^F\big(q, c(b+\epsilon)\big).$$

By triangle inequality, for any $p' \in M$, the following holds:

$$\varrho^F\big(p', c(b+\epsilon)\big) \leq \varrho^F(p', q) + \varrho^F\big(q, c\big(b+\epsilon\big)\big).$$

Then,

$$\varrho^F\big(p, c(b+\epsilon)\big) - \varrho^F(p, q) \geq \varrho^F\big(p', c(b+\epsilon)\big) - \varrho^F(p', q). \tag{7}$$

By dividing (7) to ϵ and letting $\epsilon \longrightarrow 0$, one can obtain

$$\left.\frac{d\varrho^F\big(p, c(t)\big)}{dt}\right|_{t=b} \geq \left.\frac{d\varrho^F\big(p', c(t)\big)}{dt}\right|_{t=b}.$$

Now, it is easy to see that the restriction $\lambda = 1$ does not stop any progress in the process of the proof. This completes the proof. □

Proposition 3.2. *Let (M, F) be a Finsler manifold. Suppose that (x, U) is a chart of M containing $q \in M$, such that there is a unique length minimizing geodesic that connects each two distinct elements of it. Let us put*

$$\overrightarrow{S}_r(q) := \Big\{ p \in M \;\big|\; \varrho^F(p,q) = r \Big\} \subseteq U,$$

where $r > 0$, and let $\mathrm{Ind}_q(F)$ denote the indicatrix of F at q, which is given by

$$\mathrm{Ind}_q(F) := \big\{ V_q \in T_qM \;\big|\; F(q, V_q) = 1 \big\}.$$

Then, the following hold:

(i) *Let V_q, $W_q \in \mathrm{Ind}_q(F)$, $p \in \overrightarrow{S}_r(q)$ and V_q be a tangent vector to the geodesic passed from p to q. Then,*

$$V_q.\varrho^F(p,x) \geq W_q.\varrho^F(p,x). \tag{8}$$

(ii) *The following holds:*

$$I_1^M\big(\mathrm{Ind}_q(F)\big) = \Big\{ \; d\varrho^F(p,x)\big|_{x=q} \;\Big|\; p \in \overrightarrow{S}_r(q) \; \Big\}.$$

(iii) *Let $p \in \overrightarrow{S}_r(q)$, and p_0 be an arbitrary element of U crossed by the geodesic from p to q. Then*

$$d\varrho^F(p,x)\big|_{x=q} = d\varrho^F(p_0,x)\big|_{x=q}.$$

In particular, $\frac{\partial \varrho^F(p,x)}{\partial x^i}\big|_{x=q}$ is constant along the geodesic sufficiently close to q.

Proof. (i) Let $c : [a,b] \longrightarrow M$ be a geodesic passed from p to q, and parameterized by arc length. Thus $\dot{c}(b) = V_q$. We have

$$\frac{d\varrho^F\big(p, c(t)\big)}{dt}\Big|_{t=b} = 1 = V_q.\varrho^F(p,x) = d\varrho^F(p,x)\big|_{x=q}\big(V_q\big).$$

We are going to show that (8) holds. On the contrary, suppose that the following holds:

$$V_q.\varrho^F(p,x) < W_q.\varrho^F(p,x).$$

Consider $p_0 \in \overrightarrow{S}_r(q)$ such that W_q be a tangent vector to the geodesic passed from p_0 to q. Then, by Proposition 3.1, we get

$$W_q.\varrho^F(p_0,x) \geq W_q.\varrho^F(p',x), \quad \forall p' \in M.$$

Then, we have

$$W_q.\varrho^F(p_0, x) \geq W_q.\varrho^F(p, x).$$

This means that

$$1 = W_q.\varrho^F(p_0, x) \geq W_q.\varrho^F(p, x) > V_q.\varrho^F(p, x) = 1,$$

which is a contradiction. Thus (8) holds.

(ii) Let $V_q \in \mathrm{Ind}_q(F)$. As mentioned in (i), there is a $p \in \overrightarrow{S}_r(q)$ such that V_q is tangent to the geodesic passed from p to q at point q. Then, for any other elements of $\mathrm{Ind}_q(F)$ such as W_q, the following holds:

$$V_q.\varrho^F(p, x) \geq W_q.\varrho^F(p, x).$$

It is equal to the following:

$$d\varrho^F(p, x)\big|_{x=q}\big(V_q\big) \geq d\varrho^F(p, x)\big|_{x=q}\big(W_q\big).$$

It shows that $d\varrho^F(p, x)\big|_{x=q} = \max^{-1}(V_q)$, where max is the function defined in part (viii) of Lemma 3.1. Since max is a bijection and V_q is arbitrary, we get the proof.

(iii) Let us put

$$B^+(p', s) := \Big\{q' \in M \;\big|\; \varrho^F(p', q') \leq s\Big\} \subseteq U.$$

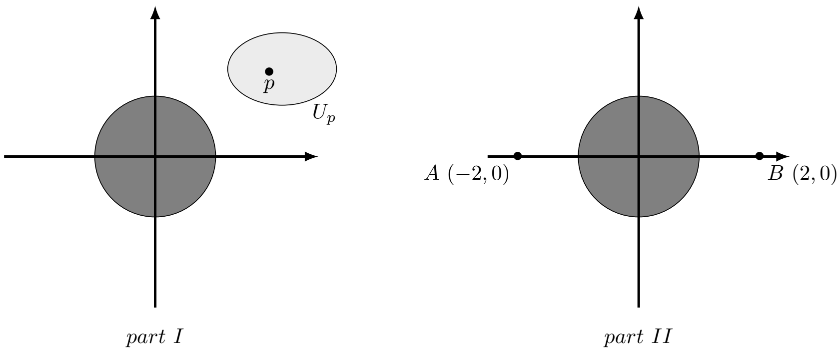
It is obvious that $B^+\big(p_0, \varrho(p_0, q)\big) \subseteq B^+\big(p, \varrho(p, q)\big)$, and q is in boundary of both sets. It follows that they have the same tangent hyper-plane at q, which is the kernel of both $d\varrho^F(p, x)\big|_{x=q}$ and $d\varrho^F(p_0, x)\big|_{x=q}$. Since $d\varrho^F(p, x)\big|_{x=q}$ and $d\varrho^F(p_0, x)\big|_{x=q}$ coincide on the tangent vectors of the geodesic from p to q at q, it follows that $d\varrho^F(p, x)\big|_{x=q} = d\varrho^F(p_0, x)\big|_{x=q}$. This completes the proof. $\square$

It is remarkable that, by the proof of part (ii) of the previous proposition, the following holds:

$$d\varrho^F(p, x)\big|_{x=q} = d\varrho^F(p_0, x)\big|_{x=q} = \max{}^{-1}(V_q). \tag{9}$$

We will use (9) in the proof of Proposition 4.1.

Remark 3.1. Let $M := \mathbb{R}^2 - \mathbf{D}$ be equipped with the Euclidean metric ϱ, where $\mathbf{D}$ is the unit disc. Suppose that F is the restriction of Riemannian metric of R^2 on M. For every point $p \in M$, there is a convex neighbourhood U_p such that $p \in U_p \subseteq M$. It is easy to see that $\varrho^F|_{U_p} = \varrho|_{U_p}$ (see Part I in the following figure). But $\varrho^F = \varrho$ does not hold, generally. If we suppose that $A = (-2, 0)$ and $B = (2, 0)$, then, obviously, $\varrho^F(A, B) \neq \varrho(A, B)$ (see Part II in the following figure).



The following result (Theorem 3.2) is the most relevant result together with [14, Theorem 3B]. Before stating it, we clarify our intention about "ϱ is obtained from F" (or "ϱ induced by F"). Consider a Finsler manifold (M, F) and a quasi-metric ϱ on M. We say that ϱ is obtained from (induced by) F if every point $q \in M$ has a neighbourhood U_q such that $\varrho^F|_{U_q} = \varrho|_{U_q}$. This point of view has at least two major advantages. First, we do not require the existence of a geodesic between any two points on the manifold. It is noted that this restrictive assumption was used by Tamássy in Theorem 3B. Second, in this case, the important relation (1) remains valid.

Let M be a smooth manifold and ϱ a quasi-metric on it. Suppose that p and q are two distinct points in M. Define

$$\overrightarrow{D}_{p,q} := \left\{ p' \in M - \{q\} \;\middle|\; d\varrho(p', x)\big|_{x=q} = d\varrho(p, x)\big|_{x=q} \right\}.$$

Then, we have the following.

Theorem 3.2. *Let ϱ be a quasi-metric on a manifold M. Then ϱ is induced by a Finsler metric $F := F_\varrho$ on M if and only if every element of M has a chart (x, U) with the following property: for every distinct $p, q \in U$ there is smooth curve $c : (a_0, b_0) \longrightarrow M$ with $c(a) = p$, $c(b) = q$ for some $a_0 < a < b < b_0$ such that*

$$\overrightarrow{D}_{c(t_0),c(t_1)} \cap U = c\big((a_0, t_1)\big), \qquad \forall\, t_0, t_1 :\ a_0 < t_0 < t_1 \leq b. \tag{10}$$

Proof. Suppose that ϱ is induced by a Finsler metric F. Then one can choose a collection Ω of charts of the manifold M such that, for every distinct pair of elements in their domains, there is a unique geodesic (of F) connecting them. Part (iii) of Proposition 3.2 shows that the above property holds for geodesics.

Conversely, suppose that there is a collection Ω consisting of charts of M which satisfies (10). Let $(x, U) \in \Omega$ and $p, q \in U$, such that $p \neq q$. Let $c : (a, b) \longrightarrow M$ be a smooth curve that satisfies $c(b) = q$, and

$$\overrightarrow{D}_{c(t_0),c(t_1)} \cap U = c\big((a, t_1)\big), \quad \forall\, t_0, t_1 :\ a_0 < t_0 < t_1 \leq b. \tag{11}$$

First, we prove that c satisfies parallelism property. By Proposition 2.1, it is sufficient to show that the following holds:

$$\begin{aligned} \frac{d\varrho\big(c(s), c(t)\big)}{dt}\Big|_{t=t_1} &= \frac{d\varrho\big(c(t_0), c(t)\big)}{dt}\Big|_{t=t_1} \\ &= \frac{d\varrho\big(c(t_1), c(t)\big)}{dt}\Big|_{t=t_1^+} \quad (a_0 < s < t_0 < t_1 \leq b). \end{aligned} \tag{12}$$

Let us rewrite the first equality of (12) as follows:

$$d\varrho\big(c(s), x\big)\big|_{x=c(t_1)}\Big(\dot{c}(t_1)\Big) = d\varrho\big(c(t_0), x\big)\big|_{x=c(t_1)}\Big(\dot{c}(t_1)\Big),$$

which is completely known by (11). Also, we have

$$\frac{d\varrho\big(c(t_1), c(t)\big)}{dt}\Big|_{t=t_1^+} = \lim_{t\to t_1^+} \frac{d\varrho\big(c(t_1), c(t)\big)}{dt} = \lim_{t\to t_1^+} d\varrho\big(c(t_1), x\big)\big|_{x=c(t)}\Big(\dot{c}(t)\Big).$$

Then, we get

$$\begin{aligned} \lim_{t\to t_1^+} d\varrho\big(c(t_1), x\big)\big|_{x=c(t)}\Big(\dot{c}(t)\Big) &= \lim_{t\to t_1^+} d\varrho\big(c(t_0), x\big)\big|_{x=c(t)}\Big(\dot{c}(t)\Big) \\ &= \lim_{t\to t_1^+} \frac{d\varrho\big(c(t_0), c(t)\big)}{dt} = \frac{d\varrho\big(c(t_0), c(t)\big)}{dt}\Big|_{t=t_1}. \end{aligned}$$

Thus c satisfies parallelism property. Now, we are going to show that $\varrho(p, q) = \varrho^F(p, q)$. By part (a) of Proposition 3 in [14], we get

$$\varrho(p, q) \leq \varrho^F(p, q).$$

Then, it is sufficient to prove that $\varrho(p, q) \geq \varrho^F(p, q)$ holds. It is easy to see that, as the following holds:

$$\varrho(p, q) = \varrho\big(c(a), c(b)\big) = \int_a^b \frac{d\varrho\big(c(a), c(t)\big)}{dt}\Big|_{t=s} ds.$$

On the other hand, by the parallelism property and definition of F, we have

$$\frac{d\varrho\big(c(a),c(t)\big)}{dt}\Big|_{t=s} = \frac{d\varrho\big(c(s),c(t)\big)}{dt}\Big|_{t=s^+} = F\big(c(s),\dot{c}(s)\big).$$

Therefore,

$$\varrho(p,q) = \int_a^b F\big(c(s),\dot{c}(s)\big)ds \geq \varrho^F(p,q).$$

This completes the proof. □

Now, we are going to find a special property of a distance function induced by a Finsler metric.

Proposition 3.3. *Let (M,F) be an n-dimensional Finsler manifold. Suppose that $\epsilon > 0$, and the curve $c : (a, b+\epsilon] \longrightarrow M$ is a length minimizing geodesic between $c(t)$ and $c(b+\epsilon)$, for each $t \in (a,b)$. Then, $\dot{c}(b)$ lies in the nullity of $\frac{\partial^2 \varrho(y,x)}{\partial y^j \partial x^i}\Big|_{x=c(b),\ y=c(t)}$, where $t \in (a,b)$. In other words, for every $t \in (a,b)$, the following holds:*

$$\sum_{i=1}^{n} \frac{dc^i(t)}{dt}\Big|_{t=b} \frac{\partial^2 \varrho(y,x)}{\partial y^j \partial x^i}\Big|_{x=c(b),\ y=c(t)} = 0, \quad (1 \leq j \leq n). \tag{13}$$

Proof. Let us consider arbitrary smooth curve $\gamma : (e,d) \longrightarrow M$ with $p = \gamma(t_0)$ $(e < t_0 < d)$. Let $V_q := \dot{c}(b)$. Clearly, by Proposition 3.1, the following map

$$\begin{cases} f : (e,d) \longrightarrow R \\ t \longmapsto V_q.\varrho^F\big(\gamma(t),x\big) \end{cases}$$

reaches its maximum at t_0, which implies that $f'(t_0) = 0$. Then, we have

$$\begin{aligned} f'(t) &= \frac{d}{dt}\left(\sum_{i=1}^{n}(V_q)^i \times \frac{\partial \varrho^F\big(\gamma(t),x\big)}{\partial x^i}\Big|_{x=q}\right) \\ &= \left(\sum_{i,j=1}^{n}(V_q)^i \times \frac{d\gamma^j(t)}{dt} \times \frac{\partial^2 \varrho^F\big(y,x\big)}{\partial y^j \partial x^i}\Big|_{x=q,y=\gamma(t)}\right). \end{aligned}$$

Therefore, we get

$$0 = \sum_{j=1}^{n}\left[\frac{d\gamma^j(t)}{dt}\Big|_{t=t_0} \times \left(\sum_{i=1}^{n}(V_q)^i \times \frac{\partial^2 \varrho^F\big(y,x\big)}{\partial y^j \partial x^i}\Big|_{x=q,y=\gamma(t_0)}\right)\right].$$

Since γ is an arbitrary smooth curve on M and $(V_q)^i = \frac{dc^i(t)}{dt}|_{t=b}$, we get (13). □

As an immediate and simple consequence of Proposition 3.3, one can obtain the following.

Corollary 3.1. *Let (M, F) be an n-dimensional Finsler manifold, and $c : [a, b) \longrightarrow M$ be a length minimizing geodesic from $p = c(a)$ to $c(t)$, for each $t \in (a, b)$. Then, $d\varrho\big(y, c(t)\big)\big|_{y=p}$ is constant along c. More precisely, $\frac{\partial \varrho\big(y, c(t)\big)}{\partial y^i}\Big|_{y=p}$, $(1 \leq i \leq n)$ are constants along the geodesic c.*

Proof. Let $V_p \in T_pM$ be an arbitrary tangent vector. Then, we have

$$\frac{d}{dt}\Big(d\varrho\big(y, c(t)\big)\big|_{y=p}\big(V_p\big)\Big) = \frac{d}{dt}\left(\sum_{i=1}^{n}(V_p)^i \frac{\partial \varrho\big(y, c(t)\big)}{\partial y^i}\Big|_{y=p}\right)$$
$$= \frac{d}{dt}\left(\sum_{i=1}^{n}(V_p)^i \left[\sum_{j=1}^{n} \frac{\partial^2 \varrho(y, x)}{\partial y^i \partial x^j}\Big|_{y=p, x=c(t)} \frac{dc^j(t)}{dt}\right]\right) = 0.$$

Since V_p is an arbitrary tangent vector, it follows that $d\varrho\big(y, c(t)\big)\big|_{y=p}$ is constant along the geodesic c. □

3.1. Dual results. In the previous section, we find some relations between length minimizing geodesics that passed from arbitrary point p in the manifold to other point such as q, and consider the map $p \mapsto d\varrho(p, x)|_{x=q}$. In this section, we are going to consider the map $q \mapsto d\varrho(y, q)|_{y=p}$ on the same geodesic and get some similar results. Indeed, this section is devoted to the dual version of the results obtained in the previous section. These results will be applied in the next section.

Definition 3.2. Let $A \subseteq \mathbb{R}^n$ be a nonempty set. Then, for an arbitrary negative real number c, let us define

$$K_c^m(A) := \big\{\Lambda \in (\mathbb{R}^n)^* | \inf_{a\in A} \Lambda(a) \geq c\big\}, \quad I_c^m(A) := \big\{\Lambda \in (\mathbb{R}^n)^* | \inf_{a\in A} \Lambda(a) = c\big\}.$$

By the same argument used in Lemma 3.1, and considering $K_{-c}^M(-A) = K_c^m(A)$, one can prove the following.

Lemma 3.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth map with $\nabla f \neq 0$ on $\mathbb{R}_0^n$, and $A = f^{-1}(b) \subseteq \mathbb{R}_0^n$ be a nonempty compact set. Suppose that $\mathrm{co}(A)$ is a strictly convex set containing 0 and $\partial\big(\mathrm{co}(A)\big) = A$. Then, for any $c < 0$, the following hold:*

(i) *For any functional $\Lambda \in (\mathbb{R}^n)_0^*$, there is a unique $x_0 \in \mathrm{co}(A)$ such that $\Lambda(x_0) = \inf_{a\in \mathrm{co}(A)} \Lambda(a) < 0$ and $x_0 \in A$.*

(ii) *For each $y \in A$, there is a unique supporting hyperplane of* $\mathrm{co}(A)$ *such as L, which crosses y, and $\nabla f(y)$ is a normal vector of L.*

(iii) *If $\Lambda_1,\ \Lambda_2 \in (\mathbb{R}^n)^*_0$ are not collinear, then they take their minimums on* $\mathrm{co}(A)$ *at different points.*

(iv) *Let $\Lambda = \sum_{i=1}^{r} \lambda_i \Lambda_i$ be a convex combination of mutually different elements of $I_c^m\big(\mathrm{co}(A)\big)$ such that $\{\lambda_i\}_{1\leq i\leq r} \neq \{0,1\}$. Then, the following holds:*

$$\inf_{a\in \mathrm{co}(A)} \Lambda(a) > c.$$

(v) *$K_c^m\big(\mathrm{co}(A)\big)$ is a nonempty compact strictly convex set, and the following holds:*

$$\partial K_c^m\big(\mathrm{co}(A)\big) = I_c^m\big(\mathrm{co}(A)\big).$$

(vi) *Let $\Lambda \in (\mathbb{R}^n)^*$ be the unique linear functional that satisfies*

$$\ker(\Lambda) = \nabla f(y)^{\perp}, \quad \Lambda(y) = c,$$

for a $y \in A$. Then the following holds:

$$\inf_{a\in \mathrm{co}(A)} \Lambda(a) = c.$$

(vii) *Let that $a_1, a_2 \in \mathbb{R}^n_0 = (\mathbb{R}^n)^{**}_0$ are not collinear. Then, they take their minimum on $K_c^m\big(\mathrm{co}(A)\big)$ at different points of $I_c^m\big(\mathrm{co}(A)\big)$.*

(viii) *The following holds:*

$$K_c^m\big(\mathrm{co}(A)\big) = K_c^m\big(A\big), \quad I_c^m\big(\mathrm{co}(A)\big) = I_c^m\big(A\big),$$

and also the following map is a bijection from $I_c^m\big(A\big)$ to A:

$$\begin{cases} \min : I_c^m\big(A\big) \longrightarrow, A \\ \Lambda \longmapsto z, \end{cases}$$

where Λ takes its minimum over $\mathrm{co}(A)$ *at z, i.e., $\Lambda(z) = c$.*

By the same argument used in the proof of Proposition 3.1, we get the following.

Proposition 3.4. *Let (M, F) be a Finsler manifold and $\epsilon > 0$, also let $c : [a-\epsilon, b] \longrightarrow M$ be the length minimizing geodesic from $c(a-\epsilon)$ to $q = c(b)$ and $p = c(a)$. Suppose that $V_p \in T_pM_0$ is tangent to c with same direction at p (i.e., $V_p = \lambda\dot{c}$, for an $\lambda > 0$). Then, the following holds*

$$V_p.\varrho^F(y, q) \leq V_p.\varrho^F(y, q'), \quad \forall\ q' \in M,$$

where ϱ^F is distance function obtained from F.

By the same method used in the proof of Proposition 3.2, one can obtain the following.

Proposition 3.5. *Let (M, F) be a Finsler manifold. Suppose that (x, U) is a chart of M containing $p \in M$, such that for each two distinct elements of U, there is a unique length minimizing geodesic connecting them. Let us put*

$$\overleftarrow{S}_r(p) := \Big\{ q \in M \mid \varrho^F(p, q) = r \Big\} \subseteq U.$$

Then the following hold:

(i) *Let $V_p, W_p \in \mathrm{Ind}_p(F)$ and $q \in \overleftarrow{S}_r(p)$. Suppose that V_p is tangent to the geodesic passed from p to q. Then,*

$$V_p.\varrho^F(y, q) \leq W_p.\varrho^F(y, q).$$

(ii) *The following holds:*

$$I^m_{-1}\big(\mathrm{Ind}_p(F)\big) = \Big\{ d\varrho^F(y, q)\big|_{y=p} \Big| q \in \overleftarrow{S}_r(p) \Big\}.$$

(iii) *Let $q \in \overleftarrow{S}_r(p)$ and $q_0 \in U$ such that the geodesic passed from p to q crosses q_0. Then,*

$$d\varrho^F(y, q)\big|_{y=p} = d\varrho^F(y, q_0)\big|_{y=q_0}.$$

In particular, $\frac{\partial \varrho^F(y,q)}{\partial y^i}\big|_{y=p}$ is constant along a geodesic sufficiently close to p.

Note that the definition of $\overrightarrow{D}_{p,q}$ and Theorem 3.2 have their respective duals, also. But we do not need those for our further studies. To prove the dual of Theorem 3.2, one should rewrite the whole of [14] with "reverse parallelism property". Here, we ignore to prove it.

Proposition 3.3 has its dual as follows.

Proposition 3.6. *Let (M, F) be an n-dimensional Finsler manifold. Suppose that $\epsilon > 0$, and the curve $c : [a - \epsilon, b) \longrightarrow M$ is a length minimizing geodesic between $c(a)$ and $c(t)$, for each $t \in (a, b)$. Then $\dot{c}(a)$ lies in the nullity of $\dfrac{\partial^2 \varrho^F(y, x)}{\partial y^j \partial x^i}\Big|_{y=c(a),\ x=c(t)}$, where $t \in (a, b)$. In other words, for every $t \in (a, b)$, the following holds:*

$$\sum_{j=1}^n \frac{dc^j(t)}{dt}\Big|_{t=a} \frac{\partial^2 \varrho^F(y, x)}{\partial y^j \partial x^i}\Big|_{y=c(a), x=c(t)} = 0, \quad (1 \leq i \leq n).$$

By Proposition 3.6, one can conclude the following.

Corollary 3.2. *Let (M,F) be an n-dimensional Finsler manifold. Suppose that $c:(a,b]\longrightarrow M$ is a length minimizing geodesic between $c(t)$ and $q=c(b)$, for each $t\in(a,b)$. Then, $d\varrho^F\big(c(t),x\big)\big|_{x=q}$ is constant along c. In other words, $\left.\dfrac{\partial\varrho^F\big(c(t),x\big)}{\partial x^i}\right|_{x=q}$, $(1\le i\le n)$, are constant along the geodesic c.*

Proposition 3.7. *Let $A:=f^{-1}(b)\subseteq R_0^n$ be a nonempty compact set, where $f:R^n\to R$ is a smooth map with $\nabla f\neq 0$ on R_0^n. Suppose that $\operatorname{co}(A)$ is a strictly convex set containing 0 and $\partial\big(\operatorname{co}(A)\big)=A$. Then, for every $c>0$, the following hold:*

(i) *If $y\in A$, then $\max^{-1}(y)=-\min^{-1}(y)$ holds, where $\max$ and $\min$ are as defined in part (viii) of Lemmas 3.1 and 3.3, respectively.*

(ii) *$I_c^M(A)=-I_{-c}^m(A)$ holds.*

Proof. (i) Obviously, the functional defined in part (vi) of Lemma 3.1 is equal to $\max^{-1}(y)$. It is clear that the function defined in part (vi) of Proposition 3.3 is equal to $-\max^{-1}(y)$ and also $\min^{-1}(y)$.

(ii) It is a simple consequence of part (i). □

By Proposition 3.7, we conclude the following.

Corollary 3.3. *Let (M,F) be a Finsler manifold. Suppose that $c:(a,b)\longrightarrow M$ is a length minimizing geodesic from $c(t_0)$ to $c(t_1)$, for each $a<t_0<t_1<b$. Then, for each $t\in(a,b)$, the following holds:*

$$d\varrho^F\big(c(t_0),x\big)\big|_{x=c(t)}=-d\varrho^F\big(y,c(t_1)\big)\big|_{y=c(t)},\quad(a<t_0<t<t_1<b).$$

Proof. By Propositions 3.2 and 3.5, for a small enough $r>0$, the following hold:

$$I_1^M\big(\operatorname{Ind}_{c(t)}(F)\big)=\left\{d\varrho^F(p,x)\big|_{x=q}\,\middle|\,p\in\overrightarrow{S}_r\big(c(t)\big)\right\},$$
$$I_{-1}^m\big(\operatorname{Ind}_{c(t)}(F)\big)=\left\{d\varrho^F(y,q)\big|_{y=p}\,\middle|\,q\in\overleftarrow{S}_r\big(c(t)\big)\right\}.$$

Let p and q belong to the range of c such that $\varrho\big(p,c(t)\big)=\varrho\big(c(t),q\big)$. Then, by Corollaries 3.1 and 3.2, it follows that

$$\begin{aligned}d\varrho^F\big(c(t_0),x\big)\big|_{x=c(t)}&=d\varrho^F\big(p,x\big)\big|_{x=c(t)}=\max^{-1}\big(\dot c(t)\big)\\&=-\min^{-1}\big(\dot c(t)\big)=-d\varrho^F\big(y,q\big)\big|_{y=c(t)}=-d\varrho^F\big(y,c(t_1)\big)\big|_{y=c(t)}.\end{aligned}$$

Thus we get the proof. □

4. Projectively flat Finsler metrics

Before mentioning our final result about the projectively flat Finsler metrics (i.e., Theorem 4.2), we should explain that all results before Theorem 3.2 that connect quasi-metrics and Finsler metrics involve the existence of special curves between any two points of the manifold. In fact, these curves are nothing other than geodesics. This situation has been changed in Theorem 3.2, from two aspects. First, Theorem 3.2 presents the geodesic. Second, it presents geodesics not as the solutions of special differential equations but as preimage of some functions which is fairly computable. For example, consider the Euclidean distance ϱ on $\mathbb{R}^n$. Then, we have

$$\varrho(y,x) = \left(\sum_{i=1}^{n}(y^i - x^i)^2\right)^{\frac{1}{2}}, \quad \frac{\partial\varrho(y,x)}{\partial x^i} = \frac{x^i - y^i}{\varrho(y,x)}.$$

Now, we are going to find the length minimizing geodesic γ from $p \in \mathbb{R}^n$ to $q \in \mathbb{R}^n - \{p\}$. By Theorem 3.2 and Corollary 3.1, if z be a point of γ, then the following holds:

$$d\varrho(z,x)|_{x=q} = d\varrho(p,x)|_{x=q}.$$

Thus, we have

$$\frac{q^i - z^i}{\varrho(z,q)} = \frac{\partial\varrho(z,x)}{\partial x^i}\Big|_{x=q} = \frac{\partial\varrho(p,x)}{\partial x^i}\Big|_{x=q} = \frac{q^i - p^i}{\varrho(p,q)}, \quad (1 \le i \le n).$$

It follows that

$$z^i = q^i - \varrho(z,q)\left[\frac{q^i - p^i}{\varrho(p,q)}\right], \quad (1 \le i \le n),$$

which obviously means that z lies on the straight line passed both of p and q. Here we find geodesics without solving any ODE.

Our result has some applications. For example, suppose that a distance function is given and one needs to find the Finsler metric associated with it. One can use the Busemann–Mayer relation

$$F\big(c(t_0),\dot{c}(t_0)\big) = \lim_{t\to t_0^+}\frac{d}{dt}\varrho\big(c(t_0),c(t)\big),$$

where $c : (a,b) \longrightarrow M$ is an arbitrary smooth curve. But in this case, there are no choices other than to calculate a derivation and get a limit at an obstacle point on a curve (i.e., ϱ is not differentiable at points of diameter of $M \times M$ unless it is

Riemannian). It deserves that one finds that curve by itself, also. Alternatively, one can calculate $I_1^M\big(\operatorname{Ind}_p(F)\big)$ just by a derivation, and find $\operatorname{Ind}_p(F)$ by the same method used in the proof of part (vi) in Lemma 3.1.

We have studied $I_1^M\big(\operatorname{Ind}_p(F)\big)$ and its properties so far. Now, we are going to consider $I_1^M\big(\operatorname{Ind}_p(F)\big)$ throughout the Finsler metric F. First, we prove the following.

Proposition 4.1. *Let (M, F) be an n-dimensional Finsler manifold, and ϱ^F its associated distance function. Suppose that $q \in M$ is in chart (x, U) and $\{dx^i\}$ are duals of $\{\frac{\partial}{\partial x^i}\}$. Then, the following holds*

$$I_1^M\big(\operatorname{Ind}_q(F)\big) = \Big\{F_i(q, y)dx^i \big| y \in T_qM, F(q, y) = 1\Big\}. \tag{14}$$

In particular, if for some $\epsilon > 0$, $c : [a, b + \epsilon] \to M$ is a length minimizing geodesic parameterized by arc length from $p = c(a)$ to $c(b + \epsilon)$, then the following holds:

$$F_i\big(q, \dot{c}(b)\big) = \frac{\partial \varrho^F(p, x)}{\partial x^i}\Big|_{x=q}, \tag{15}$$

where $q = c(b)$.

Proof. Since c is parameterized by arc length, we have $\dot{c}(b) \in \operatorname{Ind}_q(F)$. We are going to show that $\Lambda := F_i\big(q, \dot{c}(b)\big)dx^i$ is equal with $\max^{-1}\big(\dot{c}(b)\big)$, where max is the function defined in part (viii) of Lemma 3.1. Let $w \in T_qM_0$. Then, by fundamental inequality (see [1, relation 1.2.3]), we have

$$\Lambda(w) = \Lambda\left(w^i \frac{\partial}{\partial x^i}\right) = F_i\big(q, \dot{c}(b)\big)w^i \leq F\big(q, w(b)\big).$$

If we suppose that $w \in \operatorname{Ind}_q(F)$, then

$$\Lambda(w) \leq 1,$$

and $\Lambda(w) = 1$ if and only if $w = \dot{c}(b)$. This shows that $\Lambda = \max^{-1}\big(\dot{c}(b)\big)$, in particular, $\Lambda \in I_1^M\big(\operatorname{Ind}_q(F)\big)$. Now, by the same method used in part (ii) of Proposition 3.2, we get (14).

By (9) and Corollary 3.1, we get the following:

$$F_i\big(q, \dot{c}(b)\big)dx^i = \Lambda = \max{}^{-1}\big(\dot{c}(b)\big) = d\varrho(p, x)|_{x=q} = \frac{\partial \varrho^F(p, x)}{\partial x^i}\Big|_{x=q} dx^i,$$

which implies (15). $\square$

The problem of projectively flat Finsler metrics is quite old in geometry, and its origin is formulated in Hilbert's Fourth Problem: "determine the metrics on an open subset in R^n, whose geodesics are straight lines" [6]. Projectively flat Finsler metrics on a convex domain in R^n are regular solutions to Hilbert's Fourth Problem. Indeed, regular distance functions with straight geodesics are projectively flat Finsler metrics. They are characterized by a system of ODE, see [2], [8], [16].

In this section, we are going to state a result related to projectively flat Finsler metrics. Let (M, F) be an n-dimensional Finsler manifold, and ϱ^F the distance arisen from F. In Proposition 3.3, it is shown that rank of $\frac{\partial^2 \varrho^F(y,x)}{\partial y^j \partial x^i}$ is at most $n-1$. Before proving our last important result about the projectively flat metrics (i.e., Theorem 4.2), we improve Proposition 3.3, and show that the mentioned rank must be exactly $n-1$.

Lemma 4.1. *Let (M, F) be an n-dimensional Finsler manifold, and ϱ^F the distance arisen from F. Suppose that $p, q \in M$ such that q has a length minimizing geodesic to every element inside a neighbourhood of p. Then the following holds:*

$$\operatorname{rank}\left[\frac{\partial^2 \varrho^F(y,x)}{\partial y^j \partial x^i}\Big|_{y=p,x=q}\right] = n-1. \tag{16}$$

Proof. Consider $F_q : T_qM \longrightarrow R$. Since $F_i(q,y)dy^i \neq 0$ holds for every $y \neq 0$, by the preimage theorem [3, Theorem 5.5], it follows that $\mathrm{Ind}_q(F)$ is a regular submanifold of T_qM.

Define

$$\begin{cases} \psi : T_qM \to T_q^*M, \\ y \mapsto dF(q,y) = F_i(q,y)dx^i. \end{cases}$$

Then, we have $\psi|_{\mathrm{Ind}_q(F)} = \max^{-1}$, where max is the function defined in part (viii) of Lemma 3.1. Let $D\psi$ denote the differential map of ψ. Then, we get

$$D\psi\left(\frac{\partial}{\partial y^i}\right) = \frac{\partial \psi^j}{\partial y^i}\frac{\partial}{\partial \tilde{y}^j} = \sum_{j=1}^{n} F_{ij}(q,y)\frac{\partial}{\partial \tilde{y}^j},$$

where $\{\frac{\partial}{\partial y^i}\}$ and $\{\frac{\partial}{\partial \tilde{y}^j}\}$ are basis for T_qM at y and T_q^*M at $\psi(y)$, respectively. Since $\mathrm{rank}(F_{ij}) = n-1$, it follows that $\psi|_{\mathrm{Ind}_q(F)} = \max^{-1}$ is a diffeomorphism from $\mathrm{Ind}_q(M)$ to $I_1^M\big(\mathrm{Ind}_q(M)\big)$.

Now, let us consider the following map:

$$\begin{cases} f : M - \{q\} \to T_q^*M, \\ p' \mapsto -d\varrho^F(x,p')|_{x=q} = -\left.\frac{\partial \varrho^F(x,p')}{\partial x^i}\right|_{x=q} dx^i. \end{cases}$$

Existence of length minimizing geodesic from q to p means that for a $y_p \in T_q M_0$, $\exp(y_p) := \exp_{y_p}(1) = p$ holds, where exp denotes the exponential map at q. Then, by the inverse mapping theorem, exp is invertible on a neighbourhood of y_p. Therefore, on a neighbourhood of p, namely U_p, we have $f|_{U_p} = \psi \circ \exp^{-1}$. Then, we get

$$D_p f = D_{y_p}\psi \circ D_p \exp^{-1}.$$

By the equation (5.3.6) in [1], we have $\operatorname{rank}(D\exp) = n$. This means that

$$\operatorname{rank}\Big(D_p f\Big) = \operatorname{rank}\Big(D_p\big(\psi\ o\ \exp^{-1}\big)\Big) = n-1.$$

Let (x, U_1) and (z, U_2) be two charts containing q and p, respectively. Then the following holds:

$$D_p f\left(\frac{\partial}{\partial z^i}\right) = \frac{\partial f^j}{\partial z^i}\frac{\partial}{\partial \tilde{y}^j} = -\sum_{j=1}^{n} \frac{\partial^2 \varrho^F(x,z)}{\partial x^j \partial z^i}\Big|_{z=p,x=q}\frac{\partial}{\partial \tilde{y}^j},$$

where $\{\frac{\partial}{\partial \tilde{y}^j}\}$ is a basis for $T_q^* M$ at $f(p) = -d\varrho^F(x,p)|_{x=q}$. This implies (16). □

Now, we are ready to prove our final result on the projectively flat Finsler metrics. More precisely, we show the following.

Theorem 4.2. *Let F be a Finsler metric on a non-empty open convex neighbourhood $U \subset \mathbb{R}^n$. Then, the length minimizing geodesic between every two distinct points $p, q \in U$ is a straight line if and only if the following holds:*

$$\ker(A_{pq}) = \ker(A^t_{pq}),$$

where

$$A_{pq} := \left[\frac{\partial^2 \varrho^F(y,x)}{\partial y^j \partial x^i}\Big|_{y=p,\ x=q}\right]$$

and A^t denotes the transpose of A.

Proof. Suppose that for every two distinct points in U the length minimizing geodesic between them is a straight line. Suppose that $c : (a,b) \to U$ is a length minimizing geodesic parameterized by arc length between two points $p = c(t_0)$ and $q = c(t_1)$ $(a < t_0 < t_1 < b)$. By Propositions 3.3 and 3.6, we have $\dot{c}(t_1) \in \ker(A_{pq})$ and $\dot{c}(t_0) \in \ker(A^t_{pq})$. Since c is a straight line, it follows that $\dot{c}(t_1)||\dot{c}(t_0)$. On the other hand, since $c = c(s)$ is parameterized by arc length, $\dot{c}(t_1) \neq 0$ and $\dot{c}(t_0) \neq 0$. Thus, by considering $\operatorname{rank}(A_{pq}) = n-1$, we get

$$\ker(A_{pq}) = \operatorname{span}\big\{\dot{c}(t_1)\big\} = \operatorname{span}\big\{\dot{c}(t_0)\big\} = \ker(A^t_{pq}).$$

Conversely, suppose that for any two distinct points $p, q \in U$, $\ker(A_{pq}) = \ker(A^t_{pq})$ holds. Let $c : (a, b) \to U$ be a length minimizing geodesic between two points $p = c(t_0)$ and $q = c(t_1)$ $(a < t_0 < t_1 < b)$, and that c is parameterized by arc length. By Propositions 3.3 and 3.6, we have

$$\dot{c}(t_1) \in \ker(A_{pq}) \quad \text{and} \quad \dot{c}(t_0) \in \ker(A^t_{pq}),$$

and since $\operatorname{rank}(A_{pq}) = n - 1$, it follows that $\dot{c}(t_1) || \dot{c}(t_0)$. This shows that c is a curve between two points such that its velocity vector has a unique direction at any point. Obviously, c is a straight line. This completes the proof. □

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