Publ. Math. Debrecen
90/3-4 (2017), 359–371
DOI: 10.5486/PMD.2017.7518

Hilbert matrix operator on Besov spaces

By MIROLJUB JEVTIĆ (Belgrade) and BOBAN KARAPETROVIĆ (Belgrade)

Abstract. We show that if $0 < p \leqslant \infty$, $1 < q \leqslant \infty$, then the Besov spaces $H^{p,q,1}_{1+1/p}$ are not mapped by the Hilbert matrix operator H into the Bloch space $\mathcal{B}$. As a corollary, we have that the space $VMOA$ is also not mapped by H into the Bloch space $\mathcal{B}$. In [7], it is shown that if a function $f(z) = \sum_{k=0}^{\infty} \widehat{f}(k)z^k$, holomorphic in the unit disc, belongs to the logarithmically weighted Bergman space $A^2_{\log^\alpha}$, $\alpha > 2$, then $\sum_{k=0}^{\infty} \frac{|\widehat{f}(k)|}{k+1} < \infty$. We show that this implication holds only when $\alpha > 1$. In [7], it is also shown that if $\alpha > 3$, then H maps $A^2_{\log^\alpha}$ into the Bergman space A^2. We improve this result by proving that H maps $A^2_{\log^\alpha}$ into A^2 when $\alpha > 2$.

1. Introduction

The Hilbert matrix is an infinite matrix H whose entries are $a_{n,k} = \frac{1}{n+k+1}$. It can be viewed as an operator on spaces of holomorphic functions by its action on their Taylor coefficients. If

$$f(z) = \sum_{n=0}^{\infty} \widehat{f}(n)z^n$$

Mathematics Subject Classification: 47B37, 30H10, 30H20, 30H25, 30H30.
Key words and phrases: Hilbert matrix, Hardy spaces, Bergman spaces, Bloch spaces and Besov spaces.
The first author is supported by NTR Serbia, Project ON174017.
The second author is supported by NTR Serbia, Project ON174032.

is a holomorphic function in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, then we define a transformation H by

$$Hf(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\widehat{f}(k)}{n+k+1} z^n.$$

Let $\mathcal{H}(\mathbb{D})$ be the algebra of holomorphic functions in $\mathbb{D}$. For $0 < p \leqslant \infty$, Hardy space H^p is the space of all holomorphic functions $f \in \mathcal{H}(\mathbb{D})$ for which

$$||f||_{H^p} = ||f||_p = \sup_{0\leqslant r<1} M_p(r,f) < \infty,$$

where

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{it})\right|^p dt\right)^{\frac{1}{p}}, \quad 0 < p < \infty;$$

$$M_\infty(r,f) = \sup_{0\leqslant t<2\pi} \left|f(re^{it})\right|.$$

The subspace of H^∞ consisting of the functions which are also continuous on $\overline{\mathbb{D}}$, equipped with the same supremum norm, is called the disc algebra. We denote it by $\mathcal{A}$.

Recall that the space $BMOA$ consists of the functions $f \in H^1$ whose boundary values $f(e^{it})$ are of bounded mean oscillation on $\mathbb{T} = \partial\mathbb{D}$, that is,

$$\sup_I \frac{1}{|I|}\int_I \left|f(e^{it}) - f_I\right| dt < \infty,$$

where supremum is taken over all intervals $I \subset \mathbb{T}$ and

$$f_I = \frac{1}{|I|}\int_I f(e^{it})dt.$$

If

$$\lim_{|I|\to 0} \frac{1}{|I|}\int_I \left|f(e^{it}) - f_I\right| dt = 0,$$

then we say that $f \in VMOA$. Here, as usual, $|I|$ is arc length measure of the interval $I \subset \mathbb{T}$.

A function $f \in \mathcal{H}(\mathbb{D})$ is said to belong to the mixed norm space $H^{p,q,\alpha}$, $0 < p, q \leqslant \infty$, $0 < \alpha < \infty$, if

$$||f||_{H^{p,q,\alpha}} = ||f||_{p,q,\alpha} = \left(\int_0^1 M_p^q(r,f)(1-r)^{q\alpha-1}dr\right)^{\frac{1}{q}} < \infty, \quad 0 < q < \infty,$$

$$||f||_{H^{p,\infty,\alpha}} = ||f||_{p,\infty,\alpha} = \sup_{0\leqslant r<1} (1-r)^\alpha M_p(r,f) < \infty.$$

The Lebesgue measure on $\mathbb{D}$ will be denoted by A, and will be normalized so as to have $A(\mathbb{D}) = 1$, that is,

$$dA(z) = \frac{1}{\pi} dx dy = \frac{1}{\pi} r dr dt, \qquad z = x + iy = re^{it}.$$

The Bergman space A^2 is the space of holomorphic functions in $L^2(\mathbb{D}, dA)$, that is,

$$A^2 = \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{A^2}^2 = \int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty \right\}.$$

For $t \in \mathbb{R}$ we write D^t for the sequence $\{(n+1)^t\}$, for all $n \geqslant 0$. If $\lambda = \{\lambda_n\}_{n=0}^{\infty}$ is a sequence and X is a sequence space (by identifying the holomorphic function $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$ with the sequence $\{\widehat{f}(n)\}_{n=0}^{\infty}$ we may consider the spaces of holomorphic functions as sequence spaces), we write

$$\lambda X = \{\{\lambda_n x_n\} : \{x_n\} \in X\}.$$

For example, $\{a_n\} \in D^1 l^1$ if and only if $\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} < \infty$. The space $D^t H^{p,q,\alpha}$, for $t \neq 0$, will also be denoted by $H_{-t}^{p,q,\alpha}$.

Among the spaces $H_s^{p,q,\alpha}$, $0 < s < \infty$, the spaces $H_{1+s}^{p,q,1}$ are of independent interest, and are known as Besov spaces for $0 < q < \infty$, and as Lipschitz spaces when $q = \infty$.

We note that in [8] the spaces of functions $f \in \mathcal{H}(\mathbb{D})$ such that $D^n f \in H^{p,q,n-\alpha}$ (equivalently, $f^{(n)} \in H^{p,q,n-\alpha}$) for some (any) nonnegative integer n such that $n - \alpha > 0$, and where $\alpha \in \mathbb{R}$, are called Besov spaces and they are denoted by $\mathcal{B}_{\alpha}^{p,q}$. Comparing with the notations given above, $\mathcal{B}_{\alpha}^{p,q} = H^{p,q,-\alpha}$ for $\alpha < 0$, and $\mathcal{B}_{\alpha}^{p,q} = H_{1+\alpha}^{p,q,1}$ for $\alpha > 0$.

The spaces $H_{1+1/p}^{p,p,1}$, $1 < p < \infty$, can be described as spaces of functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\int_0^1 M_p^p(r, f')(1-r)^{p-2} dr < \infty,$$

or, equivalently,

$$\int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} dA(z) < \infty.$$

Obviously, $H_1^{\infty,\infty,1} = \mathcal{B}_0^{\infty,\infty}$ is the Bloch space $\mathcal{B}$.

2. Hilbert matrix operator on $VMOA$

Hilbert matrix operator is not bounded on H^∞, but it maps H^∞ into $\mathcal{B}$ (more precisely, into $BMOA$). On the other hand, H does not map $BMOA$ into $\mathcal{B}$. We improve this. We show that the Besov space $H^{p,q,1}_{1+1/p}$, that is a subspace of $BMOA$, except for $p=\infty$, $2<q\leqslant\infty$, is not mapped into the Bloch space $\mathcal{B}$ by the Hilbert matrix operator H if $1<q\leqslant\infty$. As a corollary, we have that the space $VMOA$ is also not mapped by H into the Bloch space $\mathcal{B}$.

The following well-known duality result will be needed (see [2]).

Theorem 2.1. *If $g\in\mathcal{B}$, then*

$$\varphi_g(f)=\lim_{r\to1}\sum_{n=0}^{\infty}\widehat{f}(n)\overline{\widehat{g}(n)}r^n,\quad f\in H^{1,1,1}_1,$$

defines a bounded linear functional on $H^{1,1,1}_1$ such that $||\varphi_g||\leqslant C||D^1g||_{\infty,\infty,1}$. Conversely, if $\varphi\in\left(H^{1,1,1}_1\right)'$, then there exists a unique $g\in\mathcal{B}$ such that

$$\varphi(f)=\varphi_g(f)=\lim_{r\to1}\sum_{n=0}^{\infty}\widehat{f}(n)\overline{\widehat{g}(n)}r^n,$$

for all $f\in H^{1,1,1}_1$ and $||D^1g||_{\infty,\infty,1}\leqslant C||\varphi||$.

Now we are ready to state our first result.

Theorem 2.2.
(a) *If $0<q\leqslant1$, then H maps $H^{p,q,1}_{1+1/p}$, $0<p\leqslant\infty$, into $BMOA$.*
(b) *If $1<q\leqslant\infty$, then H does not map $H^{p,q,1}_{1+1/p}$, $0<p\leqslant\infty$, into $\mathcal{B}$.*

Proof. (a) In [3], the following formula for H acting on H^p, $1\leqslant p$, was noticed:

$$Hf=P_+(M_bCf),$$

where $Cf(e^{it})=f(e^{-it})$, $M_bu=bu$, $u\in L^\infty(\mathbb{T})$, $b(e^{it})=ie^{-it}(\pi-t)$, $0\leqslant t\leqslant2\pi$, and P_+ is Szegő projection given by

$$P_+u(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{u(e^{it})}{1-ze^{-it}}dt.$$

Since the space $BMOA$ is the Szegő projection of $L^\infty(\mathbb{T})$, we have that the Hilbert matrix operator H acts as a bounded operator from H^∞ into $BMOA$ (see also Theorem 1.2 in [7]).

If $0 < q \leqslant 1$, then $H^{p,q,1}_{1+1/p} \subseteq H^{\infty,1,1}_1$. Now we prove that $H^{\infty,1,1}_1 \subseteq \mathcal{A}$. Let $f \in H^{\infty,1,1}_1$. Then we have

$$f(z) = \int_0^1 D^1 f(\rho z) d\rho.$$

To show that $f \in \mathcal{A}$, it is enough to show that the integral on the right hand side converges uniformly with respect to $z \in \overline{\mathbb{D}}$. But this follows from the estimate

$$\left| \int_r^1 D^1 f(\rho z) d\rho \right| \leqslant \int_r^1 M_\infty(\rho, D^1 f) d\rho, \quad 0 < r < 1,\, z \in \overline{\mathbb{D}},$$

and the fact that $\int_0^1 M_\infty(\rho, D^1 f) d\rho < \infty$, see [5, Theorem 4, p. 754]. Thus $H^{p,q,1}_{1+1/p} \subseteq \mathcal{A}$, for $0 < q \leqslant 1$. Therefore, if $0 < q \leqslant 1$, then $H : H^{p,q,1}_{1+1/p} \to BMOA$.

(b) It is known that if $0 < p_1 < p_2 \leqslant \infty$, then $H^{p_1,q,1}_{1+1/p_1} \subseteq H^{p_2,q,1}_{1+1/p_2}$. We use this fact below.

Case 1. $q = \infty$. In [7], it is proved that if $f \in \mathcal{B}$, then

$$|(Hf)'(z)|\,(1 - |z|) = O\left(\log \frac{2}{1 - |z|}\right), \quad z \in \mathbb{D}.$$

This estimate cannot be improved as the function $g(z) = \log \frac{2}{1-z}$ shows. Let us prove that $g \in H^{p,\infty,1}_{1+1/p}$. Since

$$\left| g^{(n)}(z) \right| \leqslant \frac{C}{|1 - z|^n}, \quad n \geqslant 1,$$

we deduce for $\frac{1}{n} < p \leqslant \frac{1}{n-1}$, $n \geqslant 2$,

$$||g^{(n)}||_{p,\infty,n-1/p} \leqslant \sup_{0 \leqslant r < 1} (1 - r)^{n-1/p} \left(\int_0^{2\pi} \frac{C dt}{|1 - re^{it}|^{np}} \right)^{1/p} \leqslant C.$$

Since $H^{p,\infty,1}_{1+1/p} \subseteq H^{\infty,\infty,1}_1 = \mathcal{B}$, we see that H does not map $H^{p,\infty,1}_{1+1/p}$ into $\mathcal{B}$, and that the estimate

$$|(Hf)'(z)|\,(1 - |z|) = O\left(\log \frac{2}{1 - |z|}\right), \quad z \in \mathbb{D},\ f \in H^{p,\infty,1}_{1+1/p},$$

cannot be improved.

Case 2. $1 < q < \infty$. We may assume that $\frac{1}{n} < p \leqslant \frac{1}{n-1}$, where $n \geqslant 2$ is a positive integer. Let $h(z) = \left(\log \frac{2}{1-z}\right)^{\gamma - 1}$, where $1 < \gamma < 2$ and $q(2-\gamma) > 1$. We show that $h \in H^{p,q,1}_{1+1/p}$ and $Hh \notin \mathcal{B}$. First, if $f \in \mathcal{H}(\mathbb{D})$, then $||D^{1+1/p}f||_{p,q,1} < \infty$ if and only if $||f^{(n)}||_{p,q,n-1/p} < \infty$. So it suffices to show that $||h^{(n)}||_{p,q,n-1/p} < \infty$. It is easy to see that

$$\left|h^{(n)}(z)\right| \leqslant \frac{C}{|1-z|^n \left(\log \frac{2}{1-|z|}\right)^{2-\gamma}}, \quad z \in \mathbb{D}.$$

Therefore,

$$\begin{aligned} ||h^{(n)}||^q_{p,q,n-1/p} &\leqslant C \int_0^1 \left(\int_0^{2\pi} \frac{dt}{|1-re^{it}|^{np} \left(\log \frac{2}{1-r}\right)^{(2-\gamma)p}} \right)^{q/p} (1-r)^{q(n-1/p)-1} dr \\ &\leqslant C \int_0^1 \frac{dr}{(1-r)\left(\log \frac{2}{1-r}\right)^{(2-\gamma)q}} < \infty. \end{aligned}$$

Here and above, we used the estimate

$$\int_0^{2\pi} \frac{dt}{|1-re^{it}|^\alpha} = O\left(\frac{1}{(1-r)^{\alpha-1}}\right), \quad \alpha > 1.$$

Let $f(z) = \frac{1}{(1-z)\left(\log \frac{2}{1-z}\right)^\gamma}$. An argument similar to that given above shows that $f \in H^{1,1,1}_1$ (see also [8] and [9]). On the other hand,

$$\begin{aligned} \lim_{r\to 1^-} \sum_{n=0}^\infty \widehat{f}(n)\overline{\widehat{Hh}(n)} r^n &= \lim_{r\to 1^-} \int_0^1 f(rt)\overline{h(t)} dt \\ &= \lim_{r\to 1^-} \int_0^1 \frac{1}{(1-rt)\left(\log \frac{2}{1-rt}\right)^\gamma} \left(\log \frac{2}{1-t}\right)^{\gamma-1} dt = \infty. \end{aligned}$$

Hence, $Hh \notin \mathcal{B}$ by Theorem 2.1. □

Corollary 2.3. *H does not map $VMOA$ into $\mathcal{B}$.*

Proof. By [8, Theorem 6.8, p. 186], we find that $H^{\infty,2,1}_1 \subseteq VMOA$. On the other hand, $H^{p,q,1}_{1+1/p} \subseteq H^{\infty,2,1}_1$, for $0 < q \leqslant 2$. By applying Theorem 2.2, we conclude that H does not map $VMOA$ into $\mathcal{B}$. □

Remark 2.4. In [7], it is proved that if $f \in H^{p,p,1}_{1+1/p}$, $1 < p < \infty$, then

$$|(Hf)'(z)|\,(1-|z|) = O\left(\left(\log \frac{2}{1-|z|}\right)^{\frac{1}{p'}}\right), \quad z \in \mathbb{D},\ p + p' = pp'.$$

We do not know whether this estimate is optimal. We note that it follows from Theorem 2.2 that it cannot be replaced with

$$|(Hf)'(z)|\,(1-|z|)=O(1),\quad z\in\mathbb{D},$$

for every $f\in H^{p,p,1}_{1+1/p}$, where $1<p<\infty$.

3. Hilbert matrix operator on H^1 (resp. $H_1^{1,1,1}$)

Hilbert matrix operator H is not bounded on H^1. We show that operator H maps continuously H^1 into the space $H^{p,\infty,1/p'}$, $1<p<\infty$, $p+p'=pp'$, but not into $H^{p,q,1/p'}$, for any $0<q<\infty$ (note that $H^1\subset H^{p,q,1/p'}\subset H^{p,\infty,1/p'}$, $1<p,q<\infty$). In fact, a little more is true.

Theorem 3.1. *Let $1<p<\infty$ and $\frac{1}{p}+\frac{1}{p'}=1$. Then*
(a) *$H:H^1\to H^{p,\infty,1/p'}$;*
(b) *H does not map $H_1^{1,1,1}$ into $H^{p,q,1/p'}$, for any $q\in(0,\infty)$.*

In the proof we will use Theorem 2.1, Theorem 2.2 and the following duality result ([8]).

Theorem 3.2. *Let $1<p,q<\infty$ and $\alpha\in\mathbb{R}$. Then the dual of space $\mathcal{B}^{p,q}_{\alpha}$ is isomorphic to the space $\mathcal{B}^{p',q'}_{-\alpha}$, $p+p'=pp'$, $q+q'=qq'$, under the pairing*

$$<f,g>=\sum_{n=0}^{\infty}\widehat{f}(n)\overline{\widehat{g}(n)},\quad f\in\mathcal{B}^{p,q}_{\alpha},\ g\in\mathcal{B}^{p',q'}_{-\alpha},$$

where the series converges in the ordinary sense.

Proof of Theorem 3.1. (a) Let $f\in H^1$. Then $Hf(z)=\int_0^1\frac{f(r)}{1-rz}dr$, $z\in\mathbb{D}$. By using Minkowski's inequality in the continuous form and Fejér–Riesz inequality, we find that

$$M_p(\rho,Hf)\leqslant\int_0^1|f(r)|dr\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{dt}{|1-r\rho e^{it}|^p}\right)^{1/p}\leqslant||f||_1\frac{C}{(1-\rho)^{1/p'}}.$$

(b) Since $H^{p,q_1,1/p'}\subseteq H^{p,q_2,1/p'}$, if $0<q_1<q_2<\infty$, we may assume that $1<q<\infty$. As usual, $q+q'=qq'$. Let $f\in H^{p',q',1}_{1+1/p'}$ and assume that $Hg\in H^{p,q,1/p'}$, for any $g\in H_1^{1,1,1}$. Then, for any $g\in H_1^{1,1,1}$ the series

$$\sum_{k=0}^{\infty}\widehat{Hg}(k)\overline{\widehat{f}(k)}$$

converges by Theorem 3.2, and therefore

$$\sum_{k=0}^{\infty} \widehat{Hg}(k)\overline{\widehat{f}(k)} = \lim_{r\to 1^-} \sum_{k=0}^{\infty} \widehat{Hg}(k)\overline{\widehat{f}(k)}r^k = \lim_{r\to 1^-} \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{\widehat{g}(n)}{k+n+1}\overline{\widehat{f}(k)}r^k.$$

Since $\sum\limits_{n=0}^{\infty} \frac{|\widehat{g}(n)|}{n+k+1} \leqslant \sum\limits_{n=0}^{\infty} \frac{|\widehat{g}(n)|}{n+1} < \infty$ and $\sum\limits_{k=0}^{\infty} |\widehat{f}(k)|r^k < \infty$, we find that

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{\widehat{g}(n)}{k+n+1}\overline{\widehat{f}(k)}r^k = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\overline{\widehat{f}(k)}r^k}{n+k+1}\widehat{g}(n),$$

for any $r \in (0,1)$. Thus

$$\begin{aligned}\sum_{k=0}^{\infty} \widehat{Hg}(k)\overline{\widehat{f}(k)} &= \lim_{r\to 1^-} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\overline{\widehat{f}(k)}r^k}{n+k+1}\widehat{g}(n)\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\overline{\widehat{f}(k)}}{n+k+1}\widehat{g}(n) = \sum_{n=0}^{\infty} \overline{\widehat{Hf}(n)}\widehat{g}(n).\end{aligned}$$

Hence, by Theorem 2.1, $Hf \in \mathcal{B}$. This contradicts to Theorem 2.2 (b). □

Remark 3.3. (1) It follows from [7, Corollary 2.2] that

$$H : H^1 \to H^{p,q,1/p'+\varepsilon},$$

where $1 < p, q < \infty$ and $\varepsilon > 0$. Since $H^{p,q,1/p'+\varepsilon} \supseteq H^{p,q,1/p'}$, Theorem 3.1 (b) shows that conclusion

$$H : H^1 \to H^{p,q,1/p'+\varepsilon}$$

does not hold for $\varepsilon = 0$.

(2) Since $H_1^{1,1,1} \subset H^1 \subset H_1^{1,2,1} \subset H^{p,2,1/p'}$, where $1 < p < \infty$, as a corollary of Theorem 3.1 we have that H does not map $H_1^{1,1,1}$ into $H_1^{1,2,1}$. The same conclusion could also be derived by using the fact that the dual of $H_1^{1,2,1}$ is isomorphic to $H_1^{\infty,2,1}$, see [1] and Theorem 2.2.

4. Hilbert matrix operator on logarithmically weighted Bergman spaces

It follows from Theorem 3.1 that H does not map H^1, a subspace of D^1l^1 by Hardy's inequality, into $H^{2,2,1/2} = A^2$. In this section, we provide some subspaces of $D^1 l^1$, the so-called logarithmically weighted Bergman spaces, that are mapped into A^2 by H. We improve the results given in [7, Section 4].

For $\alpha > 0$, we define the logarithmically weighted Bergman space $A^2_{\log^\alpha} \subset A^2$ as follows:

$$A^2_{\log^\alpha} = \left\{ f \in \mathcal{H}(\mathbb{D}) : ||f||^2_{A^2_{\log^\alpha}} = \int_{\mathbb{D}} |f(z)|^2 \left(\log \frac{2}{1-|z|^2} \right)^\alpha dA(z) < \infty \right\}.$$

The norm $||f||_{A^2_{\log^\alpha}}$ may be expressed in a different way, as the following lemma shows.

Lemma 4.1. *Let $\alpha > 0$ and $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$ be a holomorphic function in $\mathbb{D}$. Then $f \in A^2_{\log^\alpha}$ if and only if $\sum_{n=0}^{\infty} \frac{|\widehat{f}(n)|^2}{n+1} \log^\alpha(n+1) < \infty$.*

Proof. By using Parseval's formula, we find that

$$\|f\|^2_{A^2_{\log^\alpha}} = \sum_{n=0}^{\infty} |\widehat{f}(n)|^2 \int_0^1 r^n \log^\alpha \frac{2}{1-r} dr.$$

Now the lemma follows from the estimate

$$\int_0^1 r^n \log^\alpha \frac{2}{1-r} dr \asymp \frac{\log^\alpha(n+1)}{n+1}, \tag{4.1}$$

that we prove below.

A function $\phi(t) = t \log^\alpha \frac{2}{t}$, $0 < t < 1$, is normal. An argument used in the proof of [8, Theorem 5.19, p. 163], shows that

$$\int_0^1 r^n \frac{\phi(1-r)}{1-r} dr \asymp \phi\left(\frac{1}{n+1}\right).$$

Thus $\int_0^1 r^n \log^\alpha \frac{2}{1-r} dr \asymp \frac{\log^\alpha(n+1)}{n+1}$. □

Remark 4.2. We are grateful to the referee who pointed out to us that a similar argument based on the paper [10] leads to the same conclusion.

Remark 4.3. To keep the paper as self-contained as possible, we give a direct proof of (4.1). First, we find that

$$\int_0^1 r^n \log^\alpha \frac{2}{1-r} dr \geqslant \int_{1-\frac{1}{n+1}}^1 r^n \log^\alpha \frac{1}{1-r} dr \geqslant \log^\alpha(n+1) \int_{1-\frac{1}{n+1}}^1 r^n dr =$$

$$= \frac{\log^\alpha(n+1)}{n+1}\left(1-\left(1-\frac{1}{n+1}\right)^{n+1}\right) \geqslant \left(1-\frac{1}{e}\right)\frac{\log^\alpha(n+1)}{n+1}.$$

On the other hand,

$$\int_0^1 r^n \log^\alpha \frac{2}{1-r}dr = S_1 + S_2, \tag{4.2}$$

where

$$S_1 = \int_0^{1-\frac{1}{n+1}} r^n \log^\alpha \frac{2}{1-r}dr$$

and

$$S_2 = \int_{1-\frac{1}{n+1}}^1 r^n \log^\alpha \frac{2}{1-r}dr.$$

If $0 \leqslant r \leqslant 1-\frac{1}{n+1}$, then $\frac{1}{1-r} \leqslant n+1$, and therefore $\log^\alpha \frac{2}{1-r} \leqslant 2^\alpha \log^\alpha(n+1)$ for $n \geqslant 1$. Hence,

$$\begin{aligned} S_1 &= \int_0^{1-\frac{1}{n+1}} r^n \log^\alpha \frac{2}{1-r}dr \leqslant 2^\alpha \log^\alpha(n+1) \int_0^{1-\frac{1}{n+1}} r^n dr \\ &= 2^\alpha \frac{\log^\alpha(n+1)}{n+1}\left(1-\frac{1}{n+1}\right)^{n+1} \leqslant \frac{2^\alpha}{e}\frac{\log^\alpha(n+1)}{n+1}. \end{aligned} \tag{4.3}$$

It is easy to see that

$$S_2 \leqslant \int_{1-\frac{1}{n+1}}^1 \log^\alpha \frac{2}{1-r}dr = 2\int_{\log 2(n+1)}^\infty t^\alpha e^{-t}dt.$$

For $n \geqslant 2$, partial integration gives

$$\int_{\log(n+1)}^\infty t^\alpha e^{-t}dt = \frac{\log^\alpha(n+1)}{n+1} + \alpha\frac{\log^{\alpha-1}(n+1)}{n+1} + \alpha(\alpha-1)\int_{\log(n+1)}^\infty t^{\alpha-2}e^{-t}dt.$$

Continuing on this way, we find that

$$\int_{\log(n+1)}^\infty t^\alpha e^{-t}dt \leqslant C_\alpha \frac{\log^\alpha(n+1)}{n+1}.$$

Hence,

$$S_2 \leqslant C_\alpha \frac{\log^\alpha(n+1)}{n+1}. \tag{4.4}$$

From (4.2), (4.3) and (4.4), we find that $\int_0^1 r^n \log^\alpha \frac{2}{1-r}dr \leqslant C_\alpha \frac{\log^\alpha(n+1)}{n+1}$. □

In [7], it is shown that $A^2_{\log^\alpha} \subseteq D^1 l^1$ for $\alpha > 2$. Now we improve this.

Proposition 4.4. *If $\alpha > 1$, then $A^2_{\log^\alpha} \subseteq D^1 l^1$. For $\alpha = 1$, $A^2_{\log^1}$ is not a subset of $D^1 l^1$.*

Proof. Let $f \in A^2_{\log^\alpha}$, where $\alpha > 1$. By Lemma 4.1, $\sum_{n=0}^{\infty} \frac{|\widehat{f}(n)|^2}{n+1} \log^\alpha(n+1) < \infty$. Thus, by using Cauchy–Schwarz inequality, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{|\widehat{f}(n)|}{n+1} &= |\widehat{f}(0)| + \sum_{n=1}^{\infty} \frac{|\widehat{f}(n)|}{n+1} \\ &\leqslant |\widehat{f}(0)| + \left(\sum_{n=1}^{\infty} \frac{|\widehat{f}(n)|^2}{n+1} \log^\alpha(n+1) \right)^{1/2} \left(\sum_{n=1}^{\infty} \frac{1}{(n+1)\log^\alpha(n+1)} \right)^{1/2} < \infty.\end{aligned}$$

Now, let $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{\log(n+1)\log(\log(n+1))}$. Then

$$\begin{aligned}\|f\|^2_{A^2_{\log^1}} &\leqslant C \sum_{n=1}^{\infty} \frac{|\widehat{f}(n)|^2}{n+1} \log(n+1) \\ &= C \sum_{n=1}^{\infty} \frac{1}{(n+1)\log(n+1)\log^2(\log(n+1))} < \infty.\end{aligned}$$

On the other hand,

$$\sum_{n=1}^{\infty} \frac{|\widehat{f}(n)|}{n+1} = \sum_{n=1}^{\infty} \frac{1}{(n+1)\log(n+1)\log(\log(n+1))} = \infty. \qquad \square$$

In [7], it is shown that if $f \in A^2_{\log^\alpha}$, where $\alpha > 3$, then $Hf \in A^2$. We also improve this result.

Theorem 4.5. *If $f \in A^2_{\log^\alpha}$, where $\alpha > 2$, then $Hf \in A^2$.*

Proof. Since

$$Hf(z) = \widehat{f}(0) \frac{1}{z} \log \frac{1}{1-z} + \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{\widehat{f}(k)}{n+k+1} z^n,$$

and $\frac{1}{z} \log \frac{1}{1-z} \in A^2$, it suffices to show that

$$H_1 f(z) := \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{\widehat{f}(k)}{n+k+1} z^n \in A^2.$$

By using Proposition 4.4 and Lemma 4.1, we find that

$$
\begin{aligned}
||H_1 f||^2_{A^2} &= \sum_{n=0}^{\infty} \frac{1}{n+1} \left| \sum_{k=1}^{\infty} \frac{\widehat{f}(k)}{n+k+1} \right|^2 \\
&\leqslant \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=1}^{\infty} \frac{|\widehat{f}(k)|^2}{n+k+1} \log^{\alpha}(k+1) \sum_{k=1}^{\infty} \frac{1}{(n+k+1)\log^{\alpha}(k+1)} \\
&\leqslant C||f||^2_{A^2_{\log^{\alpha}}} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=1}^{\infty} \frac{1}{(n+k+1)\log^{\alpha}(k+1)} \\
&= C||f||^2_{A^2_{\log^{\alpha}}} \sum_{k=1}^{\infty} \frac{1}{k\log^{\alpha}(k+1)} \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+k+1} \right) \\
&= C||f||^2_{A^2_{\log^{\alpha}}} \sum_{k=1}^{\infty} \frac{1}{k\log^{\alpha}(k+1)} \left(1 + \frac{1}{2} + \ldots + \frac{1}{k} \right) \\
&\leqslant C||f||^2_{A^2_{\log^{\alpha}}} \sum_{k=1}^{\infty} \frac{1}{k\log^{\alpha-1}(k+1)} < \infty,
\end{aligned}
$$

because $\alpha - 1 > 1$, or equivalently $\alpha > 2$. □

Remark 4.6. We do not know whether there exists $\alpha \in (1, 2]$ such that H maps continuously $A^2_{\log^{\alpha}}$ into A^2.

References

[1] P. Ahern and M. Jevtić, Duality and multipliers for mixed norm spaces, *Michigan Math. J.* **30** (1983), 53–64.

[2] J. M. Anderson, J. Clunie and C. Pommerenke, On Bloch and normal functions, *J. Reine Angew. Math.* **270** (1974), 12–37.

[3] M. Dostanić, M. Jevtić and D. Vukotić, Norm of the Hilbert matrix on Bergman and Hardy spaces and a theorem of Nehari type, *J. Funct. Anal.* **254** (2008), 2800–2815.

[4] P. L. Duren, Theory of H^p Spaces, Pure and Applied Mathematics, Vol. **38**, *Academic Press*, *New York*, 1970.

[5] T. M. Flett, The dual of an inequality of Hardy and Littlewood and some related inequalities, *J. Math. Anal. Appl.* **38** (1972), 746–765.

[6] M. Jevtić, D. Vukotić and M. Arsenović, Taylor Coefficients and Coefficient of Multipliers of Hardy and Bergman-Type Spaces, RSME Springer Series, *Springer*, *Cham, Switzerland*, 2016.

[7] B. Lanucha, M. Nowak and M. Pavlović, Hilbert matrix operator on spaces of analytic functions, *Ann. Acad. Sci. Fenn. Math.* **37** (2012), 161–174.

[8] M. Pavlović, Function Classes on the Unit Disc. An Introduction, De Gruyter Studies in Mathematics, Vol. **52**, *De Gruyter*, *Berlin*, 2014.

[9] M. Pavlović, Analytic functions with decreasing coefficients and Hardy and Bloch spaces, *Proc. Edinb. Math. Soc. (2)* **56** (2013), 623–635.

[10] J. A. Peláez and J. Rättyä, Weighted Bergman Spaces Induced by Rapidly Increasing Weights, *Mem. Amer. Math. Soc.* **227** (2014).

MIROLJUB JEVTIĆ
FACULTY OF MATHEMATICS
UNIVERSITY OF BELGRADE
STUDENTSKI TRG 16
BELGRADE
SERBIA

E-mail: jevtic@matf.bg.ac.rs

BOBAN KARAPETROVIĆ
FACULTY OF MATHEMATICS
UNIVERSITY OF BELGRADE
STUDENTSKI TRG 16
BELGRADE
SERBIA

E-mail: bkarapetrovic@matf.bg.ac.rs

(Received October 22, 2015; revised June 23, 2016)