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Zeros and irreducibility of Stern polynomials

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Abstract. The classical Stern sequence was extended by Klavžar, Milutinović and Petr to the Stern polynomials $B_n(z)$ defined by $B_0(z) = 0$, $B_1(z) = 1$, $B_{2n}(z) = zB_n(z)$, and $B_{2n+1}(z) = B_n(z) + B_{n+1}(z)$. Ulas conjectured that $B_p(z)$ is irreducible whenever p is a prime, and verified this for the first 10^6 primes, while Schinzel proved the conjecture for a certain class of primes. In this paper, we show that the conjecture is true for various further classes of primes, which is achieved by the use of different new results on the distribution of the zeros of certain classes of $B_n(z)$, also proved in this paper. Some of these results can be seen as variants of the classical theorem of Kakeya and Eneström.

1. Introduction

The *Stern sequence* $\{a(n)\}_{n\geq 0}$ is defined by $a(0) = 0$, $a(1) = 1$, and for $n \geq 1$ by

$$a(2n) = a(n), \qquad a(2n+1) = a(n) + a(n+1). \tag{1.1}$$

The sequence starts as 0, 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3, . . .; see [4] for some historical remarks and for some properties of this sequence. Perhaps the most remarkable property is the fact that the terms $a(n), a(n+1)$ are always relatively prime, and that each positive reduced rational number occurs once and only once in the sequence $\{a(n)/a(n+1)\}_{n\geq 1}$.

In recent years, the Stern sequence was extended to two different sequences of polynomials, independently by the first author and K. B. Stolarsky [4], and

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by Klavžar, Milutinović and Petr [6]. The polynomial sequence of Klavžar et al. is defined by $B_0(z) = 0$, $B_1(z) = 1$, and for $n \geq 1$,

$$B_{2n}(z) = zB_n(z), \tag{1.2}$$

$$B_{2n+1}(z) = B_n(z) + B_{n+1}(z); \tag{1.3}$$

see Table 1 for the first 32 Stern polynomials. By comparison with (1.1), we see immediately that

$$B_n(1) = a(n) \quad (n \geq 0), \tag{1.4}$$

and an easy induction shows that

$$B_n(2) = n \quad (n \geq 0). \tag{1.5}$$

Numerous interesting properties of these polynomials were derived in [6], including connections with "hyperbinary representations" and the "standard Gray code". Also, identities and a recurrence relation for the degrees of the $B_n(z)$ are derived in that paper.

n	$B_n(z)$	n	$B_n(z)$
1	1	17	z^3+z^2+2z+1
2	z	18	z^3+2z^2+z
3	$z+1$	19	$3z^2+3z+1$
4	z^2	20	$2z^3+z^2$
5	$2z+1$	21	$3z^2+4z+1$
6	z^2+z	22	z^3+3z^2+z
7	z^2+z+1	23	z^3+2z^2+3z+1
8	z^3	24	z^4+z^3
9	z^2+2z+1	25	z^3+3z^2+2z+1
10	$2z^2+z$	26	$2z^3+2z^2+z$
11	z^2+3z+1	27	z^3+3z^2+3z+1
12	z^3+z^2	28	$z^4+z^3+z^2$
13	$2z^2+2z+1$	29	$2z^3+2z^2+2z+1$
14	z^3+z^2+z	30	$z^4+z^3+z^2+z$
15	z^3+z^2+z+1	31	$z^4+z^3+z^2+z+1$
16	z^4	32	z^5

Table 1. $B_n(z)$, $1 \leq n \leq 32$.

Further properties of the Stern polynomials were obtained in the more recent papers by M. Ulas [14], [15]; they end with a number of conjectures, including the following one [14, Conjecture 6.4], which was verified by calculation for the first million primes.

Conjecture 1.1 (Ulas). *For any prime p the Stern polynomial $B_p(z)$ is irreducible over the rationals.*

Subsequently, A. Schinzel [13] obtained various results on the factors of Stern polynomials, and he proved Conjecture 1.1 for the following special case.

Theorem 1.2 (Schinzel). *For all integers $n \geq 3$, $B_{2^n-3}(z)$ is irreducible over the rationals.*

Furthermore, Schinzel proved, without computations, that $B_p(z)$ is irreducible for all primes $p < 2017$.

It is one of the purposes of this paper to obtain further results on the irreducibility of Stern polynomials. While Schinzel's Theorem 1.2 was proved by way of Eisenstein's irreducibility criterion, we will apply a different criterion, namely that of A. Cohn, which we state here as quoted in [2].

Theorem 1.3 (Cohn). *Let $f(z) = a_0 + a_1 z + \cdots + a_n z^n \in \mathbb{Z}[z]$ have the zeros $\alpha_1, \ldots, \alpha_n$, and suppose that there is an integer b for which $f(b)$ is a prime. If $f(b-1) \neq 0$, and*

$$b > \mathrm{Re}(\alpha_j) + \frac{1}{2} \quad \text{for} \quad 1 \leq j \leq n, \tag{1.6}$$

then $f(z)$ is irreducible over the rationals.

If we take $b = 2$ and $f(z) = B_p(z)$, where p is a prime, then, by (1.5), one of the conditions in Theorem 1.3 is already satisfied. Furthermore, we clearly have $f(b-1) = B_p(1) > 0$ since the Stern polynomials have only nonnegative coefficients. We, therefore, have the following consequence of Theorem 1.3.

Corollary 1.4. *If p is a prime and the zeros of $B_p(z)$ all lie in the half-plane $\{z \in \mathbb{C} \mid \mathrm{Re(z)} < \frac{3}{2}\}$, then $B_p(z)$ is irreducible over the rationals.*

This means that we need to study the zero distribution of Stern polynomials, which will be the main part of this paper. We begin with some further properties of Stern polynomials, given in Section 2. The main purpose of Section 3 is then to derive a number of general results extending the classical theorem of Eneström and Kakeya, and apply it to classes of Stern polynomials. In Section 4, we obtain results on the zeros of different but related classes of Stern polynomials. While Sections 3 and 4 are mainly concerned with zeros in a half-plane, in Section 5 we prove results on zeros in an open disk, and then return to irreducibility questions. We conclude this paper with a few further remarks in Section 6.

2. Some properties of Stern polynomials

In this brief section we quote, or derive, some properties of the Stern polynomials that will be used in later sections. We begin with a pair of identities, the first of which is due to Schinzel [13, Lemma 1].

Lemma 2.1. *For all nonnegative integers a, m, and r with $0 \le r \le 2^a$, we have*

$$B_{m2^a+r}(z) = B_{2^a-r}(z)B_m(z) + B_r(z)B_{m+1}(z), \tag{2.1}$$

and for odd $m \ge 1$,

$$B_{m2^a+r}(z) = B_{2^a+r}(z)B_m(z) - B_r(z)B_{m-1}(z). \tag{2.2}$$

Proof. It remains to prove (2.2). If we multiply both sides of (1.3) by z and use (1.2), then with $m = 2n+1$ we get

$$zB_m(z) = B_{m+1}(z) + B_{m-1}(z). \tag{2.3}$$

We then multiply both sides of this by $B_r(z)$ and note that (2.1) with $m = 1$ gives

$$B_{2^a+r}(z) - B_{2^a-r}(z) = zB_r(z). \tag{2.4}$$

Hence we get, with (2.3),

$$B_m(z)\left(B_{2^a+r}(z) - B_{2^a-r}(z)\right) = B_r(z)\left(B_{m+1}(z) + B_{m-1}(z)\right),$$

or

$$B_{2^a-r}(z)B_m(z) + B_r(z)B_{m+1}(z) = B_{2^a+r}(z)B_m(z) - B_r(z)B_{m-1}(z).$$

This, together with (2.1), immediately gives (2.2). □

Next, we consider two subsequences of the Stern polynomials that are related to the following interesting property of the Stern sequence (1.1). In each interval $2^{n-2} \le m \le 2^{n-1}$ the maximum value of $a(m)$ is the Fibonacci number F_n. It was apparently first shown by LEHMER [7] that this maximum occurs at

$$\alpha_n := \frac{1}{3}\left(2^n - (-1)^n\right) \quad \text{and} \quad \beta_n := \frac{1}{3}\left(5 \cdot 2^{n-2} + (-1)^n\right) \quad (n \ge 2), \tag{2.5}$$

where α_n is also defined for $n = 0, 1$; see Table 2 for the first few values of both sequences.

n	0	1	2	3	4	5	6	7	8	9	10
α_n	0	1	1	3	5	11	21	43	85	171	341
β_n			2	3	7	13	27	53	107	213	427

Table 2. α_n, β_n, $1 \le n \le 10$.

Numerous properties of these sequences can be found in [9] under A001045 and A048573, respectively. Here we mention only the recurrence relations

$$\alpha_{n+1} = 2\alpha_n + (-1)^n, \qquad \beta_{n+1} = 2\beta_n - (-1)^n, \tag{2.6}$$

which immediately follow from (2.5). Also, by (1.4) and the remark preceding (2.5), we have

$$B_{\alpha_n}(1) = B_{\beta_n}(1) = F_n \quad (n \ge 2). \tag{2.7}$$

We now state and prove two recurrence relations.

Lemma 2.2. *The following identities hold:*

$$B_{\alpha_{n+1}}(z) = B_{\alpha_n}(z) + zB_{\alpha_{n-1}}(z) \quad (n \ge 1), \tag{2.8}$$
$$B_{\beta_{n+1}}(z) = B_{\beta_n}(z) + zB_{\beta_{n-1}}(z) \quad (n \ge 3), \tag{2.9}$$

with the initial conditions $B_{\alpha_0}(z) = 0$, $B_{\alpha_1}(z) = 1$, $B_{\beta_2}(z) = z$, *and* $B_{\beta_3}(z) = z + 1$.

PROOF. Using the recurrence relations (1.3) and (2.6), we get

$$B_{\alpha_{n+1}}(z) = B_{2\alpha_n + (-1)^n}(z) = B_{\alpha_n}(z) + B_{\alpha_n + (-1)^n}(z). \tag{2.10}$$

Using (2.6) again, we see that with (1.2) we have

$$B_{\alpha_n + (-1)^n}(z) = B_{2\alpha_{n-1}}(z) = zB_{\alpha_{n-1}}(z).$$

This, combined with (2.10), gives (2.8). The proof of (2.9) is almost identical, and the initial conditions are easily obtained from Tables 1 and 2. $\square$

Lemma 2.2 can now be used to derive explicit expansions for the two sequences $B_{\alpha_n}(z)$, $B_{\beta_n}(z)$. The recurrence relation (2.8), along with its initial conditions, is a special case of the well-known sequence of the bivariate Fibonacci polynomials (also known as Lucas sequences) defined by $F_0(x,y) = 0$, $F_1(x,y) = 1$, and

$$F_k(x,y) = xF_{k-1}(x,y) + yF_{k-2}(x,y) \quad (k \geq 1). \tag{2.11}$$

These polynomials go back to at least Lucas [8], and they are known to have the explicit expansion

$$F_{n+1}(x,y) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{j} x^{n-2j} y^j. \tag{2.12}$$

Comparing (2.11) with (2.8), we see that $B_{\alpha_n}(z) = F_n(1,z)$, and therefore (2.12) immediately gives the first of the following two identities.

Lemma 2.3. *For all $n \geq 2$, we have*

$$B_{\alpha_n}(z) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-j}{j} z^j, \tag{2.13}$$

$$B_{\beta_n}(z) = 1 + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n-1-j}{j-1} + \binom{n-3-j}{j} \right) z^j, \tag{2.14}$$

where (2.13) *also holds for* $n = 0, 1$.

Proof. Only (2.14) remains to be proven. To do so, we note that

$$B_{\beta_n}(z) = zB_{\alpha_{n-1}}(z) + B_{\alpha_{n-2}}(z) \quad (n \geq 2). \tag{2.15}$$

This comes from the fact that both sides of (2.15) satisfy the same recurrence relation, and we have the initial conditions $B_{\beta_2}(z) = z \cdot 1 + 0$ and $B_{\beta_3}(z) = z \cdot 1 + 1$, as required. Finally, we obtain (2.14) from (2.15) and (2.13) after some easy manipulations. $\square$

3. Zeros of Stern polynomials, I.

The following notation will be used for a half-plane that occurs repeatedly throughout the remainder of this paper:

$$\mathbb{D} := \{z \in \mathbb{C} \mid \mathrm{Re(z)} < 1\}.$$

We begin this section with an observation which we formulate as a conjecture.

Conjecture 3.1. *The zeros of all Stern polynomials $B_m(z)$, $m \geq 2$, lie in $\mathbb{D}$.*

By the identity (1.2), we only need to consider odd indices m. We verified this conjecture numerically for all $m \leq 10^7$. See also Figure 1, which shows most, but not all, of the zeros for $m \leq 2^{16}$ (see Proposition 3.3 in this connection). A proof of Conjecture 3.1 would immediately imply Conjecture 1.1, by Corollary 1.4. In this section and the next, we will prove a variety of partial results.

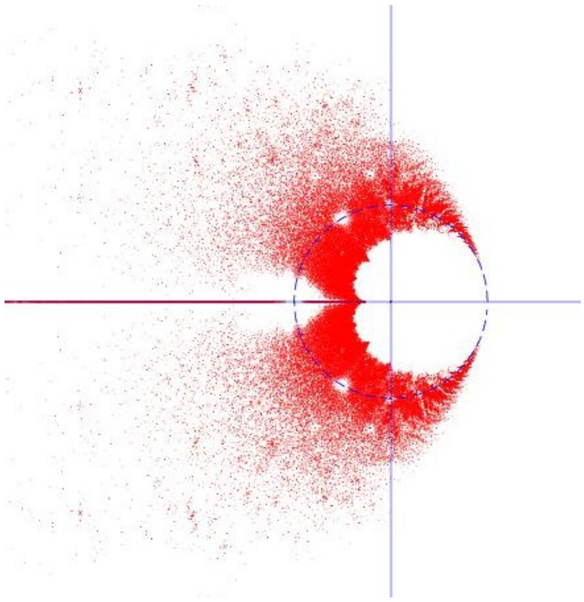

Figure 1. Zeros of $B_m(z)$, $m \leq 2^{16}$, with unit circle.

3.1. Polynomials with explicit zeros. We begin with an easy result which also shows that Conjecture 3.1 is best possible.

Proposition 3.2. *For every $\nu \geq 2$, all zeros of $B_{2^\nu-1}(z)$ lie in $\mathbb{D}$. Furthermore, the supremum of the set of real parts of the zeros of $B_{2^\nu-1}(z)$, for all $\nu \geq 2$, is* 1.

Proof. We can say more about the zeros of $B_{2^\nu-1}(z)$. It is easy to see by induction, using the recurrences (1.2) and (1.3), that

$$B_{2^\nu-1}(z) = z^{\nu-1} + z^{\nu-2} + \cdots + z + 1 = \frac{z^\nu - 1}{z-1}. \tag{3.1}$$

This shows that the zeros are all the ν-th roots of unity, except $z = 1$. Both parts of the proposition follow immediately. $\square$

We now consider a second class of Stern polynomials with an explicit expansion, for which we can determine the zeros explicitly. The Fibonacci polynomials defined by (2.11) are closely related to the Chebyshev polynomials of the second kind defined by the recurrence relation $U_0(x) = 1$, $U_1(x) = 2x$, and $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$. Among numerous properties (see, e.g., [10, Chapter 18]) is the explicit expansion

$$U_n(x) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n-j}{j} (2x)^{n-2j}.$$

Comparing this with (2.13), we see that

$$B_{\alpha_{n+1}}(\tfrac{-1}{4x^2}) = \frac{1}{(2x)^n} U_n(x). \tag{3.2}$$

Another well-known property of the polynomials $U_n(x)$ is the fact that all their zeros are real and lie in the interval $(-1, 1)$; moreover, they are explicitly given by $x_j := \cos(k\pi/(n+1))$, $j = 1, \ldots, n$. This fact, with (3.2), leads to the following result.

Proposition 3.3. *For $\alpha_n = (2^n - (-1)^n)/3$, the zeros of $B_{\alpha_n}(z)$ are real and negative, and are given by*

$$z_j := -\frac{1}{4} \sec^2\left(\frac{\pi j}{n}\right), \qquad j = 1, 2, \ldots, \lfloor \tfrac{n-1}{2} \rfloor.$$

In particular, this shows that zeros of Stern polynomials can be negative numbers with arbitrarily large absolute values. One can also observe that among the zeros z_j above there are the rational values $-\frac{1}{3}$, $-\frac{1}{2}$, and -1. Interestingly, it was recently proved by Gawron [5] that these three numbers, along with 0, are the only possible rational zeros of Stern polynomials, and that each of them is a zero of infinitely many Stern polynomials. This was earlier conjectured by Ulas [14].

While the polynomials $B_{\beta_n}(z)$ also have an explicit expansion, their zeros cannot be given explicitly, and it appears that the polynomials in this class all have a pair of nonreal zeros, in addition to negative real zeros. We did not pursue this further.

3.2. The Eneström–Kakeya theorem and variants. A different approach involves the relative sizes of the coefficients of $B_m(z)$; the key is the following remarkable theorem of Eneström and Kakeya; see, e.g., [12, III.22].

Theorem 3.4 (Eneström, Kakeya)**.** *If $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ satisfies*

$$0 < a_0 \le a_1 \le \cdots \le a_n, \tag{3.3}$$

then all the zeros of $f(z)$ lie on the unit disk $|z| \le 1$.

To apply this theorem, we note that for $\nu \ge 3$ we have

$$B_{2^\nu - 3}(z) = 1 + 2z + 2z^2 + \cdots + 2z^{\nu-2}. \tag{3.4}$$

To see this, we write $2^\nu - 3 = (2^{\nu-1} - 2) + (2^{\nu-1} - 1) = 2(2^{\nu-2} - 1) + (2^{\nu-1} - 1)$, so that with (1.3) and (1.2) we get

$$B_{2^\nu - 3}(z) = zB_{2^{\nu-2}-1}(z) + B_{2^{\nu-1}-1}(z).$$

The identity (3.4) then follows from (3.1). Theorem 3.4 now gives the following result, if we note that $z = 1$ can never be a zero.

Proposition 3.5. *For any $\nu \ge 3$, all the zeros of $B_{2^\nu - 3}(z)$ lie inside or on the unit circle, and in particular in $\mathbb{D}$.*

Theorem 3.4 also applies to $B_{2^\nu - 1}(z)$ and its reciprocal, not giving us anything new. While it appears that these two classes are the only ones to which this theorem directly applies, the idea behind the usual proof of Theorem 3.4 can be adapted to make it more applicable to our situation. Our first result in this direction is as follows.

Theorem 3.6. *Let $n \ge 1$. If $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ satisfies*

$$0 < a_0 \le a_1 \le \cdots \le a_{n-1} > a_n > 0, \tag{3.5}$$

then all the zeros of $f(z)$ lie in $\mathbb{D}$.

Proof. We multiply $f(z)$ by $z - 1$, obtaining

$$\begin{aligned}(z-1)f(z) = a_n z^{n+1} &+ (a_{n-1} - a_n)z^n \\ &- \left[(a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_1 - a_0) + a_0\right].\end{aligned} \tag{3.6}$$

Let $N(z)$ be the expression in square brackets on the right-hand side of (3.6). Then, for $|z| \geq 1$ we have, keeping (3.5) in mind,

$$\begin{aligned} |N(z)| &\leq |a_{n-1} - a_{n-2}| \cdot |z|^{n-1} + \cdots + |a_1 - a_0| \cdot |z| + a_0 \\ &\leq ((a_{n-1} - a_{n-2}) + \cdots + (a_1 - a_0) + a_0)\, |z|^{n-1} = a_{n-1}|z|^{n-1}. \end{aligned} \tag{3.7}$$

On the other hand, if we assume that $\mathrm{Re}(z) \geq 1$, then

$$\begin{aligned} \left|a_n z^{n+1} + (a_{n-1} - a_n) z^n\right| &= |a_n z + (a_{n-1} - a_n)| \cdot |z|^n \\ &\geq |a_n + (a_{n-1} - a_n)| \cdot |z|^n = a_{n-1}|z|^n. \end{aligned} \tag{3.8}$$

Now, (3.6) together with (3.7) and (3.8) shows that

$$|(z-1)f(z)| \geq a_{n-1}(|z| - 1)|z|^{n-1} > 0, \tag{3.9}$$

whenever $|z| > 1$ and $\mathrm{Re}(z) \geq 1$. These conditions are satisfied for all $z \notin \mathbb{D}$, with the exception of $z = 1$. But certainly $f(1) > 0$ by (3.5), and so by (3.9) we have $f(z) \neq 0$, whenever $z \notin \mathbb{D}$. This completes the proof. □

This result can be applied to a few classes of Stern polynomials. Indeed, we can show the following.

Lemma 3.7. *For all ν as indicated, we have*

$$B_{2^\nu-5}(z) = 1 + 3z + \cdots + 3z^{\nu-3} + z^{\nu-2} \quad (\nu \geq 4), \tag{3.10}$$

$$B_{2^\nu-7}(z) = 1 + 2z + 3z^2 + \cdots + 3z^{\nu-3} + z^{\nu-2} \quad (\nu \geq 5), \tag{3.11}$$

$$B_{2^\nu-11}(z) = 1 + 4z + 5z^2 + \cdots + 5z^{\nu-4} + 3z^{\nu-3} \quad (\nu \geq 6), \tag{3.12}$$

$$B_{2^\nu-13}(z) = 1 + 3z + 5z^2 + \cdots + 5z^{\nu-4} + 3z^{\nu-3} \quad (\nu \geq 6), \tag{3.13}$$

where the dots indicate constant coefficients.

Proof. We use the same argument as that following (3.4), and write

$$\begin{aligned} 2^\nu - 5 &= 2(2^{\nu-2} - 1) + (2^{\nu-1} - 3), & 2^\nu - 11 &= 2(2^{\nu-2} - 3) + (2^{\nu-1} - 5), \\ 2^\nu - 7 &= 4(2^{\nu-3} - 1) + (2^{\nu-1} - 3), & 2^\nu - 13 &= 2(2^{\nu-2} - 3) + (2^{\nu-1} - 7). \end{aligned}$$

Then we use the identities (1.3) and (1.2), together with (3.1) and (3.4), to obtain (3.10) and (3.11). Finally, we use these last two identities, along with (3.4) again, to get (3.12) and (3.13). □

Theorem 3.6, applied to (3.10)–(3.13), immediately leads to the following consequences.

Corollary 3.8. *If $m = 2^\nu - k$ for $k \in \{5, 7, 11, 13\}$ and ν is such that $2^\nu > k$, then all the zeros of $B_m(z)$ lie in $\mathbb{D}$.*

This is clear for all ν satisfying the bounds given in Lemma 3.7, while for small values of ν the statement of Corollary 3.8 is easy to verify by computation.

We now prove a variant of Theorem 3.6.

Theorem 3.9. *Let $n \geq 2$. If $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ satisfies*

$$0 < a_0 \leq a_1 \leq \cdots \leq a_{n-2} > a_{n-1} \geq a_n > 0, \quad \text{and} \tag{3.14}$$

$$(a_n + a_{n-1})^2 \geq 2 a_n a_{n-2}, \tag{3.15}$$

then all the zeros of $f(z)$ lie in $\mathbb{D}$.

Proof. We proceed as in the proof of Theorem 3.6, and consider

$$\begin{aligned}(z-1)f(z) &= a_n z^{n+1} + (a_{n-1} - a_n) z^n + (a_{n-2} - a_{n-1}) z^{n-1} \\ &\quad - \left[(a_{n-2} - a_{n-3}) z^{n-2} + \cdots + (a_1 - a_0) z + a_0\right].\end{aligned} \tag{3.16}$$

We denote the expression in square brackets again by $N(z)$, and with the same analysis as in (3.7) we get

$$|N(z)| \leq a_{n-2} |z|^{n-2}, \quad \text{whenever} \quad |z| \geq 1. \tag{3.17}$$

To consider the first three terms on the right of (3.16), we set $g(z) := az^2 + bz + c$, with

$$a := a_n, \qquad b := a_{n-1} - a_n, \qquad c := a_{n-2} - a_{n-1}. \tag{3.18}$$

Now, with $z = x + iy$, where $x, y \in \mathbb{R}$, a straightforward calculation gives

$$\begin{aligned}|g(z)|^2 &= \left(ax^2 - ay^2 + bx + c\right)^2 + (2axy + by)^2 \\ &= \left(a^2x^4 + 2abx^3 + (2ac + b^2)x^2 + 2bcx + c^2\right) \\ &\quad + y^2\left(a^2y^2 + b^2 - 2ac + 2a^2x^2 + 2abx\right) \\ &\geq \left(a^2 + 2ab + 2ac + b^2 + 2bc + c^2\right) + y^2\left(b^2 - 2ac + 2a^2 + 2ab\right)\end{aligned}$$

for $x \geq 1$. So, if

$$2a^2 + 2ab + b^2 \geq 2ac, \tag{3.19}$$

then

$$|g(z)|^2 \geq (a+b+c)^2, \quad \text{or} \quad |g(z)| \geq a+b+c = a_{n-2}, \tag{3.20}$$

where we have used (3.18) for the last identity. Now, combining (3.16), (3.17) and (3.20), the conclusion of the proof is exactly as in the proof of Theorem 3.6, with $n-1$ replaced by $n-2$ in (3.9). Finally, we note that by (3.18) the condition (3.19) is equivalent to (3.15). □

Theorem 3.9 can be applied to more classes of Stern polynomials.

Lemma 3.10. *If $m = 2^\nu - k$ for $k \in \{9, 15, 19, 21, 23, 25, 27, 29, 43, 45, 51, 53\}$, then $B_m(z)$ satisfies (3.14) for $\nu \geq \nu_k$, where ν_k and the coefficients are as listed in Table 3.*

k	deg	ν_k	coefficients $a_0, a_1, \ldots$
9	$n-2$	6	1, 3, 4, …, 4, 2, 1
15	$n-2$	7	1, 2, 3, 4, …, 4, 2, 1
19	$n-3$	7	1, 4, 7, …, 7, 5, 2
21	$n-3$	7	1, 5, 8, …, 8, 6, 1
23	$n-3$	8	1, 4, 6, 7, …, 7, 5, 2
25	$n-3$	8	1, 3, 6, 7, …, 7, 5, 2
27	$n-3$	8	1, 4, 7, 8, …, 8, 6, 1
29	$n-3$	8	1, 3, 5, 7, …, 7, 5, 2
43	$n-4$	9	1, 6, 12, 13, …, 13, 11, 4
45	$n-4$	9	1, 5, 10, 12, …, 12, 10, 5
51	$n-4$	9	1, 4, 9, 12, …, 12, 10, 5
53	$n-4$	9	1, 5, 10, 13, …, 13, 11, 4

Table 3. $B_{2^\nu-k}(z)$, $\nu \geq \nu_k$.

The entries in Table 3, and thus Lemma 3.10, can be obtained in the same way as the identities in Lemma 3.7. It is now easy to check that the last three coefficients in each of the entries in Table 3 satisfy the condition (3.15). Therefore, Theorem 3.9 leads to the following extension of Corollary 3.8.

Corollary 3.11. *If $m = 2^\nu - k$ for $k \in \{9, 15, 19, 21, 23, 25, 27, 29, 43, 45, 51, 53\}$ and ν is such that $2^\nu > k$, then all the zeros of $B_m(z)$ lie in $\mathbb{D}$.*

For each k as in this corollary, if $\nu < \nu_k$, then the conclusion is easy to check numerically. Alternatively, one can check whether Theorems 3.4, 3.6 or 3.9 apply.

We can go one step further and prove another variant of Theorems 3.6 and 3.9.

Theorem 3.12. *Let $n \geq 3$. If $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ satisfies*

$$0 < a_0 \leq a_1 \leq \cdots \leq a_{n-3} > a_{n-2} \geq a_{n-1} \geq a_n > 0, \quad \text{and} \tag{3.21}$$

$$(a_{n-1} - a_n)(2a_n - a_{n-1} + a_{n-2}) \geq \frac{7}{4}(a_{n-3} - a_{n-2})^2, \tag{3.22}$$

then all the zeros of $f(z)$ lie in $\mathbb{D}$.

Proof. Once again we proceed as before. We consider

$$(z-1)f(z) = g(z)z^{n-2} - N(z), \tag{3.23}$$

where

$$g(z) := a_n z^3 + (a_{n-1} - a_n)z^2 + (a_{n-2} - a_{n-1})z + (a_{n-3} - a_{n-2}), \tag{3.24}$$

$$N(z) := (a_{n-3} - a_{n-4})z^{n-3} + \cdots + (a_1 - a_0)z + a_0, \tag{3.25}$$

and we set

$$a := a_n, \quad b := a_{n-1} - a_n, \quad c := a_{n-2} - a_{n-1}, \quad d := a_{n-3} - a_{n-2}. \tag{3.26}$$

In analogy to the proof of Theorem 3.9, we have

$$\begin{aligned} |g(z)|^2 &= \left(ax^3 - 3axy^2 + bx^2 - by^2 + cx + d\right)^2 + \left(3ax^2y - ay^3 + 2bxy + cy\right)^2 \\ &= A(x) + y^2 B(x,y), \end{aligned} \tag{3.27}$$

where

$$\begin{aligned} A(x) &= a^2x^6 + 2abx^5 + (2ac + b^2)x^4 + (2ad + 2bc)x^3 + (2bd + c^2)x^2 + 2cdx + d^2 \\ &\geq a^2 + b^2 + c^2 + d^2 + 2(ab + ac + ad + bc + bd + cd) = (a+b+c+d)^2, \end{aligned} \tag{3.28}$$

provided that $x \geq 1$, and where

$$\begin{aligned} B(x,y) :&= a^2y^4 + \left(3a^2x^2 + 2abx + b^2 - 2ac\right)y^2 \\ &\quad + 3a^2x^4 + 4abx^3 + 2b^2x^2 + (2bc - 6ad)x + c^2 - 2bd \\ &= \left(a^2y^4 - 2acy^2 + c^2\right) + \left(b^2 + 3a^2x^2 + 2abx\right)y^2 + C(x) \\ &= \left(ay^2 - c\right)^2 + \left(b^2 + 3a^2x^2 + 2abx\right)y^2 + C(x). \end{aligned}$$

Here, we have set

$$C(x) := 3a^2x^4 + 4abx^3 + 2b^2x^2 + (2bc - 6ad)x - 2bd.$$

For $x \geq 1$, we use the obvious facts that $x^4 \geq x^2$ and $x^2 \geq 1$, and completing the square in two instances, we get

$$\begin{aligned} C(x) &\geq \left(3a^2x^2 - 6adx + 3d^2\right) - 3d^2 + \left(2b^2 - 2bd + \tfrac{1}{2}d^2\right) - \tfrac{1}{2}d^2 + 4ab + 2bc \\ &= 3(ax-d)^2 + 2(b - \tfrac{1}{2}d)^2 + 4ab + 2bc - \tfrac{7}{2}d^2 \geq 2b(2a+c) - \tfrac{7}{2}d^2. \end{aligned}$$

Therefore, if

$$4b(2a+c) \geq 7d^2, \tag{3.29}$$

then $C(x) \geq 0$, and, consequently, $B(x,y) \geq 0$ for all $y \in \mathbb{R}$ and $x \geq 1$. Hence, by (3.27) and (3.28) we have

$$|g(z)| \geq a + b + c + d = a_{n-3},$$

where we have also used (3.26). Finally, the conclusion follows from (3.23) and (3.25) as in the previous proofs; it only remains to note that (3.22) is equivalent to (3.29), via (3.26). □

In analogy to Theorem 3.9, we now apply Theorem 3.12 to the following classes of Stern polynomials.

Lemma 3.13. *If $m = 2^\nu - k$, with k as listed in Table 4, then $B_m(z)$ satisfies (3.21) for $\nu \geq \nu_k$, where ν_k and the highest coefficients $a_{n-3}, a_{n-2}, a_{n-1}$ and a_n are shown in Table 4.*

k	ν_k	$a_{n-3}, \dots, a_n$	k	ν_k	$a_{n-3}, \dots, a_n$	k	ν_k	$a_{n-3}, \dots, a_n$
17	8	5, 3, 2, 1	75	10	18, 16, 9, 3	107	11	21, 19, 10, 1
31	9	5, 3, 2, 1	77	10	17, 15, 10, 3	109	11	19, 17, 10, 2
35	9	9, 7, 4, 2	83	10	19, 17, 10, 2	115	11	17, 15, 10, 3
37	9	11, 9, 4, 1	85	10	21, 19, 10, 1	117	11	18, 16, 9, 3
39	9	10, 8, 5, 1	87	11	18, 16, 9, 3	171	12	34, 32, 21, 5
41	9	11, 9, 4, 1	89	11	17, 15, 10, 3	173	12	31, 29, 20, 7
47	10	9, 7, 4, 2	91	11	19, 17, 10, 2	179	12	29, 27, 20, 8
49	10	9, 7, 4, 2	93	11	16, 14, 9, 4	181	12	31, 29, 20, 7
55	10	11, 9, 4, 1	99	11	16, 14, 9, 4	203	12	31, 29, 20, 7
57	10	10, 8, 5, 1	101	11	19, 17, 10, 2	205	12	29, 27, 20, 8
59	10	11, 9, 4, 1	103	11	17, 15, 10, 3	211	12	31, 29, 20, 7
61	10	9, 7, 4, 2	105	11	18, 16, 9, 3	213	12	34, 32, 21, 5

Table 4. $B_{2^\nu - k}(z)$, $\nu \geq \nu_k$.

Although we listed the first few coefficients in each of the entries of Table 3, they are not needed, and for the sake of compactness, we do not include them in Table 4. The respective degrees of the entries in Table 4 are $n = \nu - 2$ for $k = 17$ and 31, $n = \nu - 3$ for $35 \leq k \leq 61$, $n = \nu - 4$ for $75 \leq k \leq 117$, and $n = \nu - 5$ for $k \geq 171$. As a specific example, written out in full, we mention

$$B_{2^\nu-17}(z) = 1 + 3z + 4z^2 + 5z^3 + \cdots + 5z^{\nu-5} + 3z^{\nu-4} + 2z^{\nu-3} + z^{\nu-2},$$

valid for $\nu \geq \nu_{17} = 8$. The entries in Table 4 can again be obtained as in the proof of Lemma 3.7. We can now state an extension of Corollaries 3.8 and 3.11.

Corollary 3.14. *If $m = 2^\nu - k$ with k as in Table 4, except for $k = 17$ and $k = 31$, and if ν is such that $2^\nu > k$, then all the zeros of $B_m(z)$ lie in $\mathbb{D}$.*

This result is obtained by checking that the coefficients $a_{n-3}, \ldots, a_n$ in Table 4 satisfy the condition (3.22) in Theorem 3.12. In practice it is easier to consider the polynomials $(z-1)B_m(z)$ for checking the condition (3.21) along with the condition (3.29). With the exception of $k = 17$ and $k = 31$, all other values of k in Table 4 satisfy these two conditions. While Theorem 3.12 does not apply to these two cases, computations indicate that they still satisfy Conjecture 3.1.

We conclude this section with a few remarks. First, we note that all Stern polynomials we considered here are unimodal. In fact, we believe that this is always true:

Conjecture 3.15. *The coefficients of any Stern polynomial $B_m(z)$ form a unimodal sequence of nonnegative integers.*

We verified this conjecture numerically with PARI [11] for all $m \leq 10^{11}$. In the next section, we will encounter further infinite classes of Stern polynomials which are seen to be unimodal.

We also saw in this section that Stern polynomials of the form $B_{2^\nu-k}(z)$, for fixed small integers k, are such that the decrease in the coefficient sequence occurs only near the end, i.e., near the leading coefficient. Let n' be the largest coefficient for which $a_{n'-1} \leq a_{n'}$ and $n = \deg B_{2^\nu-k}(z)$. Then we saw in (3.1) and (3.4) that $n' = n$ occurs when $k = 1$ and 3, while by Lemma 3.7 we have $n' = n - 1$ when $k \in \{5, 7, 11, 13\}$. Furthermore, Lemmas 3.10 and 3.13 show that $n' = n - 2$ for 12 odd values of k, and $n' = n - 3$ for 36 odd values of k. Our computations seem to indicate that there are no further odd values of k in each of these categories.

We now summarize the results concerning Stern polynomials obtained in this section:

Corollary 3.16. *Let k be odd and $k \in \{1 \le k \le 61 \mid k \ne 17, 31, 33\}$, or $k = 75, 77, 83, \dots, 93, 99, \dots, 109, 115, 117, 171, 173, 179, 181, 203, 205, 211$, or 213. Then, for all ν such that $2^\nu > k$, the zeros of $B_{2^\nu - k}(z)$ all lie in $\mathbb{D}$.*

Finally, with regards to the "gap" $k = 33$, we note that with the usual methods one can obtain

$$B_{2^\nu-33}(z) = 1+3z+4z^2+5z^3+6z^4+\dots+6z^{\nu-6}+4z^{\nu-5}+3z^{\nu-4}+2z^{\nu-3}+z^{\nu-2},$$

valid for $\nu \ge 10$. Dealing with this class, and with numerous others, would require a further extension of the general theorems of Section 3, with increasingly complicated conditions. However, in Section 5 we will obtain some weaker results for this case, and for some other gaps in Corollary 3.16.

4. Zeros of Stern polynomials, II.

The main purpose of this section is to show that all the zeros of $B_{2^\nu+k}(z)$, for all k as treated in the previous section, and for all $\nu \ge 1$, also lie in $\mathbb{D}$. In this case, the proofs are specific to Stern polynomials, and therefore we do not have analogues of the general Theorems 3.6, 3.9, and 3.12.

The main connection between the "$+k$" and the "$-k$" case is the identity

$$B_{2^\nu+k}(z) = B_{2^\nu-k}(z) + zB_k(z) \quad (k \ge 0, 2^\nu \ge k), \tag{4.1}$$

which follows from (2.1) by setting $m = 1$. As a first application, we take $k = 1$; then (4.1) together with (3.1) gives

$$B_{2^\nu+1}(z) = z^{\nu-1} + \dots + z^2 + 2z + 1 \quad (\nu \ge 3), \tag{4.2}$$

where the dots once again indicate constant coefficients. Similarly, (4.1) with $k = 3$, combined with (3.4) and the fact that $B_3(z) = z + 1$, gives

$$B_{2^\nu+3}(z) = 2z^{\nu-2} + \dots + 2z^3 + 3z^2 + 3z + 1 \quad (\nu \ge 5). \tag{4.3}$$

Multiplying both sides of the identities (4.2) and (4.3) by $z - 1$, we get

$$(z-1)B_{2^\nu+1}(z) = z^\nu + z^2 - z - 1 \quad (\nu \ge 3), \tag{4.4}$$

$$\begin{aligned}(z-1)B_{2^\nu+3}(z) &= 2z^{\nu-1} + z^3 - 2z - 1 \\ &= 2z^{\nu-1} + (z^2 - z - 1)(z+1) \quad (\nu \ge 5).\end{aligned} \tag{4.5}$$

These identities show that the quadratic polynomial $z^2 - z - 1$ plays a special role, which, as we shall see, extends to other classes of Stern polynomials. The following two inequalities are important tools for this and for the following section, respectively.

Lemma 4.1. *With $z = x + iy$, where $x, y \in \mathbb{R}$, we have*

$$|z^2 - z - 1| \le |z|^3 \quad \textit{when} \quad x \ge 1, \tag{4.6}$$

$$|z^2 - z - 1| \le \tfrac{19}{9}|z|^2 \quad \textit{when} \quad |z| \ge \tfrac{3}{2}. \tag{4.7}$$

Proof. Let $h(z) := z^2 - z - 1$. Easy calculations show that

$$\begin{aligned} |h(z)|^2 &= x^4 - 2x^3 - x^2 + 2x + 1 + y^2 \left(y^2 + 2x^2 - 2x + 3\right), \\ |z|^6 &= x^6 + 3x^4y^2 + 3x^2y^4 + y^6, \end{aligned}$$

so that

$$\begin{aligned} |h(z)|^2 - |z|^6 = {} & (x^4 - x^6) + 2(x - x^3) + (1 - x^2) \\ & + y^2 \left(-3x^4 + 2x^2 - 2x + 3 + y^2(-y^2 - 3x^2 + 1)\right). \end{aligned} \tag{4.8}$$

Now, we note that

$$-3x^4 + 2x^2 - 2x + 3 = (1 - x)(3x^3 + 3x^2 + x + 3) \le 0 \quad \text{for} \quad x \ge 1,$$

and that clearly $-y^2 - 3x^2 + 1 < 0$ for $x \ge 1$. Hence, all terms on the right of (4.8) are nonpositive when $x \ge 1$, which immediately gives (4.6).

Finally, we note that for $|z| \ge \frac{3}{2}$ we have

$$\frac{|z^2 - z - 1|}{|z|^2} \le \frac{|z|^2 + |z| + 1}{|z|^2} = 1 + \frac{1}{|z|} + \frac{1}{|z|^2} \le 1 + \frac{2}{3} + \frac{4}{9} = \frac{19}{9},$$

which proves (4.7). $\square$

We are now ready to deal with the first two classes of Stern polynomials in this section.

Proposition 4.2. *For integers $\nu \ge 1$, all zeros of $B_{2^\nu+1}(z)$ and $B_{2^\nu+3}(z)$ lie in $\mathbb{D}$.*

Proof. By (4.4) and (4.6) we have, for $\nu \ge 3$,

$$|(z - 1)B_{2^\nu+1}(z)| \ge |z|^\nu - |z^2 - z - 1| \ge |z|^\nu - |z|^3 \quad (x \ge 1).$$

Hence, for $\nu \ge 4$, the right-hand side is positive, with the exception of $z = 1$. However, we know that $B_n(1) > 0$ for any n. This proves the first statement for $\nu \ge 4$.

Next, by (4.5) and (4.6) we have, for $\nu \geq 5$,

$$|(z-1)B_{2^\nu+3}(z)| \geq 2|z|^{\nu-1} - |z|^3|z+1| \quad (x \geq 1). \tag{4.9}$$

Now, $|z+1| \leq 2|z|$ when $|z| \geq 1$. Hence, the right-hand side of (4.9) is positive when $x \geq 1$, with the exception of $z = 1$, a case that is resolved as before. This proves the second statement for $\nu \geq 5$. The remaining cases in both statements for small $\nu \geq 1$ are easy to verify by computation. □

To motivate our next results, we begin by extending the identities (4.4) and (4.5).

Lemma 4.3. *For all ν as indicated, we have*

$$(z-1)B_{2^\nu+5}(z) = (z+2)z^{\nu-2} + (2z^3 - z^2 - 3z - 1) \quad (\nu \geq 6), \tag{4.10}$$
$$(z-1)B_{2^\nu+7}(z) = (z+2)z^{\nu-2} + (z^4 - z^2 - 2z - 1) \quad (\nu \geq 7), \tag{4.11}$$
$$(z-1)B_{2^\nu+11}(z) = (3z+2)z^{\nu-3} + (z^4 + 2z^3 - 3z^2 - 4z - 1) \quad (\nu \geq 8), \tag{4.12}$$
$$(z-1)B_{2^\nu+13}(z) = (3z+2)z^{\nu-3} + (2z^4 - 3z^2 - 3z - 1) \quad (\nu \geq 8). \tag{4.13}$$

Proof. From (3.10), with (4.1) and the corresponding entry in Table 1, we get

$$B_{2^\nu+5}(z) = z^{\nu-2} + 3z^{\nu-3} + \cdots + 3z^3 + 5z^2 + 4z + 1 \quad (\nu \geq 6),$$

with similar expressions for $B_{2^\nu+k}(z)$ with $k = 7$, 11 and 13, which follow from (3.11)–(3.13), respectively. Upon multiplying each of these by $z-1$, we get the desired identities (4.10)–(4.13). □

Upon factoring the low-degree polynomials on the right-hand sides of (4.10)–(4.13), we can see that they are equal to $(z^2-z-1)B_k(z)$, with $k = 5, 7, 11$ and 13, respectively. The polynomials in (4.4) and (4.5) follow the same pattern. These are special cases of the following result, which we state and prove together with an analogue relevant to the previous section.

Proposition 4.4. *Let $\mu \geq 0$ and $0 \leq \ell \leq 2^\mu - 1$ be integers such that $k := 2^\mu + \ell$ is odd. Then, for $\nu \geq \mu$,*

$$(z-1)B_{2^\nu+k}(z) = \big(B_k(z) - (z-1)B_\ell(z)\big)z^{\nu-\mu} + (z^2 - z - 1)B_k(z), \tag{4.14}$$
$$(z-1)B_{2^\nu-k}(z) = \big(B_k(z) - (z-1)B_\ell(z)\big)z^{\nu-\mu} - B_k(z). \tag{4.15}$$

PROOF. We use (2.2) with $m = 2^{\nu-\mu}+1$, $a = \mu$, and $r = \ell$; then we get

$$B_{(2^{\nu-\mu}+1)2^\mu+\ell}(z) = B_{2^{\nu-\mu}+1}(z)B_{2^\mu+\ell}(z) - B_\ell(z)B_{2^{\nu-\mu}}(z).$$

Now, we note that by (1.2) and (4.2), respectively, we have

$$B_{2^{\nu-\mu}}(z) = z^{\nu-\mu}, \qquad B_{2^{\nu-\mu}+1}(z) = \frac{z^{\nu-\mu}-1}{z-1} + z,$$

both of which hold for all $\nu \ge \mu$. Then, with $k = 2^\mu + \ell$, we have

$$B_{2^\nu+k}(z) = \left(\frac{z^{\nu-\mu}-1}{z-1} + z\right) B_k(z) - B_\ell(z)z^{\nu-\mu}.$$

Multiplying both sides by $z-1$, we immediately get (4.14).

Finally, we obtain (4.15) by multiplying both sides of (4.1) by $z-1$, and equating the left-hand side with that of (4.14). □

Before continuing, we note that the polynomials $B_k(z)-(z-1)B_\ell(z)$, which occur in (4.14) and (4.15), have nonnegative coefficients. Indeed, by (4.1) we have

$$\begin{aligned} B_{2^\mu+\ell}(z) - (z-1)B_\ell(z) &= (B_{2^\mu-\ell}(z) + zB_\ell(z)) - (z-1)B_\ell(z) \\ &= B_{2^\mu-\ell}(z) + B_\ell(z). \end{aligned} \tag{4.16}$$

Since both polynomials on the right have nonnegative coefficients, this proves the claim. If $\mu \ge 1$ and ℓ is odd, then Conjecture 3.15 implies that the polynomial in question actually has positive coefficients.

We now use Proposition 4.4 to show that Corollaries 3.8, 3.11 and 3.14 also hold in the cases $m = 2^\nu + k$. In fact, we have the following analogue to Corollary 3.16.

Corollary 4.5. *The zeros of $B_{2^\nu+k}(z)$ all lie in $\mathbb{D}$ for all $\nu \ge 1$ and all odd $k \in \{1 \le k \le 61 \mid k \ne 17, 31, 33\}$, and for $k = 75, 77, 83, \dots, 93, 99, \dots, 109, 115, 117, 171, 173, 179, 181, 203, 205, 211,$ and 213.*

PROOF. For $k = 1$ and 3 this is just Proposition 4.2. The proof of all other cases follow the same general outline as that of Proposition 4.2. In particular, by (4.14) we have

$$|(z-1)B_{2^\nu+k}(z)| \ge |B_k(z) - (z-1)B_\ell(z)| \cdot |z|^{\nu-\mu} - |z^2 - z - 1| \cdot |B_k(z)|. \tag{4.17}$$

Now, for those k listed in the statement of the corollary we have, by (4.15), exactly one of the situations of the proofs of Theorem 3.6, 3.9, or 3.12. The only difference is the additional term $|z^2 - z - 1|$. However, by Lemma 4.1 we have $|z^2-z-1| \le |z|^3$ when $\mathrm{Re}(z) \ge 1$. Therefore, everything carries through as before, with the restrictions $\nu \ge \nu_k + 3$, where ν_k is given in Lemma 3.7 and in Tables 3 and 4. Finally, the small cases $1 \le \nu \le \nu_k + 2$ can be verified numerically. □

5. Irreducibility

While Sections 3 and 4 are devoted to those polynomials whose zeros lie in the half-plane $\mathbb{D}$, Corollary 1.4 indicates that for the purpose of proving irreducibility, some weaker results on the zero distribution will suffice. Before we state and prove two such results, we require the following inequalities.

Lemma 5.1. *For $|z| \geq \frac{3}{2}$ we have*

$$|B_{2^\mu+1}(z)| \leq \tfrac{157}{27}|z|^{\mu-1} \quad (\mu \geq 4), \tag{5.1}$$
$$|B_{2^\mu-1}(z)| \leq \tfrac{97}{27}|z|^{\mu-1} \quad (\mu \geq 4), \tag{5.2}$$
$$|B_{2^\mu+3}(z)| \leq \tfrac{722}{81}|z|^{\mu-2} \quad (\mu \geq 6), \tag{5.3}$$
$$|B_{2^\mu-3}(z)| \leq \tfrac{566}{81}|z|^{\mu-2} \quad (\mu \geq 6). \tag{5.4}$$

Proof. With (4.2), (3.1), (4.3) and (3.4), we have, respectively,

$$\begin{aligned}
B_{2^\mu+1}(z) &= \frac{z^\mu-1}{z-1}+z = \frac{z^\mu+z^2-z-1}{z-1},\\
B_{2^\mu-1}(z) &= \frac{z^\mu-1}{z-1},\\
B_{2^\mu+3}(z) &= 2\frac{z^{\mu-1}-1}{z-1}+z^2+z-1 = \frac{2z^{\mu-1}+z^3-2z-1}{z-1},\\
B_{2^\mu-3}(z) &= 2\frac{z^{\mu-1}-1}{z-1}-1 = \frac{2z^{\mu-1}-z-1}{z-1};
\end{aligned}$$

they actually hold for all $\mu \geq 2$. The first of these identities, together with the triangle inequality, now leads to

$$\frac{1}{|z|^{\mu-1}}\,|B_{2^\mu+1}(z)| \leq \frac{|z|^\mu+|z|^2+|z|+1}{|z|^{\mu-1}(|z|-1)} = \frac{1+\frac{1}{|z|^{\mu-2}}+\frac{1}{|z|^{\mu-1}}+\frac{1}{|z|^{\mu}}}{1-\frac{1}{|z|}}.$$

Using the fact that the right-hand term is a decreasing function both in $|z|$ and in μ, we get for $|z| \geq \frac{3}{2}$ and $\mu \geq 4$,

$$\frac{1}{|z|^{\mu-1}}\,|B_{2^\mu+1}(z)| \leq \frac{1+(\frac{2}{3})^2+(\frac{2}{3})^3+(\frac{2}{3})^4}{1-\frac{2}{3}} = \frac{157}{27},$$

which is equivalent to (5.1). Analogously, the other three identities give

$$\frac{1}{|z|^{\mu-1}}\,|B_{2^\mu-1}(z)| \leq \frac{|z|^\mu+1}{|z|^{\mu-1}(|z|-1)} \leq \frac{97}{27} \quad (\mu \geq 4),$$

$$\frac{1}{|z|^{\mu-2}}\,|B_{2^\mu+3}(z)| \le \frac{2|z|^{\mu-1}+|z|^3+2|z|+1}{|z|^{\mu-2}(|z|-1)} \le \frac{722}{81} \quad (\mu\ge 6),$$
$$\frac{1}{|z|^{\mu-2}}\,|B_{2^\mu-3}(z)| \le \frac{2|z|^{\mu-1}+|z|+1}{|z|^{\mu-2}(|z|-1)} \le \frac{566}{81} \quad (\mu\ge 6),$$

and these are equivalent to (5.2)–(5.4). □

We are now ready to prove two results on the zero distribution of certain classes of Stern polynomials.

Proposition 5.2. *Let $k\in\{17,31,33,63,65,67,125,127,129,131,253,255\}$. Then the zeros of $B_{2^\nu\pm k}(z)$, for all $\nu\ge 1$ for which $2^\nu\pm k$ is positive, all lie in the open disk $|z|<\frac{3}{2}$.*

Proof. We use the identities (4.14) and (4.15), with $k=2^\mu+\ell$ and $\ell=1,2,2^\mu-1$, and $2^\mu-3$, for appropriate exponents μ.

(1) First, with the right-hand side of (4.14) in mind, we note that by (4.2) we have

$$B_{2^\mu+1}(z)-(z-1)B_1(z)=z^{\mu-1}+\cdots+z^2+z+2=\frac{z^\mu+z-2}{z-1},$$

and thus, for $|z|\ge\frac{3}{2}$ and $\mu\ge 4$,

$$\frac{1}{|z|^{\mu-1}}\,|B_{2^\mu+1}(z)-(z-1)B_1(z)| \ge \frac{|z|^\mu-|z|-2}{|z|^{\mu-1}(|z|+1)} \ge \frac{5}{27}, \tag{5.5}$$

where the number on the right is obtained by substituting $\mu=4$ and $|z|=\frac{3}{2}$ in the middle term. Now (4.14), together with (5.5), (4.7) and (5.1) gives, with $k=2^\mu+1$,

$$|(z-1)B_{2^\nu+k}(z)| \ge \frac{5}{27}|z|^{\mu-1}|z|^{\nu-\mu}-\frac{19}{9}|z|^2\frac{157}{27}|z|^{\mu-1} = \frac{5}{27}|z|^{\mu+1}\left(|z|^{\nu-\mu-2}-\frac{2983}{45}\right).$$

This is strictly positive for all z with $|z|\ge\frac{3}{2}$, whenever $|\frac{3}{2}|^{\nu-\mu-2}>2983/45$, or, solving for ν,

$$\nu\ge\nu_1:=\left\lceil\frac{\log(2983/45)}{\log(3/2)}+\mu+2\right\rceil=\mu+13 \quad (\mu\ge 4), \tag{5.6}$$

where $\lceil x\rceil$ is the ceiling of $x\in\mathbb{R}$, i.e., the smallest integer $\ge x$.

To obtain the analogue for $B_{2^\nu-k}(z)$, we only need to note that (4.15) differs from (4.14) only in the absence of the term z^2-z-1, so that the corresponding modulus is also positive when (5.6) holds.

(2) We now consider $k = 2^{\mu+1} - 1$. Since $2^{\mu+1} - 1 = 2^\mu + (2^\mu - 1)$, we see by (4.16) with $\ell = 2^\mu - 1$ that

$$B_{2^{\mu+1}-1}(z) - (z-1)B_{2^\mu-1}(z) = B_{2^\mu+1}(z) - (z-1)B_1(z). \tag{5.7}$$

Using this, we see that (4.14), again with (5.5) and (4.7), and this time with (5.2) gives, for $k = 2^{\mu+1} - 1$,

$$|(z-1)B_{2^\nu+k}(z)| \geq \frac{5}{27}|z|^{\mu-1}|z|^{\nu-\mu} - \frac{19}{9}|z|^2\frac{97}{27}|z|^\mu = \frac{5}{27}|z|^{\mu+2}\left(|z|^{\nu-\mu-3} - \frac{1843}{45}\right),$$

and just as before, we see that this is strictly positive for $|z| \geq \frac{3}{2}$, whenever

$$\nu \geq \nu_2 := \left\lceil \frac{\log(1843/45)}{\log(3/2)} + \mu + 3 \right\rceil = \mu + 13 \quad (\mu \geq 4). \tag{5.8}$$

Furthermore, the analogue for $B_{2^\nu-k}(z)$ also holds when (5.8) is satisfied, by the same argument as before.

(3) Next, we note that by (4.3) we have

$$\begin{aligned} B_{2^\mu+3}(z) - (z-1)B_3(z) &= 2z^{\mu-2} + \cdots + 2z^2 + 3z + 2 = 2\frac{z^{\mu-1}-1}{z-1} + z \\ &= \frac{2z^{\mu-1} + z^2 - z - 2}{z-1}, \end{aligned}$$

and thus, for $|z| \geq \frac{3}{2}$ and $\mu \geq 6$,

$$\frac{1}{|z|^{\mu-2}}\,|B_{2^\mu+3}(z) - (z-1)B_3(z)| \geq \frac{2|z|^{\mu-1} - |z|^2 - |z| - 2}{|z|^{\mu-2}(|z|+1)} \geq \frac{302}{405}, \tag{5.9}$$

where the number on the right is obtained by substituting $\mu = 6$ and $|z| = \frac{3}{2}$ in the middle term. Now (4.14), together with (5.9), (4.7) and (5.3) gives, with $k = 2^\mu + 3$,

$$\begin{aligned} |(z-1)B_{2^\nu+k}(z)| &\geq \frac{302}{405}|z|^{\mu-2}|z|^{\nu-\mu} - \frac{19}{9}|z|^2\frac{722}{81}|z|^{\mu-2} \\ &= \frac{302}{405}|z|^\mu\left(|z|^{\nu-\mu-2} - \frac{34295}{1359}\right). \end{aligned}$$

This is strictly positive for all z with $|z| \geq \frac{3}{2}$, whenever

$$\nu \geq \nu_3 := \left\lceil \frac{\log(34295/1359)}{\log(3/2)} + \mu + 2 \right\rceil = \mu + 10 \quad (\mu \geq 6). \tag{5.10}$$

The analogue for $B_{2^\nu-k}(z)$ also holds, as before.

(4) Finally, with $k = 2^{\mu+1} - 3$, everything carries through as in part 2 above, with an analogue of (5.7) and using (5.4) with $\mu + 1$ in place of μ. Then we have

$$\begin{aligned}|(z-1)B_{2^\nu+k}(z)| &\geq \frac{302}{405}|z|^{\mu-2}|z|^{\nu-\mu} - \frac{19}{9}|z|^2\frac{566}{81}|z|^{\mu-1} \\ &= \frac{302}{405}|z|^{\mu+1}\left(|z|^{\nu-\mu-3} - \frac{26885}{1359}\right),\end{aligned}$$

and once again we see that for $|z| \geq \frac{3}{2}$ this is positive, whenever

$$\nu \geq \nu_4 := \left\lceil \frac{\log(26885/1359)}{\log(3/2)} + \mu + 3 \right\rceil = \mu + 11 \quad (\mu \geq 6), \tag{5.11}$$

with the same bound again for $B_{2^\nu-k}(z)$.

(5) We summarize the relevant k and the bounds $\nu_1, \ldots, \nu_4$ in Table 5:

μ	$2^\mu + 1$	ν_1	$2^{\mu+1} - 1$	ν_2	$2^\mu + 3$	ν_3	$2^{\mu+1} - 3$	ν_4
4	17	17	31	17				
5	33	18	63	18				
6	65	19	127	19	67	16	125	17
7	129	20	255	20	131	17	253	18

Table 5. The bounds ν_j in (5.6), (5.8), (5.10) and (5.11).

The result is now proved for all ν at least equal to the corresponding bounds in Table 5. All smaller cases are easy to verify by computation using Maple. □

We can now combine Corollary 3.16, Corollary 4.5 and Proposition 5.2 with Corollary 1.4 to obtain the following irreducibility result.

Theorem 5.3. *Suppose that the prime p is of the form $p = 2^\nu \pm k$, $\nu \geq 1$, where k is odd and $1 \leq k \leq 67$, or k is one of $75, 77, 83, \ldots, 93, 99, \ldots, 109, 115, 117,$ $125, 127,$ 129, 131, 171, 173, 179, 181, 203, 205, 211, 213, 253, 255. Then $B_p(z)$ is irreducible over $\mathbb{Q}$.*

It is clear from the proof that Proposition 5.2 can be extended as follows.

Proposition 5.4. *Let $\mu \geq 1$ and $\nu \geq \mu + 9$ be integers. Then all the zeros of $B_{2^\nu\pm2^\mu\pm1}(z)$ and $B_{2^\nu\pm2^\mu\pm3}(z)$ lie in the open disk $|z| < \frac{3}{2}$, where the two instances of "±" are independent in both cases.*

Proof. The idea of the proof is essentially the same as that of Proposition 5.2; we only need to improve some of the estimates in order to get the improved condition $\nu \geq \mu + 9$. We may now assume that $\mu \geq 8$ since the smaller cases were dealt with in Proposition 5.2. First, in analogy to (5.1) and its proof, we have, for $|z| \geq \frac{3}{2}$,

$$\frac{1}{|z|^{\mu-1}}\,|B_{2^\mu+1}(z)| \leq \frac{(\frac{3}{2})^8 + (\frac{3}{2})^2 + \frac{3}{2} + 1}{(\frac{3}{2})^7(\frac{1}{2})} = \frac{7777}{2187} \quad (\mu \geq 8).$$

Similarly, in analogy to (5.5), we have, again for $|z| \geq \frac{3}{2}$,

$$\frac{1}{|z|^{\mu-1}}\,|B_{2^\mu+1}(z) - (z-1)B_1(z)| \geq \frac{(\frac{3}{2})^8 - \frac{3}{2} - 2}{(\frac{3}{2})^7(\frac{3}{2}+1)} = \frac{1133}{2187} \quad (\mu \geq 8).$$

This leads to the estimate

$$\begin{aligned}|(z-1)B_{2^\nu+2^\mu+1}(z)| &\geq \frac{1133}{2187}|z|^{\mu-1}|z|^{\nu-\mu} - \frac{19}{9}\cdot\frac{7777}{2187}|z|^2|z|^{\mu-1}\\ &= \frac{1133}{2187}|z|^{\mu+1}\left(|z|^{\nu-\mu-2} - \frac{13433}{927}\right),\end{aligned}$$

which is positive, whenever

$$\nu \geq \left\lceil \frac{\log(13433/927)}{\log(3/2)} + \mu + 2 \right\rceil = \mu + 9 \quad (\mu \geq 8).$$

This proves the result for subscripts $2^\nu + 2^\mu + 1$ and $\mu \geq 8$, while $\mu \leq 7$ is covered by Proposition 5.2.

All the other cases are obtained by analogous modifications of the proof of Proposition 5.2; we leave the details to the reader. □

As an immediate consequence of Proposition 5.4, we now get our last irreducibility result, once again by Corollary 1.4.

Theorem 5.5. *Suppose that the prime p is of the form $p = 2^\nu \pm 2^\mu \pm 1$ or $p = 2^\nu \pm 2^\mu \pm 3$, where $\mu \geq 1$ and $\nu \geq \mu + 9$ are integers, and the instances of "±" are independent. Then $B_p(z)$ is irreducible over $\mathbb{Q}$.*

6. Further remarks

(1) The proofs in Section 5 indicate that various improvements and modifications of the results are possible. First, the radius $\frac{3}{2}$ of the disk in Propositions 5.2

and 5.4 was mainly chosen for simplicity, and with the aim of applying Corollary 1.4 in mind. The way we derived the estimates (5.1)–(5.4), as well as (5.5) and (5.9), indicates that the proof of Proposition 5.2 can be adapted to prove the following: *For any $\varepsilon > 0$, the zeros of $B_{2^\nu \pm 2^\mu \pm 1}(z)$ and $B_{2^\nu \pm 2^\mu \pm 3}(z)$ have modulus less than $1 + \varepsilon$, for ν sufficiently large depending on ε.*

Second, much of the proof of Proposition 5.2 is based on the particularly simple structure of the four classes of polynomials $B_{2^\mu \pm 1}(z)$ and $B_{2^\mu \pm 3}(z)$. Lemma 3.7 and Table 3 indicate that estimates similar to those in (5.1)–(5.4) and (5.5), (5.9) could also be obtained for other classes of polynomials.

(2) It is easy to verify that the fractional linear transformation $z \mapsto 2z/(z+1)$ maps the unit circle to the vertical line $\{z \in \mathbb{C} \mid \mathrm{Re}(z) = 1\}$. Therefore, if we define the transformed polynomials

$$b_n(z) := (1+z)^{d_n} B_n(\tfrac{2z}{z+1}),$$

where $d_n := \deg B_n(z)$, then Conjecture 3.1 is equivalent to the conjecture that all zeros of $b_n(z)$ lie inside the unit circle. This is illustrated by Figure 2.

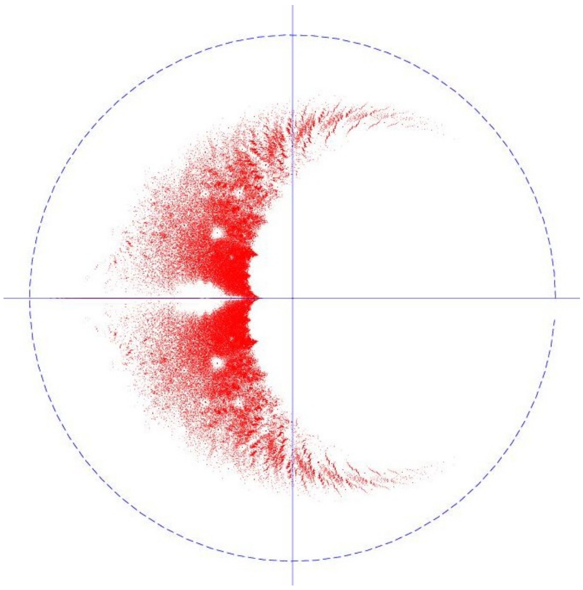

Figure 2. Zeros of $B_n(\frac{2z}{z+1})$, $n \le 2^{16}$, inscribed in the unit circle.

For $n \le 36$, the polynomials $b_n(z)$ satisfy the Eneström–Kakeya condition (3.3) of Theorem 3.4, but we have

$$b_{37}(z) = 1 + 11z + 39z^2 + 37z^3,$$

with increasing numbers of the polynomials $b_n(z)$ violating the condition (3.3). However, based on computations using PARI [11] up to $n = 10^{11}$, we propose:

Conjecture 6.1. *The coefficients of any transformed Stern polynomial $b_n(z)$ form a unimodal sequence of nonnegative integers.*

Eneström–Kakeya type criteria for unimodal polynomials [3] apply to some of the polynomials $b_n(z)$, but fail in general.

(3) The zero distribution of the Stern polynomials, as illustrated in Figure 1, has some features in common with the zero distribution of Littlewood polynomials, i.e., polynomials with coefficients -1 or 1. For instance, Figure 11 in [1, p. 908], though more symmetric than our Figure 1, also has a large interior zero-free region with an apparent fractal boundary, as well as "holes" of different sizes along the unit circle.

(4) Of the classes of polynomials considered in Section 3, we see that only $B_{2^k-1}(z)$ and $B_{2^k-5}(z)$ are self-reciprocal (or "palindromic"). In this connection it is interesting to note that Gawron [5] recently proved the following: For two recurrence-generated integer sequences u_n and v_n, with $u_0 = 1$ and $v_0 = 5$, the polynomials $B_{2^k-u_n}(z)$ and $B_{2^k-v_n}(z)$, for sufficiently large k, are all self-reciprocal.

References

[1] P. Borwein and L. Jörgenson, Visible structures in number theory, *Amer. Math. Monthly* **108** (2001), 897–910.

[2] J. Brillhart, M. Filaseta and A. Odlyzko, On an irreducibility theorem of A. Cohn, *Canad. J. Math.* **33** (1981), 1055–1059.

[3] K. Dilcher, A generalization of the Eneström–Kakeya theorem, *J. Math. Anal. Appl.* **116** (1986), 473–488.

[4] K. Dilcher and K. B. Stolarsky, A polynomial analogue to the Stern sequence, *Int. J. Number Theory* **3** (2007), 85–103.

[5] M. Gawron, A note on the arithmetic properties of Stern polynomials, *Publ. Math. Debrecen* **85** (2014), 453–465.

[6] S. Klavžar, U. Milutinović and C. Petr, Stern polynomials, *Adv. in Appl. Math.* **39** (2007), 86–95.

[7] D. H. Lehmer, On Stern's diatomic series, *Amer. Math. Monthly* **36** (1929), 59–67.

[8] E. Lucas, Théorie des fonctions numériques simplement périodiques, *Amer. J. Math.* **1** (1878), no. 2, 184–196; no. 3, 197–240, and no. 4, 289–321. English translation available at `http://www.fq.math.ca/Books/Complete/simply-periodic.pdf`.

[9] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, 2011, http://oeis.org.

[10] F. W. J. Olver et al. (eds.), NIST Handbook of Mathematical Functions, *Cambridge University Press*, *New York*, 2010.

[11] The PARI Group, Bordeaux, PARI/GP, version 2.5.2, *Université de Bordeaux*, 2014, http://pari.math.u-bordeaux.fr/.

[12] G. Pólya and G. Szegő, Aufgaben und Lehrsätze aus der Analysis, *Springer-Verlag*, *Berlin*, 1925.

[13] A. Schinzel, On the factors of Stern polynomials (remarks on the preceding paper of M. Ulas), *Publ. Math. Debrecen* **79** (2011), 83–88.

[14] M. Ulas, On certain arithmetic properties of Stern polynomials, *Publ. Math. Debrecen* **79** (2011), 55–81.

[15] M. Ulas, Arithmetic properties of the sequence of degrees of Stern polynomials and related results, *Int. J. Number Theory* **8** (2012), 669–687.

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