

A note on the weighted strong law of large numbers under general conditions

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Abstract. In 2003, R. Jajte proved the almost sure convergence for a large class of weighted sums of independent random variables. In this note we show that this result remains valid for dependent random variables satisfying certain general maximal inequality.

1. Introduction

In 2003, JAJTE [7] obtained a version of the strong law of large numbers (SLLN) for a large class of means. His result generalized considerably two classical theorems for sequences of independent and identically distributed (i.i.d.) random variables (r.v.'s): the SLLN of Kolmogorov and that of Marcinkiewicz–Zygmund. For completeness, let us recall this result.

Theorem 1 (Jajte, 2003). *Let $\{X_k\}_{k \in \mathbb{N}}$ be a sequence of i.i.d. r.v.'s. Let also f be a positive increasing function and g a positive function, such that $\varphi(y) = f(y)g(y)$ satisfies the following conditions:*

- (i) *for some $d \geq 0$, φ is strictly increasing on $[d, \infty)$ with range $[0, \infty)$,*
- (ii) *there exist C and a positive integer k_0 such that $\varphi(y+1)/\varphi(y) \leq C$, for all $y \geq k_0$,*

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(iii) *there exist constants a and b such that*

$$\varphi^2(s) \int_s^\infty \frac{1}{\varphi^2(x)} dx \leq as + b, \quad \text{for all } s > d.$$

Then the following two conditions are equivalent:

- (a) $\mathbb{E}(\varphi^{-1}(|X_1|)) < \infty$,
- (b) $\frac{1}{f(n)} \sum_{i=1}^n \frac{X_i - m_i}{g(i)} \rightarrow 0$, *almost surely*, as $n \rightarrow \infty$,

where φ^{-1} is the inverse function of φ and $m_i = \mathbb{E}(X_i \mathbb{I}_{[|X_i| \leq \varphi(i)]})$.

Jajte points out that in the above class of sequence transformations, one can find Cesaro means ($g(y) = 1, f(y) = y$), logarithmic means ($g(y) = y, f(y) = \log y$), and the means related to the Marcinkiewicz–Zygmund theorem ($g(y) = 1, f(y) = y^{1/\alpha}$, with $\alpha \in (0, 2)$).

Many authors extended Jajte's result to the cases of specific types of dependence. JING and LIANG [8] considered negatively associated r.v.'s with the same distribution. MENG and LIN [12] focused on $\tilde{\rho}$ -mixing r.v.'s, whereas WANG [15] studied the case of nonidentically distributed negatively associated r.v.'s. Recently, TANG [14] presented Jajte's type sufficient condition for the SLLN for the family of asymptotically almost negatively associated (AANA) r.v.'s. Assumption of equidistribution was also weakened and replaced by stochastic domination. The result of Jajte was also generalized to the random field setting by ŁAGODOWSKI and MATUŁA [11].

The aim of this paper is to generalize the result of Jajte to the case of equidistributed (or stochastically dominated) r.v.'s regardless of any dependence structure. The original necessary and sufficient condition for SLLN (see (a) in Theorem 1) becomes, however, the sufficient one, in general. Only in a special case of ρ^- -mixing sequences, we manage to show that the equivalence of (a) and (b) holds. The formulation of our result is motivated by the general approach to the strong law of large numbers, developed by FAZEKAS and KLESOV (see [4] and [5], where further references are given).

Throughout the paper, we shall denote by \mathbb{N} the set of positive integers.

2. Main result

The key result of the paper is the following Jajte's type sufficient condition for the SLLN. In the formulation, we adopt the assumptions on the weighting functions introduced by TANG [14], which are slightly simpler than imposed in Theorem 1.

Theorem 2. Let $\{X_k\}_{k \in \mathbb{N}}$ be a sequence of equidistributed r.v.'s. Let also $f : [0, \infty) \rightarrow \mathbb{R}$ and $g : [0, \infty) \rightarrow \mathbb{R}$ be positive functions, such that $\varphi(y) \equiv f(y)g(y)$ is a function satisfying the following conditions:

- (A1) φ is strictly increasing and $\varphi([0, \infty)) = [0, +\infty)$,
- (A2) there exist $a, b \in \mathbb{R}$, such that for any $s > 0$,

$$\varphi^2(s) \int_s^\infty \frac{1}{\varphi^2(x)} dx \leq as + b.$$

Furthermore, we assume that for a sequence $\{Y_k\}_{k \in \mathbb{N}}$ of truncated r.v.'s defined by

$$Y_k := X_k \mathbb{I}_{[|X_k| \leq \varphi(k)]} + \varphi(k) \mathbb{I}_{[X_k > \varphi(k)]} - \varphi(k) \mathbb{I}_{[X_k < -\varphi(k)]}, \tag{1}$$

there exists an absolute constant $C > 0$, such that

$$\mathbb{E} \left(\max_{m \leq l \leq n} \left| \sum_{k=m}^l \frac{Y_k - \mathbb{E}Y_k}{\varphi(k)} \right| \right)^2 \leq C \sum_{k=m}^n \frac{\text{Var}(Y_k)}{\varphi^2(k)} \quad \text{for all } n, m \in \mathbb{N}, m \leq n. \tag{2}$$

Then, the condition

$$\mathbb{E}\varphi^{-1}(|X_1|) < \infty \tag{3}$$

implies that the series

$$\sum_{k=1}^\infty \frac{X_k - \mathbb{E}Y_k}{\varphi(k)} \quad \text{is almost surely convergent.} \tag{4}$$

If, in addition, f is increasing and $\lim_{y \rightarrow \infty} f(y) = \infty$, then the following weighted SLLN holds:

$$\frac{1}{f(n)} \sum_{k=1}^n \frac{X_k - \mathbb{E}Y_k}{g(k)} \rightarrow 0, \quad \text{almost surely, as } n \rightarrow \infty. \tag{5}$$

Remark 1. From the assumption (A1) it follows that φ is a continuous function with $\varphi(0) = 0$.

Remark 2. Inequality of the type (2), in [4], is called “the second Kolmogorov type maximal inequality for moments” for the r.v.'s $(Y_k - \mathbb{E}Y_k)/\varphi(k)$.

The key step in our proof is the following variant of the two series theorem.

Lemma 1. Assume that an arbitrary sequence $\{Y_k\}_{k \in \mathbb{N}}$ and the function φ defined in Theorem 2 satisfy (2), and

$$\sum_{k=1}^{\infty} \frac{\text{Var}(Y_k)}{\varphi^2(k)} < \infty. \quad (6)$$

Then the series

$$\sum_{k=1}^{\infty} \frac{Y_k - \mathbb{E}Y_k}{\varphi(k)} \text{ is almost surely convergent.}$$

PROOF. According to the standard procedure (see Lemma 6.9 in the book of PETROV [13]), we check that $S_k = \sum_{i=1}^k \frac{Y_i - \mathbb{E}Y_i}{\varphi(i)}$ is a Cauchy sequence with an almost surely finite limit. Indeed, for fixed $m \in \mathbb{N}$,

$$\sup_{k, l \geq m} |S_k - S_l| = \sup_{k, l \geq m} |S_k - S_m + S_m - S_l| \leq 2 \sup_{k \geq m} |S_k - S_m|,$$

thus, by (6) we have

$$\begin{aligned} P\left(\sup_{k, l \geq m} |S_k - S_l| > \epsilon\right) &\leq P\left(\sup_{k \geq m} |S_k - S_m| > \epsilon/2\right) \\ &= \lim_{n \rightarrow \infty} P\left(\max_{m \leq k \leq n} |S_k - S_m| > \epsilon/2\right) \\ &\leq \frac{4C}{\epsilon^2} \sum_{k=m}^{\infty} \frac{\text{Var}(Y_k)}{\varphi^2(k)} \rightarrow 0, \text{ as } m \rightarrow \infty. \end{aligned}$$

Therefore, the series $\sum_{k=1}^{\infty} \frac{Y_k - \mathbb{E}Y_k}{\varphi(k)}$ is almost surely convergent. \square

PROOF OF THEOREM 2. We proceed along the lines of the original paper of JAJTE [7] and the proof of TANG [14]; therefore, we only briefly present the main steps. In the first step, we observe that the sequences $\{X_k\}_{k \in \mathbb{N}}$ and $\{Y_k\}_{k \in \mathbb{N}}$ are equivalent in the sense that $P(X_k \neq Y_k \text{ i.o.}) = 0$. It is easy to see that, by the assumption (3),

$$\sum_{k=1}^{\infty} P(X_k \neq Y_k) = \sum_{k=1}^{\infty} P(\varphi^{-1}(|X_1|) > k) < \infty,$$

and it suffices to use the first Borel–Cantelli lemma.

In the second step, we prove that the series $\sum_{k=1}^{\infty} \text{Var} \left(\frac{Y_k}{\varphi(k)} \right)$ is convergent and apply our Lemma 1. In what follows, $\lfloor \cdot \rfloor$ will denote the integer part of a number (floor function). By the definition of Y_k 's, we can write

$$\sum_{k=1}^{\infty} \frac{\mathbb{E}Y_k^2}{\varphi^2(k)} = \sum_{k=1}^{\infty} \frac{\mathbb{E} \left(X_1^2 \mathbb{I}_{\{|X_1| \leq \varphi(k)\}} \right)}{\varphi^2(k)} + \sum_{k=1}^{\infty} P(|X_1| > \varphi(k)) =: S_1 + S_2. \tag{7}$$

Evoking (3), we easily get $S_2 = \sum_{k=1}^{\infty} P(\varphi^{-1}(|X_1|) > k) < \infty$. With a view to estimating S_1 , let us split the sum in the following way:

$$\begin{aligned} S_1 &= \mathbb{E} \left(\sum_{k=1}^{\lfloor \varphi^{-1}(|X_1|) \rfloor + 1} \frac{X_1^2 \mathbb{I}_{\{|X_1| \leq \varphi(k)\}}}{\varphi^2(k)} + \sum_{k=\lfloor \varphi^{-1}(|X_1|) \rfloor + 2}^{\infty} \frac{X_1^2 \mathbb{I}_{\{|X_1| \leq \varphi(k)\}}}{\varphi^2(k)} \right) \\ &\leq \mathbb{E}(\lfloor \varphi^{-1}(|X_1|) \rfloor + 1) + \mathbb{E} \left(\sum_{k=\lfloor \varphi^{-1}(|X_1|) \rfloor + 2}^{\infty} \frac{X_1^2}{\varphi^2(k)} \right) \end{aligned}$$

since $1/\varphi^2(x)$ is decreasing, we can approximate sums by integrals

$$\leq \mathbb{E}(\varphi^{-1}(|X_1|) + 1) + \mathbb{E} \left(X_1^2 \int_{\varphi^{-1}(|X_1|)}^{\infty} \frac{1}{\varphi^2(x)} dx \right),$$

which, in the light of the assumptions (3) and (A2),

$$\leq \mathbb{E}\varphi^{-1}(|X_1|) + 1 + \mathbb{E}(a\varphi^{-1}(|X_1|) + b) < \infty.$$

Thus $\sum_{k=1}^{\infty} \frac{\mathbb{E}Y_k^2}{\varphi^2(k)} < \infty$, and by Lemma 1, the conclusion follows, i.e. the series $\sum_{k=1}^{\infty} \frac{X_k - \mathbb{E}Y_k}{\varphi(k)}$ converges almost surely. In order to arrive at the weighted SLLN (5), it suffices to use Kronecker's lemma. \square

In order to relax the assumption that the r.v.'s have the same distributions, let us recall the following definition.

Definition 1. We say that the sequence $\{X_k\}_{k \in \mathbb{N}}$ of r.v.'s is stochastically dominated by a random variable X if

$$P(|X_n| > x) \leq CP(|X| > x),$$

for all $x \geq 0$ and $n \in \mathbb{N}$, where C is a fixed constant.

Remark 3. Using standard methods, one can prove Theorem 2 for stochastically dominated random variables. More precisely, Theorem 2 remains true if the assumption “let $\{X_k\}_{k \in \mathbb{N}}$ be a sequence of equidistributed r.v.’s” is replaced by the assumption “let $\{X_k\}_{k \in \mathbb{N}}$ be a sequence of random variables being stochastically dominated by some random variable X ”. Then, moreover, instead of the condition (3), we suppose that $\mathbb{E}\varphi^{-1}(|X|) < \infty$.

Now, we turn to the problem of eliminating $\mathbb{E}Y_k$ from the expressions (4) and (5). Applying the same calculations as in the proof of Theorem 3.2 in TANG [14], we can prove the following theorem.

Theorem 3. *Assume that the conditions of Theorem 2 are valid. Moreover:*

- (a) *in the case $\int_1^\infty \frac{1}{\varphi(x)} dx < \infty$, we assume additionally that there exists $C_1 > 0$, such that $\int_r^\infty \frac{1}{\varphi(x)} dx \leq \frac{C_1 r}{\varphi(r)}$, for every $r \geq 1$;*
- (b) *in the case $\int_1^\infty \frac{1}{\varphi(x)} dx = \infty$, we assume additionally that $\frac{x}{\varphi(x)}$ is nondecreasing and there exists $C_2 > 0$, such that $\int_1^t \frac{1}{\varphi(x)} dx \leq \frac{C_2 t}{\varphi(t)}$, for every $t \geq 1$, and furthermore, $\mathbb{E}X_k = 0$, for every $k \in \mathbb{N}$.*

If (a) or (b) is satisfied, then the series

$$\sum_{k=1}^{\infty} \frac{X_k}{\varphi(k)}$$

is almost surely convergent. If, in addition, f is increasing and $\lim_{y \rightarrow \infty} f(y) = \infty$, then

$$\frac{1}{f(n)} \sum_{k=1}^n \frac{X_k}{g(k)} \rightarrow 0,$$

almost surely, as $n \rightarrow \infty$.

Remark 4. Theorem 3 is also true for stochastically dominated r.v.’s.

3. Examples

In this section, we present how Theorem 2 works for some sequences of dependent r.v.’s.

3.1. AANA sequences. Let us begin with the definition of asymptotically almost negatively associated (AANA) random variables, introduced by CHANDRA and GHOSAL in [1].

Definition 2. A sequence $\{X_k\}_{k \in \mathbb{N}}$ of r.v.'s is called asymptotically almost negatively associated (AANA), if there exists a sequence $\{q(m)\}_{m \in \mathbb{N}}$ of nonnegative numbers, converging to zero, such that

$$\begin{aligned} & \text{Cov}(f(X_m), g(X_{m+1}, \dots, X_{m+k})) \\ & \leq q(m) (\text{Var}(f(X_m)) \text{Var}(g(X_{m+1}, \dots, X_{m+k})))^{1/2}, \end{aligned} \tag{8}$$

for all $m, k \geq 1$ and for all coordinatewise increasing continuous functions f and g whenever the right side of (8) is finite.

AANA sequences contain negatively associated (NA) sequences (see [1] and [9]) and hence, in particular, independent sequences.

In the proof of the Marcinkiewicz–Zygmund SLLN for AANA sequence of nonidentically distributed r.v.'s, CHANDRA and GHOSAL [1] used a maximal inequality at a crucial step. Precisely, they proved that if $A := \sum_{k=1}^{\infty} q^2(k) < \infty$ and $\mathbb{E}X_k = 0, k = 1, 2, \dots$, then

$$\mathbb{E} \left(\max_{1 \leq l \leq n} \sum_{k=1}^l X_k \right)^2 \leq C \sum_{k=1}^n \mathbb{E}X_k^2, \quad \text{where } C = \left(A + (1 + A^2)^{1/2} \right)^2. \tag{9}$$

YUAN and AN in [17] showed that if $\{X_k\}_{k \in \mathbb{N}}$ is a sequence of AANA r.v.'s with mixing coefficients $\{q(k)\}_{k \in \mathbb{N}}$, and f_1, f_2, \dots are all nondecreasing (nonincreasing) continuous functions, then $\{f_k(X_k)\}_{k \in \mathbb{N}}$ is also AANA with the same mixing coefficients. Furthermore, under the same assumptions as in (9), they obtained a more general maximal inequality for AANA r.v.'s:

$$\mathbb{E} \left(\max_{1 \leq l \leq n} \left| \sum_{k=1}^l X_k \right| \right)^p \leq C_{p,A} \sum_{k=1}^n \mathbb{E}|X_k|^p, \tag{10}$$

where $C_{p,A}$ is a constant dependent on $p \in (1, 2]$, and $A = \sum_{k=1}^{\infty} q^2(k) < \infty$.

Hence, for fixed $m, n \in \mathbb{N}, m < n$, considering r.v.'s $Y_k \equiv 0$ for $1 \leq k \leq m-1$ and Y_k defined according to formula (1), for $m \leq k \leq n$, we can write for $p = 2$

$$\begin{aligned} & \mathbb{E} \left(\max_{m \leq l \leq n} \left| \sum_{k=m}^l \frac{Y_k - \mathbb{E}Y_k}{\varphi(k)} \right| \right)^2 = \mathbb{E} \left(\max_{1 \leq l \leq n} \left| \sum_{k=1}^l \frac{Y_k - \mathbb{E}Y_k}{\varphi(k)} \right| \right)^2 \\ & \leq C_{p,A} \sum_{k=1}^n \mathbb{E} \left(\frac{Y_k - \mathbb{E}Y_k}{\varphi(k)} \right)^2 = C_{p,A} \sum_{k=m}^n \mathbb{E} \left(\frac{Y_k - \mathbb{E}Y_k}{\varphi(k)} \right)^2. \end{aligned}$$

As a result, the second Kolmogorov type maximal inequality (2), for moments of r.v.'s $\frac{Y_k - \mathbb{E}Y_k}{\varphi(k)}$, is satisfied. Therefore, the result of TANG (see [14, Theorem 3.1]) follows as the corollary.

Corollary 1. *Let $\{X_k\}_{k \in \mathbb{N}}$ be a sequence of equidistributed AANA r.v.'s satisfying the summability condition $\sum_{k=1}^{\infty} q^2(k) < \infty$. Let the functions f, g and r.v.'s Y_k be defined as in Theorem 2, f be increasing, and $\lim_{y \rightarrow \infty} f(y) = \infty$. If conditions (A1), (A2) and (3) of Theorem 2 hold, then*

$$\frac{1}{f(n)} \sum_{k=1}^n \frac{X_k - \mathbb{E}Y_k}{g(k)} \rightarrow 0, \quad \text{almost surely, as } n \rightarrow \infty. \tag{11}$$

Remark 5. The above Corollary 1 remains true in the stochastically dominated case.

3.2. ρ^- -mixing sequences. In this subsection, we consider ρ^- -mixing sequences of random variables and prove that for such sequences (3) is not only a sufficient but also a necessary condition for the weighted strong law of large numbers in the Jajte's form.

Let us begin with the definition which is due to ZHANG and WANG (see [18]).

Definition 3. A sequence $\{X_k\}_{k \in \mathbb{N}}$ is called ρ^- -mixing, if

$$\rho^-(s) = \sup \{ \rho^-(S, T) : S, T \subset \mathbb{N}, \text{dist}(S, T) \geq s \} \rightarrow 0, \quad s \rightarrow \infty,$$

where $\text{dist}(S, T) = \min \{ |m - n| : m \in S, n \in T \}$, and

$$\rho^-(S, T) := \max \left(0; \sup \left\{ \frac{\text{Cov}(f(X_i, i \in S), g(X_j, j \in T))}{\sqrt{\text{Var}(f(X_i, i \in S)) \text{Var}(g(X_j, j \in T))}} \right\} \right),$$

with the supremum (in the definition of the mixing coefficient $\rho^-(\cdot, \cdot)$) running over all coordinatewise nondecreasing functions f and g .

Remark 6. ZHANG and WANG (see [18]) noted that ρ^- -mixing r.v.'s include negatively associated sequences and ρ^* -mixing r.v.'s ($\rho^-(s) \leq \rho^*(s)$). Furthermore, increasing functions defined on disjoint subsets of ρ^- -mixing sequence $\{X_k\}_{k \in \mathbb{N}}$ with mixing coefficients $\rho^-(s)$ are also ρ^- -mixing with coefficients not greater than $\rho^-(s)$. Let us also note that ρ^* -mixing r.v.'s (also called ρ -mixing r.v.'s), introduced by Kolmogorov and Rozanov in 1960 (see [6] for the references), have been studied extensively and have found a lot of applications.

In [16], WANG and LU studied the properties of ρ^- -mixing r.v.'s satisfying the following condition:

$$\lim_{n \rightarrow \infty} \rho^-(n) \leq r, \quad 0 \leq r < \left(\frac{1}{6p} \right)^{p/2}, \quad \text{for some } p \geq 2. \tag{12}$$

They obtained the Rosenthal-type maximal inequality as follows.

Lemma 2. For a positive integer $N \geq 1$, positive real numbers $p \geq 2$ and $0 \leq r < \left(\frac{1}{6p}\right)^{p/2}$, if $\{X_k\}_{k \in \mathbb{N}}$ is a sequence of r.v.'s with $\rho^-(N) \leq r$, $\mathbb{E}X_k = 0$ and $\mathbb{E}|X_k|^p < \infty$ for every $k \geq 1$, then there exists a positive constant $D_{p,N,r} = D(p, N, r)$ such that, for all $n \geq 1$,

$$\mathbb{E} \left(\max_{1 \leq k \leq n} \left| \sum_{j=1}^k X_j \right| \right)^p \leq D_{p,N,r} \left(\sum_{k=1}^n \mathbb{E}|X_k|^p + \left(\sum_{k=1}^n \mathbb{E}X_k^2 \right)^{p/2} \right). \tag{13}$$

It is easy to see that for $p = 2$ (and thus $0 \leq r < 1/12$) we have an inequality of the form (10). Proceeding exactly as in the case of AANA r.v.'s, we show that the second Kolmogorov type maximal inequality (2) for moments of r.v.'s $\frac{Y_k - \mathbb{E}Y_k}{\varphi(k)}$ is, again, satisfied.

Now, we are in a position to prove that the sufficient condition (3) for the weighted SLLN in Theorem 2 is indeed the necessary one, as well. In other words, we can finally prove the following result.

Theorem 4. Let $\{X_k\}_{k \in \mathbb{N}}$ be a sequence of ρ^- -mixing equidistributed r.v.'s with mixing coefficients satisfying condition $\sum_{n=1}^\infty \rho^-(n) < \infty$. Let also the functions f, g and r.v.'s Y_k be defined as in Theorem 2; furthermore, f is increasing, and $\lim_{y \rightarrow \infty} f(y) = \infty$. Under assumptions (A1) and (A2) of Theorem 2, condition (3), i.e. $\mathbb{E}\varphi^{-1}(|X_1|) < \infty$, is equivalent to

$$\frac{1}{f(n)} \sum_{k=1}^n \frac{X_k - \mathbb{E}Y_k}{g(k)} \rightarrow 0, \quad \text{almost surely, as } n \rightarrow \infty. \tag{14}$$

We shall need the following version of the second Borel–Cantelli lemma.

Lemma 3. Let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of events such that $\sum_{n=1}^\infty P(A_n) = \infty$, and for all $n, k \in \mathbb{N}$,

$$P(A_k \cap A_{n+k}) \leq P(A_k)P(A_{n+k}) + \rho^-(k) \frac{P(A_n) + P(A_{n+k})}{2}, \tag{15}$$

where $\{\rho^-(k)\}_{k \in \mathbb{N}}$ is a sequence of nonnegative numbers such that $\sum_{j=1}^\infty \rho^-(j) < \infty$. Then $P(\limsup A_n) = 1$.

PROOF. Without loss of generality, we may and do assume that $P(A_1) > 0$, then $s := \sum_{i=1}^n P(A_i) > 0$, and moreover,

$$P(\cup_{i=1}^n A_i) > 0, \quad \text{for each } n \in \mathbb{N}. \tag{16}$$

In 1952, CHUNG and ERDŐS (see e.g. [2]) showed that for a sequence $\{A_n\}_{n \in \mathbb{N}}$ of events satisfying condition (16),

$$P(\cup_{i=1}^n A_i) \geq \frac{(\sum_{i=1}^n P(A_i))^2}{\sum_{i=1}^n \sum_{j=1}^n P(A_i \cap A_j)} = \frac{s^2}{s + 2 \sum_{1 \leq i < j \leq n} P(A_i \cap A_j)}. \tag{17}$$

In order to proceed further, we need to derive the following estimation.

$$\sum_{1 \leq i < j \leq n} P(A_i \cap A_j) = \sum_{i=1}^{n-1} \sum_{j=i+1}^n P(A_i \cap A_j) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} P(A_i \cap A_{j+i}),$$

which, by (15),

$$\begin{aligned} &\leq \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} P(A_i)P(A_{j+i}) + \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \frac{\rho^-(j)}{2} (P(A_i) + P(A_{j+i})) \\ &= \frac{s^2 - \sum_{i=1}^n P^2(A_i)}{2} + \sum_{i=1}^{n-1} \frac{P(A_i)}{2} \sum_{j=1}^{n-i} \rho^-(j) + \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \frac{\rho^-(j)}{2} P(A_{j+i}) \\ &\leq \frac{s^2 - \sum_{i=1}^n P^2(A_i)}{2} + \frac{s}{2} \sum_{j=1}^{\infty} \rho^-(j) + \sum_{1 \leq i < j \leq n} \frac{\rho^-(i)}{2} P(A_j), \end{aligned}$$

and denoting $c := \sum_{j=1}^{\infty} \rho^-(j) < \infty$,

$$\leq \frac{s^2 - \sum_{i=1}^n P^2(A_i)}{2} + \frac{s}{2} \sum_{j=1}^{\infty} \rho^-(j) + \frac{s}{2} \sum_{j=1}^{\infty} \rho^-(j) \leq \frac{s^2}{2} + sc.$$

Returning to inequality (17), we get $P(\cup_{i=1}^n A_i) \geq \frac{s}{1+s+2c}$.

Approaching the generalized second Borel–Cantelli lemma, let us now assume that $\sum_{n=1}^{\infty} P(A_n) = \infty$. Then, for $m \in \mathbb{N}$,

$$P(\cup_{i=m}^{\infty} A_i) \geq P(\cup_{i=m}^{m+n} A_i) \geq \frac{s_n}{1 + s_n + 2c},$$

where $s_n := \sum_{i=m}^{m+n} P(A_i) \rightarrow \infty, n \rightarrow \infty$. Hence we have $P(\cup_{i=m}^{\infty} A_i) \geq 1$. Thus,

$$P(\limsup A_i) = P(\cap_{m=1}^{\infty} \cup_{i=m}^{\infty} A_i) = \lim_{m \rightarrow \infty} P(\cup_{i=m}^{\infty} A_i) = 1. \quad \square$$

PROOF OF THEOREM 4. Sufficiency follows from Theorem 2. To prove necessity, let us observe that as in [7], from (14) it follows that $X_n/\varphi(n) \rightarrow 0$, almost surely. Define the events $A_n = \{X_n/\varphi(n) \geq 1\}$, then $\mathbb{I}_{A_n} = \mathbb{I}_{(\varphi(n), \infty)}(X_n)$ is a nondecreasing function of X_n , and by the ρ^- -mixing property, we have

$$\begin{aligned} \text{Cov}(\mathbb{I}_{A_n}, \mathbb{I}_{A_{n+k}}) &= P(A_n \cap A_{n+k}) - P(A_n)P(A_{n+k}) \\ &\leq \rho^-(k) \sqrt{P(A_n)(1 - P(A_n))} \sqrt{P(A_{n+k})(1 - P(A_{n+k}))} \\ &\leq \rho^-(k) \sqrt{P(A_n)} \sqrt{P(A_{n+k})} \leq \rho^-(k) \frac{P(A_n) + P(A_{n+k})}{2}. \end{aligned}$$

Similarly, for $B_n = \{-X_n/\varphi(n) \geq 1\}$, we see that $-\mathbb{I}_{B_n} = -\mathbb{I}_{(-\infty, -\varphi(n))}(X_n)$ is a nondecreasing function of X_n , thus

$$P(B_n \cap B_{n+k}) - P(B_n)P(B_{n+k}) \leq \rho^-(k) \frac{P(B_n) + P(B_{n+k})}{2}.$$

Consequently, by the standard technique, we can apply Lemma 3. If it were $\sum_{n=1}^\infty P(A_n) = \infty$, then $P(A_n, \text{i.o.}) = 1$, which contradicts $X_n/\varphi(n) \rightarrow 0$, almost surely. Thus, $\sum_{n=1}^\infty P(A_n) < \infty$, and similarly, $\sum_{n=1}^\infty P(B_n) < \infty$. We therefore conclude that $\sum_{n=1}^\infty P(|X_n/\varphi(n)| \geq 1) = \sum_{n=1}^\infty P(|X_1| \geq \varphi(n)) < \infty$, implying $\mathbb{E}\varphi^{-1}(|X_1|) < \infty$. \square

The Marcinkiewicz-Zygmund strong law of large numbers for negatively associated or AANA sequences has been proved only in its direct part (see [14], for the classical i.i.d. case we refer the reader to [3]). In the next theorem, we prove also the converse part in the case of ρ^- -mixing sequences. We derive this result directly from Theorem 2 and Lemma 3.

Theorem 5. *Let $\{X_k\}_{k \in \mathbb{N}}$ be a sequence of ρ^- -mixing, equidistributed r.v.'s with mixing coefficients satisfying the condition $\sum_{n=1}^\infty \rho^-(n) < \infty$. Then, for any $0 < p < 2$,*

$$\frac{1}{n^{1/p}} \sum_{k=1}^n (X_k - c) \rightarrow 0, \quad \text{almost surely, as } n \rightarrow \infty \tag{18}$$

if and only if

$$\mathbb{E}|X_1|^p < \infty. \tag{19}$$

If (19) is satisfied, then $c = \mathbb{E}X_1$ in the case $1 \leq p < 2$; while c is arbitrary (and may be taken as equal to 0) in the case $0 < p < 1$.

PROOF. Similarly, like in [3], from (18) it follows that $X_n/n^{1/p} \rightarrow 0$, almost surely, as $n \rightarrow \infty$. Proceeding as in the proof of Theorem 4, by Lemma 3 we get $\sum_{n=1}^\infty P(|X_1| \geq n^{1/p}) < \infty$, thus (19) holds.

From (19) and Theorem 4 it follows that

$$\frac{1}{n^{1/p}} \sum_{k=1}^n (X_k - \mathbb{E}Y_k) \longrightarrow 0, \quad \text{almost surely, as } n \rightarrow \infty, \quad (20)$$

with $\mathbb{E}Y_k = -k^{1/p}P(X_k < -k^{1/p}) + \mathbb{E}X_k \mathbb{I}[|X_k| \leq k^{1/p}] + k^{1/p}P(X_k > k^{1/p})$.

We start with the case $0 < p < 1$. Let us define the following functions:

$$f_k(x) = k^{1/p} \mathbb{I}_{(-\infty, -k^{1/p}) \cup (k^{1/p}, +\infty)}(x) + |x| \mathbb{I}_{(-k^{1/p}, k^{1/p})}(x).$$

The sequence $\{f_k\}_{k \in \mathbb{N}}$ is nondecreasing, and therefore

$$0 \leq f_k(x) \leq f_n(x), \quad \text{for } 1 \leq k \leq n \text{ and every } x \in \mathbb{R},$$

furthermore,

$$\frac{f_n(x)}{n^{1/p-1}} \leq |x|^p.$$

Thus

$$\begin{aligned} \left| \frac{1}{n^{1/p}} \sum_{k=1}^n \mathbb{E}Y_k \right| &\leq \frac{1}{n^{1/p}} \sum_{k=1}^n \mathbb{E}|Y_k| = \frac{1}{n^{1/p}} \sum_{k=1}^n \mathbb{E}f_k(X_k) = \frac{1}{n^{1/p}} \sum_{k=1}^n \mathbb{E}f_k(X_1) \\ &\leq \frac{\mathbb{E}f_n(X_1)}{n^{1/p-1}}, \end{aligned} \quad (21)$$

on account that $\frac{f_n(X_1)}{n^{1/p-1}} \rightarrow 0$, almost surely, $\frac{f_n(X_1)}{n^{1/p-1}} \leq |X_1|^p$ and $\mathbb{E}|X_1|^p < \infty$, from (21) and Lebesgue's dominated convergence theorem, we get $|\frac{1}{n^{1/p}} \sum_{k=1}^n \mathbb{E}Y_k| \rightarrow 0$, as $n \rightarrow \infty$. Thus $\frac{1}{n^{1/p}} \sum_{k=1}^n (X_k - c) \rightarrow 0$, almost surely, as $n \rightarrow \infty$, for any constant c .

In the remaining case $1 \leq p < 2$, we have to prove that

$$\frac{1}{n^{1/p}} \sum_{k=1}^n (\mathbb{E}Y_k - \mathbb{E}X_1) \longrightarrow 0.$$

Let us observe that

$$\begin{aligned} \left| \frac{1}{n^{1/p}} \sum_{k=1}^n (\mathbb{E}Y_k - \mathbb{E}X_1) \right| &\leq \frac{1}{n^{1/p}} \sum_{k=1}^n \left(k^{1/p} P(|X_1| > k^{1/p}) + \mathbb{E}|X_1| \mathbb{I}[|X_1| > k^{1/p}] \right) \\ &\leq \frac{2}{n^{1/p}} \sum_{k=1}^n k^{1/p-1} \mathbb{E}|X_1|^p \mathbb{I}[|X_1|^p > k] \rightarrow 0, \quad \text{as } n \rightarrow \infty, \end{aligned}$$

by the Toeplitz theorem on regular transformation of sequences into sequences (see [10]), which may be applied since $\frac{1}{n^{1/p}} \sum_{k=1}^n k^{1/p-1} \rightarrow p$, as $n \rightarrow \infty$ and $\mathbb{E}|X_1|^p \mathbb{I} [|X_1|^p > k] \rightarrow 0$, as $k \rightarrow \infty$. \square

We complete this section with constructing an example of a sequence of negatively associated r.v.'s with the same distribution, for which the results of this section may be applied.

Example 1. Let us consider a gaussian sequence $\{\xi_k\}_{k \in \mathbb{N}}$ such that ξ_k 's have the standard normal distribution, and $\text{Cov}(\xi_i, \xi_j) = -a^{i+j}$, $i \neq j$ with $0 < a \leq \frac{1}{2}$. These r.v.'s are negatively correlated gaussian, and according to [9], are negatively associated. To show that such a sequence exists, it suffices to prove that the matrix $\mathbf{A}_n = [a_{ij}]$ where $a_{ij} = -a^{i+j}$, for $i \neq j$ and $a_{ii} = 1$ is positive-definite, and it is indeed the covariance matrix of the vector $[\xi_1, \xi_2, \dots, \xi_n]$. Let us write $\mathbf{A}_n = \mathbf{I} - \mathbf{B} + \mathbf{D}$, where \mathbf{I} is a unit matrix, \mathbf{D} is a diagonal matrix with entries $d_{ii} = a^{2i}$, and $\mathbf{B} = [b_{ij}]$ with $b_{ij} = a^{i+j}$.

Let us focus on the matrix \mathbf{B} . We put $\mathbf{x} = [x_1, \dots, x_n]$, by the Cauchy-Schwarz inequality we have

$$\mathbf{x}\mathbf{B}\mathbf{x}^T = \left(\sum_{i=1}^n a^i x_i \right)^2 \leq \sum_{i=1}^n a^{2i} \sum_{i=1}^n x_i^2 \leq \frac{a^2}{1-a^2} \|\mathbf{x}\|^2 \leq \|\mathbf{x}\|^2.$$

Thus

$$\mathbf{x}\mathbf{A}_n\mathbf{x}^T = \|\mathbf{x}\|^2 - \mathbf{x}\mathbf{B}\mathbf{x}^T + \mathbf{x}\mathbf{D}\mathbf{x}^T > 0,$$

provided $\mathbf{x} \neq \mathbf{0}$.

Now, let us denote by Φ the standard normal distribution, and let F be any continuous distribution with the quasi-inverse F^{-1} . Define $X_k = F^{-1}(\Phi(\xi_k))$, then $\{X_k\}_{k \in \mathbb{N}}$ is a sequence of negatively associated r.v.'s with the same distribution F . Therefore, for such a sequence $\{X_k\}_{k \in \mathbb{N}}$, (18) holds iff $\int |x|^p dF(x) < \infty$.

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