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Trispectrum and higher order spectra for non-Gaussian homogeneous and isotropic random field on the 2D-plane

By GYÖRGY TERDIK (Debrecen)

Abstract. In this paper, we study the non-Gaussian homogeneous and isotropic random field on the plane in frequency domain. The trispectrum and higher order spectra of such a random field are described in terms of Bessel functions. Some particular integrals of Bessel functions are considered as well.

1. Introduction

In several random fields of sciences like geophysics, astrophysics, climatology etc. we come across observations which are non-Gaussian. A Gaussian process is characterized by its first two moments, namely, mean, variance and autocorrelations (or equivalently, second-order spectrum). There are several clearly different processes having identical second-order properties, but their distributions are not Gaussian and they are clearly different [SR97], [IT97], [Dig13], therefore it is necessary to study higher order structures. Although second-order properties of Gaussian random fields are well established, see [Yag87], [Yad83], [AT09], [Bri01], [Ros00], [Ros85], [Pri88], [Leo89], [BH86], [LS12], [Mok07], there are only a few results concerning non-Gaussian random fields. Characterizations of non-Gaussian random fields which require study of higher order moments (or equivalently, higher order spectra) are not well known. The isotropy of stochastic phenomenons in two

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dimensions has been established in several applications [NY97], [OT96]. Only recently has been made quite a number of steps toward the statistical investigation of non-Gaussian isotropic random fields, mainly for understanding the Cosmic Microwave Background (CMB) anisotropies [OH02], [AH12], [AAAC$^+$14], in particular flat sky approximation, i.e. projecting an observed fraction of the sky onto a plane rather than a sphere [Pla14].

In this paper, our object is to continue the frequency domain investigations published in our paper [Ter14], where, additional to the covariance and the spectrum, the connection between the bicovariance and the bispectrum has been studied in details. Here the trispectrum and all higher order spectra of such random fields are described in terms of Bessel functions. The connection between the tricovariance and the trispectrum is similar to the one between the bicovariance and the bispectrum. It was necessary to prove some particular integrals of Bessel functions, given in terms of sides and angles of a multilateral.

We now summarize the contents of the paper. Some basic results of homogeneous and isotropic random field as properties of spectrum and bispectrum are included in Introduction. In Section 2, we show a connection between the trispectrum and tricovariance in terms of a kernel function. The generalization of these results for higher order spectra and covariances are given in Section 3. Some technical Lemmas concerning integrals of Bessel functions, Dirac-function in polar coordinates and higher order cumulants of spectral measures are set in Appendix.

1.1. Isotropy. A homogeneous measurable real valued stochastic field $X(\underline{x})$, $\underline{x} \in \mathbb{R}^2$, which is continuous (in mean square sense), has the spectral representation

$$X(\underline{x}) = \int_{\mathbb{R}^2} e^{i\underline{x}\cdot\underline{\omega}} Z(d\underline{\omega}), \quad \underline{\omega}, \underline{x} \in \mathbb{R}^2,$$

with $EX(\underline{x}) = 0$, and the orthogonal complex random spectral measure $Z(d\underline{\omega})$ has $E|Z(d\underline{\omega})|^2 = F_0(d\underline{\omega})$. Homogeneity is defined in strict sense, i.e. all the finite dimensional distributions of $X(\underline{x})$ are translation invariant, see [Yag87] for details. We can rewrite $X(\underline{x})$ in terms of polar coordinates:

$$X(r,\varphi) = \int_0^\infty \int_0^{2\pi} e^{i\rho r\cos(\varphi-\eta)} Z(\rho d\rho d\eta),$$

where $\underline{x} = (r,\varphi)$, $\underline{\omega} = (\rho,\eta)$ are polar coordinates, $r = |\underline{x}| = \sqrt{x_1^2 + x_2^2}$, $\rho = |\underline{\omega}|$, $\underline{x}\cdot\underline{\omega} = r\rho\cos(\varphi-\eta)$. This representation provides an isotropic random field if $F_0(d\underline{\omega})$ is isotropic, i.e. $F_0(d\underline{\omega}) = E|Z(d\underline{\omega})|^2 = E|Z(\rho d\rho d\eta)|^2 = F(\rho d\rho)\,d\eta$.

The isotropy is usually defined through the invariance of the covariance structure under rotations. A rotation $g \in SO(2)$ is characterized by an angle γ. We consider rotations g about the origin of the coordinate system. If $\underline{x} \in \mathbb{R}^2$ is given in polar coordinates $\underline{x} = (r, \varphi)$, then $g\underline{x} = (r, \varphi - \gamma)$, and as usual, the operator $\Lambda(g)$ acts on functions $f(r, \varphi)$, such that $\Lambda(g) f(r, \varphi) = f\left(g^{-1}(r, \varphi)\right) = f(r, \varphi + \gamma)$.

The invariance of the covariance function is satisfactory for Gaussian cases, but for non-Gaussian random fields we need invariance of higher order cumulants as well.

Definition 1. A homogeneous stochastic field $X(\underline{x})$ is strictly isotropic if all finite dimensional distributions of $X(\underline{x})$ are invariant under all rotations $g \in SO(2)$, i.e. all finite dimensional distributions of $X(\underline{x})$ and $\Lambda(g) X(\underline{x})$ are the same.

As far as the homogeneous random field $X(\underline{x})$ is Gaussian, the isotropy of the spectral measure $F_0(d\underline{\omega})$, i.e. in polar coordinates $F_0(d\underline{\omega}) = F(\rho d\rho)\, d\eta$, implies

$$\operatorname{Cov}(\Lambda(g) X(\underline{x}_1), \Lambda(g) X(\underline{x}_2)) = \operatorname{Cov}(X(\underline{x}_1), X(\underline{x}_2)),$$

for each $\underline{x}_1$, $\underline{x}_2$ and for every $g \in SO(2)$. It will be convenient for us later if we assume the existence of all moments of the random field $X(\underline{x})$, in this way from the isotropy follows that all higher order moments and cumulants are also invariant under rotations.

Example 1. Consider a Gaussian homogeneous and isotropic random field $X(\underline{x})$, then $X(\underline{x}) + X^2(\underline{x})$ is clearly a homogeneous and isotropic non-Gaussian random field.

Let us consider a homogeneous and isotropic stochastic field $X(\underline{x}) = X(r, \varphi)$, $(r > 0,\ \varphi \in [0, 2\pi))$ on the plane, and put it into spectral representation, see [Ter14],

$$X(r, \varphi) = \sum_{\ell=-\infty}^{\infty} e^{i\ell\varphi} \int_0^\infty J_\ell(\rho r)\, Z_\ell(\rho d\rho), \tag{1.1}$$

where J_ℓ denotes the Bessel function of the first kind, and

$$Z_\ell(\rho d\rho) = \int_0^{2\pi} i^\ell e^{-i\ell\eta} Z(\rho d\rho d\eta). \tag{1.2}$$

Z_ℓ are complex-valued random measures, orthogonal to each other:

$$\operatorname{Cov}(Z_{\ell_1}(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2)) = \delta_{\ell_1-\ell_2} F(\rho d\rho),$$

where δ_ℓ denotes the Kronecker-δ. Note that the spectral measure $F(\rho d\rho)$ of the stochastic spectral measure $Z_\ell(\rho d\rho)$ does not depend on ℓ. In representation (1.1) $e^{i\ell\varphi}$ plays a role of spherical harmonics of degree ℓ with complex values on the plane. It follows that an isotropic random field $X(\underline{x})$ can be decomposed into a countable number of mutually uncorrelated spectral measures with a one-dimensional parameter.

The isotropy of $X(r,\varphi)$ implies that the distribution of $X(r,\varphi)$ does not change under rotations $g \in SO(2)$

$$\Lambda(g)\,X(r,\varphi) = \sum_{\ell=-\infty}^{\infty} e^{i\ell\varphi}\int_0^\infty J_\ell(\rho r)\,e^{i\ell\gamma}Z_\ell(\rho d\rho) = \sum_{\ell=-\infty}^{\infty} e^{i\ell\varphi}\int_0^\infty J_\ell(\rho r)\,Z_\ell(\rho d\rho),$$

hence the distribution of $Z_\ell(\rho d\rho)$ and $e^{i\ell\gamma}Z_\ell(\rho d\rho)$ should be the same. Therefore, under isotropy assumption we have

$$\operatorname{Cum}(Z_{\ell_1}(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2)) = e^{i(\ell_1+\ell_2)\gamma}\operatorname{Cum}(Z_{\ell_1}(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2)),$$

for each γ, hence either $\ell_1+\ell_2=0$ or $\operatorname{Cum}(Z_{\ell_1}(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2))=0$. In general, under isotropy assumption we have

$$\begin{aligned}&\operatorname{Cum}\left(Z_{\ell_1}(\rho_1 d\rho_1),\ldots,Z_{\ell_p}(\rho_p d\rho_p)\right)\\&\quad= e^{i(\ell_1+\ell_2\cdots+\ell_p)\gamma}\operatorname{Cum}\left(Z_{\ell_1}(\rho_1 d\rho_1),\ldots,Z_{\ell_p}(\rho_p d\rho_p)\right),\end{aligned}$$

that is either $\ell_1+\ell_2\cdots+\ell_p=0$, or $\operatorname{Cum}\left(Z_{\ell_1}(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2),\ldots,Z_{\ell_p}(\rho_p d\rho_p)\right) = 0$, should be fulfilled. In turn, if this assumption fulfils for each p, then all cumulants $\operatorname{Cum}\left(Z_{\ell_1}(\rho_1 d\rho_1),\ldots,Z_{\ell_p}(\rho_p d\rho_p)\right)$ are invariant under rotations, and if in addition the distributions of $X(r,\varphi)$ are determined by the moments, then the random field is isotropic.

1.2. Spectrum and bispectrum. It is well known from the theory of Gaussian random fields that

$$\operatorname{Cov}\left(X(\underline{x}), X(\underline{y})\right) = \int_0^\infty J_0(\rho r)\,F(\rho d\rho),$$

where $r = \left|\underline{x}-\underline{y}\right|$, see [Yad83], [Yag87], [Bri74]. The covariances and the spectral measure uniquely define each other since the Hankel transform gives the inverse. For absolutely continuous spectral measure we have $F(\rho d\rho) = \sigma^2 |A(\rho)|^2 \rho d\rho$, and therefore

$$\mathcal{C}_2(r) = \int_0^\infty J_0(\rho r)\,\sigma^2 |A(\rho)|^2 \rho d\rho,$$

$$\sigma^2 |A(\rho)|^2 = \frac{1}{2\pi}\int_0^\infty J_0(\rho r)\,\mathcal{C}_2(r)\,r dr,$$

where $\mathcal{C}_2(r) = \operatorname{Cov}\left(X(\underline{x}), X(\underline{y})\right)$, $r = \left|\underline{x}-\underline{y}\right|$.

The third-order structure of a homogeneous and isotropic stochastic field $X(\underline{x})$ is described by either the third-order cumulants (bicovariances) in spatial domain or the bispectrum in frequency domain, see [Ter14] for details. The bicovariance of $X(\underline{x})$ is

$$\begin{aligned}&\operatorname{Cum}\left(X\left(\underline{x}_1\right), X\left(\underline{x}_2\right), X\left(\underline{x}_3\right)\right)\\ &\quad = \operatorname{Cum}\left(X(0), X\left(g\left(\underline{x}_2-\underline{x}_1\right)\right), X\left(\left|\underline{x}_3-\underline{x}_1\right| \underline{n}\right)\right),\end{aligned} \tag{1.3}$$

where g denotes the rotation carrying $\underline{x}_3-\underline{x}_1$ into $\underline{n}=(0,1)$. The third-order cumulant of the stochastic spectral measure $Z(d\underline{\omega})$ of the homogeneous random field $X(\underline{x})$ is given by

$$\begin{aligned}\operatorname{Cum}\left(Z\left(d\underline{\omega}_1\right), Z\left(d\underline{\omega}_2\right), Z\left(d\underline{\omega}_3\right)\right) &= \delta\left(\Sigma_1^3 \underline{\omega}_k\right) S_3\left(\underline{\omega}_1, \underline{\omega}_2, \underline{\omega}_3\right) d\underline{\omega}_1 d\underline{\omega}_2 d\underline{\omega}_3\\ &= \delta\left(\Sigma_1^3 \rho_k \widehat{\underline{\omega}}_k\right) S_3\left(\rho_1, \rho_2, \alpha_3\right) \prod_{k=1}^3 \Omega\left(d\widehat{\underline{\omega}}_k\right) \rho_k d\rho_k,\end{aligned}$$

where $\widehat{\underline{\omega}}_k=\underline{\omega}_k /\left|\underline{\omega}_k\right|$. Now $S_3\left(\alpha, \rho_2, \rho_3\right)$ depends on $0<\alpha<\pi$, in other words, depends on $\left(\rho_1, \rho_2, \rho_3\right)$, such that these positive numbers form a triangle, see Figure 2.

The bicovariance $\operatorname{Cum}\left(X(0), X\left(\underline{x}_2\right), X\left(r_3 \underline{n}\right)\right)$ depends on the lengths r_2, $r_3=\left|\underline{x}_3\right|$, and the angle φ between them, this way a triangle is defined with length of the third side r_1, such that $r_1^2=r_2^2+r_3^2-2 r_2 r_3 \cos(\varphi)$. According to this definition of r_1, we introduce $\mathcal{C}_3\left(r_1, r_2, r_3\right)=\operatorname{Cum}\left(X(0), X\left(\underline{x}_2\right), X\left(r_3 \underline{n}\right)\right)$. Similarly, the bispectrum S_3 (possible with complex values in general) of the homogeneous and isotropic stochastic field $X(\underline{x})$ depends on wave numbers $\left(\rho_1, \rho_2, \rho_3\right)$ such that ρ_1, ρ_2, ρ_3 should form a triangle. It has been shown, see [Ter14], that

$$\mathcal{C}_3\left(\varphi, r_2, r_3\right)=2 \iint_0^{\infty} \int_0^\pi \mathcal{T}_3\left(\alpha, \rho_2, \rho_3 \mid \varphi, r_2, r_3\right) S_3\left(\alpha, \rho_2, \rho_3\right) d\alpha \prod_{k=2}^3 \rho_k d\rho_k,$$

where the function

$$\mathcal{T}_3\left(\alpha, \rho_2, \rho_3 \mid \varphi, r_2, r_3\right)=\sum_{\ell=-\infty}^{\infty} \cos(\ell \varphi) J_{\ell}\left(\rho_2 r_2\right) J_{\ell}\left(\rho_3 r_3\right) \cos(\ell \alpha) \tag{1.4}$$

gives the transformation of the bispectrum $S_3\left(\rho_1, \rho_2, \alpha\right)$ into the bicovariance $\mathcal{C}_3\left(\varphi, r_2, r_3\right)$. Notice that both angles φ and α are related to the third sides ρ_1 and r_1 of the triangles, defined by the wave numbers $\left(\rho_1, \rho_2, \rho_3\right)$ and distances $\left(r_1, r_2, r_3\right)$. Distances $\left(r_1, r_2, r_3\right)$ *are not* the norm of $\left(\underline{x}_1, \underline{x}_2, \underline{x}_3\right)$ in (1.3) but defined by $|\underline{x}_2-\underline{x}_1|$, $|\underline{x}_3-\underline{x}_1|$, and the angle φ is the one between the differences. By inversion of the bicovariance function, we obtain the bispectrum

$$S_3(\rho_1,\rho_2,\rho_3)=\frac{1}{2\pi}\iint_0^\infty\int_0^\pi \mathcal{T}_3(\alpha,\rho_2,\rho_3|\,\varphi,r_2,r_3)\,\mathcal{C}_3(r_1,r_2,r_3)\,d\varphi\prod_{k=2}^3 r_k dr_k. \quad (1.5)$$

Note here that the bispectrum of a homogeneous and isotropic stochastic field is real valued.

2. Trispectrum and tricovariance

The spectrum and bispectrum of a homogeneous and isotropic stochastic field have particular form, and as it will be seen, considering trispectrum, they do not show the general pattern for higher order spectra.

From now on, we introduce a short notation for vectors using sets for indices, for instance, $\underline{\omega}_{1:4}$, where $1:4=(1,2,3,4)$, denotes $(\underline{\omega}_1,\underline{\omega}_2,\underline{\omega}_3,\underline{\omega}_4)$, and so on.

Consider the spectral representation of the fourth-order cumulant of a homogeneous random field

$$\begin{aligned}&\operatorname{Cum}(X(\underline{x}_1),X(\underline{x}_2),X(\underline{x}_3),X(\underline{x}_4))\\&=\underbrace{\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2}}_{4}e^{i\left(\Sigma_1^4\underline{x}_k\cdot\underline{\omega}_k\right)}S_4(\underline{\omega}_{1:4})\,\delta\left(\Sigma_1^4\underline{\omega}_k\right)\prod_{k=1}^4 d\underline{\omega}_k,\end{aligned}$$

and under isotropy assumption for each $g\in SO(2)$ we have in addition

$$\begin{aligned}&\operatorname{Cum}(X(g\underline{x}_1),X(g\underline{x}_2),X(g\underline{x}_3),X(g\underline{x}_4))\\&=\underbrace{\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2}}_{4}e^{i\left(\Sigma_1^4\underline{x}_k\cdot\underline{\omega}_k\right)}S_4(g\underline{\omega}_{1:4})\,\delta\left(\Sigma_1^4\underline{\omega}_k\right)\prod_{k=1}^4 d\underline{\omega}_k\\&=\operatorname{Cum}(X(\underline{x}_1),X(\underline{x}_2),X(\underline{x}_3),X(\underline{x}_4)),\end{aligned}$$

hence $S_4(\underline{\omega}_{1:4})=S_4\left(\underline{\omega}_{1:3},-\Sigma_1^3\underline{\omega}_k\right)$, and at the same time $S_4(g\underline{\omega}_{1:4})=S_4(\underline{\omega}_{1:4})$. Now $\underline{\omega}_{1:4}$ is defined by eight coordinates, and if $\Sigma_1^4\underline{\omega}_k=0$, then these vectors form a quadrilateral, in general. This quadrilateral has invariants under rotations. In this way, $S_4(\underline{\omega}_{1:4})=S_4(\alpha_1,\rho_2,\rho_3,\rho_4,\beta_2)$, see Figure 3.

Now, let us shift the vectors $(\underline{x}_1,\underline{x}_2,\underline{x}_3,\underline{x}_4)$ by the vector $-\underline{x}_1$, and rotate this new set of vectors $(\underline{x}_2-\underline{x}_1,\underline{x}_3-\underline{x}_1,\underline{x}_4-\underline{x}_1)=\left(\underline{y}_2,\underline{y}_3,\underline{y}_4\right)$ such that $\underline{y}_4=$

$\left|\underline{y}_4\right| \underline{n}$, where $\underline{n} = (1,0)$, the locations $\left(\underline{0}, \underline{y}_2, \underline{y}_3, \underline{y}_4\right)$ will be given by $r_2 = \left|\underline{y}_2\right|$, $r_3 = \left|\underline{y}_3\right|$, $r_4 = \left|\underline{y}_4\right|$, together with angles (r_2, φ_2), (r_3, φ_3). From now on, we shall consider the cumulants at locations defined by their invariants $\left((r,\varphi)_{2:3}, r_4\right) = \left((r_2,\varphi_2),(r_3,\varphi_3),r_4\right)$.

We use the invariance of the cumulants under the shift and rotation to obtain

$$\begin{aligned}
&\operatorname{Cum}\left(X\left(\underline{x}_1\right), X\left(\underline{x}_2\right), X\left(\underline{x}_3\right), X\left(\underline{x}_4\right)\right)\\
&\quad = \operatorname{Cum}\left(X\left(0\right), X\left(\underline{x}_2-\underline{x}_1\right), X\left(\underline{x}_3-\underline{x}_1\right), X\left(\underline{x}_4-\underline{x}_1\right)\right)\\
&\quad = \operatorname{Cum}\left(X\left(0\right), X\left(g\left(\underline{x}_2-\underline{x}_1\right)\right), X\left(g\left(\underline{x}_3-\underline{x}_1\right)\right), X\left(g\left(\underline{x}_4-\underline{x}_1\right)\right)\right),
\end{aligned}$$

where g denotes the rotation carrying the $\underline{x}_4 - \underline{x}_1$ into the x-axis. The general form of cumulants is $\operatorname{Cum}\left(X\left(0\right), X\left(\underline{x}_2\right), X\left(\underline{x}_3\right), X\left(r_4\underline{n}\right)\right)$, where $\underline{x}_2$ and $\underline{x}_3$ are arbitrary locations, and $\underline{n} = (1,0)$. The fourth-order cumulants of a homogeneous and isotropic stochastic field $X\left(\underline{x}\right)$ are determined by the quantities r_4, $\underline{x}_2$ and $\underline{x}_3$, in other words, by $\left((r,\varphi)_{2:3}, r_4\right)$, see Figure 1. Let

$$\mathcal{C}_4\left(\left(r_2,\varphi_2\right),\left(r_3,\varphi_3\right),r_4\right) = \operatorname{Cum}\left(X\left(\underline{x}_1\right), X\left(\underline{x}_2\right), X\left(r_3\underline{n}\right), X\left(0\right)\right),$$

and the trispectrum $S_4\left(\alpha_1, \rho_{2:4}, \beta_2\right)$ be given on the domain of variables $(\alpha_1, \rho_2, \rho_3, \rho_4, \beta_2)$, where $\alpha_1, \beta_2 \in (0,\pi)$, and $0 < \rho_2, \rho_3, \rho_4$.

Now let

$$\begin{aligned}
&\mathcal{T}_4\left(\alpha_1, \rho_{2:4}, \beta_2 \middle| (r,\varphi)_{2:3}, r_4\right)\\
&= \sum_{\ell_2,\ell_3=-\infty}^{\infty} e^{i(\ell_2\varphi_2+\ell_3\varphi_3)} J_{\ell_2}(\rho_2 r_2) J_{\ell_3}(\rho_3 r_3) J_{\ell_2+\ell_3}(\rho_4 r_4) \cos\left(\ell_2\alpha_1\right)\cos\left(\ell_2\alpha_3 - \ell_3\beta_2\right),
\end{aligned}$$

where the angle α_3 is determined by ρ_2, ρ_3, ρ_4, i.e. $\left(\rho_2^2+\rho_4^2-\rho_3^2\right)/\left(2\rho_2\rho_4\right) = \cos\alpha_3$, see Figure 3.

Theorem 1. *Let $X\left(\underline{x}\right)$ be a homogeneous and isotropic stochastic field on the plane, then the trispectrum S_4 of $X\left(\underline{x}\right)$ is real valued, and the tricovariance function $\mathcal{C}_4$ and the trispectrum S_4 are connected by the kernel function $\mathcal{T}_4$, namely,*

$$\begin{aligned}
\mathcal{C}_4\left((r,\varphi)_{2:3}, r_4\right) = 4 \iiint_0^\infty \iint_0^\pi & \mathcal{T}_4\left(\alpha_1, \rho_{2:4}, \beta_2 \middle| (r,\varphi)_{2:3}, r_4\right)\\
&\times S_4\left(\alpha_1, \rho_{2:4}, \beta_2\right) \prod_{k=2}^{4} \rho_k d\rho_k d\alpha_1 d\beta_2,
\end{aligned}$$

conversely,

$$S_4\left(\alpha_1, \rho_{2:4}, \beta_2\right) = \frac{1}{(2\pi)^4} \iiint\limits_0^\infty \iint_0^{2\pi} \mathcal{T}_4\left(\alpha_1, \rho_{2:4}, \beta_2 \middle| (r,\varphi)_{2:3}, r_4\right)$$
$$\times\, \mathcal{C}_4\left((r,\varphi)_{2:3}, r_4\right) \prod_{k=2}^{4} r_k dr_k d\varphi_2 d\varphi_3, \tag{2.1}$$

provided these integrals exist.

Proof. We apply the series representation (1.1) of $X\left(\underline{x}\right)$, and rewrite it for particular cases $\underline{n} = (1, 0)$:

$$X\left(r\underline{n}\right) = \sum_{\ell=-\infty}^{\infty} \int_0^\infty J_\ell\left(\rho r\right) Z_\ell\left(\rho d\rho\right), \tag{2.2}$$

$$X\left(\underline{0}\right) = \int_{\mathbb{R}^2} Z\left(d\underline{\omega}\right) = \int_0^\infty Z_0\left(\rho d\rho\right). \tag{2.3}$$

We obtain

$$\begin{aligned}
&\operatorname{Cum}\left(X\left(0\right), X\left(\underline{x}_2\right), X\left(\underline{x}_3\right), X\left(r_4\underline{n}\right)\right)\\
&= \sum_{\ell_2,\ell_3,\ell_4=-\infty}^{\infty} e^{i(\ell_2\varphi_2+\ell_3\varphi_3)} \iiiint\limits_0^\infty J_{\ell_2}\left(\rho_2 r_2\right) J_{\ell_3}\left(\rho_3 r_3\right) J_{\ell_4}\left(\rho_4 r_4\right)\\
&\quad \times \operatorname{Cum}\left(Z_0\left(\rho_1 d\rho_1\right), Z_{\ell_2}\left(\rho_2 d\rho_2\right), Z_{\ell_3}\left(\rho_3 d\rho_3\right), Z_{\ell_4}\left(\rho_4 d\rho_4\right)\right)\\
&= \sum_{\ell_2,\ell_3=-\infty}^{\infty} e^{i(\ell_2\varphi_2+\ell_3\varphi_3)} \iiiint\limits_0^\infty J_{\ell_2}\left(\rho_2 r_2\right) J_{\ell_3}\left(\rho_3 r_3\right) J_{-(\ell_2+\ell_3)}\left(\rho_4 r_4\right)\\
&\quad \times \operatorname{Cum}\left(Z_0\left(\rho_1 d\rho_1\right), Z_{\ell_2}\left(\rho_2 d\rho_2\right), Z_{\ell_3}\left(\rho_3 d\rho_3\right), Z_{-(\ell_2+\ell_3)}\left(\rho_4 d\rho_4\right)\right)\\
&= \sum_{\ell_2,\ell_3=-\infty}^{\infty} e^{i(\ell_2\varphi_2+\ell_3\varphi_3)} \iiiint\limits_0^\infty J_{\ell_2}\left(\rho_2 r_2\right) J_{\ell_3}\left(\rho_3 r_3\right) J_{\ell_2+\ell_3}\left(\rho_4 r_4\right) (-1)^{\ell_2+\ell_3}\\
&\quad \times \operatorname{Cum}\left(Z_0\left(\rho_1 d\rho_1\right), Z_{\ell_2}\left(\rho_2 d\rho_2\right), Z_{\ell_3}\left(\rho_3 d\rho_3\right), Z_{-(\ell_2+\ell_3)}\left(\rho_4 d\rho_4\right)\right),
\end{aligned}$$

in polar coordinates. The fourth-order cumulant of the stochastic spectral measure $Z\left(d\underline{\omega}\right)$ according to a homogeneous random field $X\left(\underline{x}\right)$ fulfils the following equation:

$$\operatorname{Cum}\left(Z\left(d\underline{\omega}_1\right), Z\left(d\underline{\omega}_2\right), Z\left(d\underline{\omega}_3\right), Z\left(d\underline{\omega}_4\right)\right) = \delta\left(\Sigma_1^4 \underline{\omega}_k\right) S_4\left(\underline{\omega}_{1:4}\right) \prod_{k=1}^{4} d\underline{\omega}_k;$$

and the stochastic spectral measures $Z_\ell(\rho d\rho)$ are connected to $Z(d\underline{\omega})$ by (1.2) in frequency domain, hence

$$\begin{aligned}
&\operatorname{Cum}\left(Z_0(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2), Z_{\ell_3}(\rho_3 d\rho_3), Z_{-(\ell_2+\ell_3)}(\rho_4 d\rho_4)\right) \\
&\quad = 4(-1)^{\ell_2+\ell_3} \int_0^\pi \frac{\delta(\triangle|\rho_1,\rho_2,\kappa)}{\rho_1 \kappa \sin\alpha_2} \cos(\ell_2\alpha_1) \\
&\quad\quad \times \cos(\ell_2\alpha_3 + \ell_3\beta_2) S_4(\alpha_1, \rho_{2:4}, \beta_2) d\beta_2 \prod_{k=1}^4 \rho_k d\rho_k, \qquad (2.4)
\end{aligned}$$

where $\widehat{\underline{\omega}}_k = \underline{\omega}_k / |\underline{\omega}_k| = (\cos\eta_k, \sin\eta_k)$ defines the angle η_k, and $\delta(\triangle|\rho_1,\rho_2,\kappa)$ is zero if (ρ_1,ρ_2,κ) does not form a triangle, otherwise it is 1. Notice that the cumulants $\operatorname{Cum}(X(0), X(\underline{x}_2), X(\underline{x}_3), X(r_4\underline{n}))$ are given in terms of three distances and two angles $((r,\varphi)_{2:3}, r_4)$, see Figure 1.

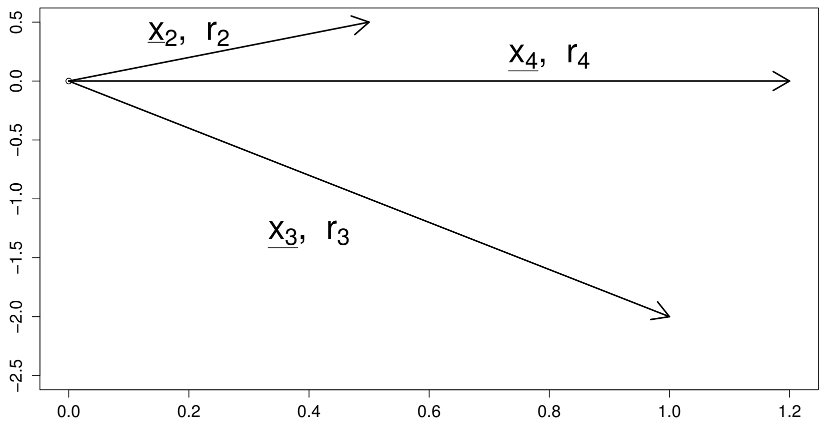

Figure 1. Locations on the plane.

The function $\mathcal{C}_4(((r,\varphi)_{2:3}, r_4))$ is expressed as

$$\begin{aligned}
&\operatorname{Cum}(X(0), X(\underline{x}_2), X(\underline{x}_3), X(r_4\underline{n})) \\
&\quad = 4 \sum_{\ell_2,\ell_3=-\infty}^{\infty} e^{i(\ell_2\varphi_2+\ell_3\varphi_3)} \iiiint_0^\infty J_{\ell_2}(\rho_2 r_2) J_{\ell_3}(\rho_3 r_3) J_{\ell_2+\ell_3}(\rho_4 r_4) \\
&\quad\quad \times \frac{\delta(\triangle|\rho_1,\rho_2,\kappa)}{\rho_2\kappa\sin\alpha_1} e^{-i\ell_2(\alpha_1-\alpha_3)-i\ell_3\beta_2} S_4(\alpha_1,\rho_{2:4},\beta_2) d\beta_2 \prod_{k=1}^4 \rho_k d\rho_k =
\end{aligned}$$

$$= 4 \sum_{\ell_2, \ell_3 = -\infty}^{\infty} e^{i(\ell_2 \varphi_2 + \ell_3 \varphi_3)} \iiint_0^{\infty} J_{\ell_2}(\rho_2 r_2) J_{\ell_3}(\rho_3 r_3) J_{\ell_2+\ell_3}(\rho_4 r_4)$$

$$\times \iint_0^{\pi} \cos(\ell_2 \alpha_1) \cos(\ell_2 \alpha_3 + \ell_3 \beta_2) S_4(\alpha_1, \rho_{2:4}, \beta_2) \, d\alpha_1 d\beta_2 \prod_{k=2}^{4} \rho_k d\rho_k,$$

$\rho_1 d\rho_1 = \kappa \rho_2 \sin(\alpha_1) \, d\alpha_1$, where $\kappa = \sqrt{\rho_3^2 + \rho_4^2 - 2\rho_3 \rho_4 \cos \beta_2}$, and $\alpha_2 = \arccos \left[\left(\rho_3^2 - \rho_4^2 - \kappa^2\right)/2\kappa\rho_4\right]$, hence α_3 is determined by ρ_3, ρ_4 and β_2. The result of the above summation is real, therefore the imaginary part is zero.

Now, for proving (2.1), consider the integral

$$\iint_0^{2\pi} \iiint_0^{\infty} \mathcal{T}_4\left(\alpha_1, \rho_{2:4}, \beta_2 \middle| (r, \varphi)_{2:3}, r_4\right) \mathcal{C}_4\left((r, \varphi)_{2:3}, r_4\right) d\varphi_2 d\varphi_3 \prod_{k=2}^{4} r_k dr_k$$

$$= 4 \iint_0^{2\pi} \iiint_0^{\infty} \mathcal{T}_4\left(\alpha_1, \rho_{2:4}, \beta_2 \middle| (r, \varphi)_{2:3}, r_4\right) \iint_0^{2\pi} \iiint_0^{\infty} \mathcal{T}_4\left(\alpha_1', \rho_{2:4}', \beta_2' \middle| (r, \varphi)_{2:3}, r_4\right)$$

$$\times S_4(\alpha_1', \rho_{2:4}', \beta_2') \, d\alpha_1' d\beta_2' \prod_{k=2}^{4} \rho_k' d\rho_k' d\varphi_2 d\varphi_3 \prod_{k=2}^{4} r_k dr_k$$

$$= 4 (2\pi)^2 \sum_{\ell_2, \ell_3 = -\infty}^{\infty} \iiint_0^{\infty} \iiint_0^{\infty} J_{\ell_2}(\rho_2' r_2) J_{\ell_3}(\rho_3' r_3) J_{\ell_2+\ell_3}(\rho_4' r_4) J_{\ell_2}(\rho_2 r_2)$$

$$\times J_{\ell_3}(\rho_3 r_3) J_{\ell_2+\ell_3}(\rho_4 r_4) \prod_{k=2}^{4} r_k dr_k \iint_0^{\pi} \cos(\ell_2 \alpha_1') \cos(\ell_2 \alpha_3' + \ell_3 \beta_2')$$

$$\times \cos(\ell_2 \alpha_1) \cos(\ell_2 \alpha_3 + \ell_3 \beta_2) S_4(\alpha_1', \rho_{2:4}', \beta_2') \, d\alpha_1' d\beta_2' \prod_{k=2}^{4} \rho_k' d\rho_k'$$

$$= 4 (2\pi)^2 \iint_0^{\pi} \sum_{\ell_2, \ell_3 = -\infty}^{\infty} \cos(\ell_2 \alpha_1') \cos(\ell_2 \alpha_3' + \ell_3 \beta_2') \cos(\ell_2 \alpha_1) \cos(\ell_2 \alpha_3 + \ell_3 \beta_2)$$

$$\times S_4(\alpha_1', \rho_{2:4}, \beta_2') \, d\alpha_1' d\beta_2' = (2\pi)^4 S_4(\alpha_1, \rho_{2:4}, \beta_2).$$

To show the last equality, one can turn cosine to exponential and get the result, since both β_2 and β_2' are positive, similarly α_3 and α_3'. If $\beta_2 = \beta_2'$, then $\alpha_3 = \alpha_3'$ follows, and finally, $\alpha_1 = \alpha_1'$. □

3. Expression for higher order spectra

Theorem 1 can be generalized for higher order spectra. Put $\rho_{2:p}=(\rho_2,\ldots,\rho_p)$, $\beta_{1:p-3,2}=(\beta_{1,2},\ldots,\beta_{p-3,2})$, $(r,\varphi)_{2:p-1}, = ((r_2,\varphi_2),\ldots,(r_{p-1},\varphi_{p-1}))$, and define the transformation

$$\mathcal{T}_p\left(\alpha_1,\rho_{2:p},\beta_{1:p-3,2}|\,(r,\varphi)_{2:p-1},r_p\right) = \sum_{\ell_2,\ldots,\ell_{p-1}=-\infty}^{\infty} J_{\Sigma_1^{p-1}\ell_k}(\rho_p r_p)\prod_{k=2}^{p-1} e^{i\ell_k\varphi_k} J_{\ell_k}(\rho_k r_k)\cos\left(\alpha_{k-1}\textstyle\sum_{j=2}^{k-1}\ell_j-\ell_k\beta_{k+1}\right),$$

where $\alpha_1\sum_{j=2}^{1}\ell_j=0$.

Theorem 2. *Let $X(\underline{x})$ be a homogeneous and isotropic stochastic field on the plane, then the $p-$spectrum S_p of $X(\underline{x})$ is real valued and the $p-$covariance function $\mathcal{C}_p$ and the $p-$spectrum S_p are connected by the kernel function $\mathcal{T}_p$, namely,*

$$\mathcal{C}_p\left((r,\varphi)_{2:p-1},r_p\right) = 2^{p-2}\int_0^\infty\cdots\int_0^\infty\int_0^\pi\cdots\int_0^\pi \mathcal{T}_p\left(\alpha_1,\rho_{2:p},\beta_{1:p-3,2}|\,(r,\varphi)_{2:p-1},r_p\right) \times S_p(\alpha_1,\rho_{2:p},\beta_{1:p-3,2})\prod_{k=2}^{p}\rho_k d\rho_k d\alpha_1\prod_{k=1}^{p-3}d\beta_{k,2}.$$

Conversely,

$$S_p(\alpha_1,\rho_{2:p},\beta_{1:p-3,2}) = \frac{1}{(2\pi)^{p-2}}\int_0^\infty\cdots\int_0^\infty\int_0^\pi\cdots\int_0^\pi \mathcal{T}_p\left(\alpha_1,\rho_{2:p},\beta_{1:p-3,2}|\,(r,\varphi)_{2:p-1},r_p\right) \times \mathcal{C}_p\left((r,\varphi)_{2:p-1},r_p\right) r_p dr_p\prod_{k=2}^{p-1} r_k dr_k d\varphi_k,$$

provided these integrals exist.

Proof. The argument of obtaining higher order spectra is similar to the evaluation of the trispectrum, the only difference is that instead of using Lemma 2 one has to use Theorem 3. □

Appendix A. Some integrals of Bessel functions

One of the key formula necessary for deriving an expression for the bispectrum (see [Ter14]), is the integral

$$\int_0^\infty J_0(\rho_1\lambda) J_\ell(\rho_2\lambda) J_\ell(\rho_3\lambda) \lambda d\lambda = \frac{\cos(\ell \arccos(R))}{\pi\rho_2\rho_3\sqrt{1-R^2}} = \frac{\cos(\ell\alpha_1)}{\pi\rho_2\rho_3 \sin\alpha_1},$$

where $\rho_1^2 = \rho_2^2 + \rho_3^2 - 2\rho_2\rho_3\cos\alpha_1$ and $R = \left(\rho_2^2 + \rho_3^2 - \rho_1^2\right)/(2\rho_2\rho_3) = \cos\alpha_1$ (see [PBM86, 2.12.41.16]). This expression is a special case of the following result.

Lemma 1. *Let $\rho_k > 0$, $|\rho_2 - \rho_3| \le \rho_1 \le \rho_2 + \rho_3$, and $\rho_1^2 = \rho_2^2 + \rho_3^2 - 2\rho_2\rho_3\cos\alpha_1$, then*

$$\int_0^\infty J_{\ell_1}(\rho_1\lambda) J_{\ell_2}(\rho_2\lambda) J_{\ell_1+\ell_2}(\rho_3\lambda) \lambda d\lambda = \frac{\cos(\ell_1\alpha_2 - \ell_2\alpha_1)}{\pi\rho_2\rho_3\sin\alpha_1},$$

otherwise, if ρ_1, ρ_2 and ρ_3 do not form a triangle, then the integral is zero.

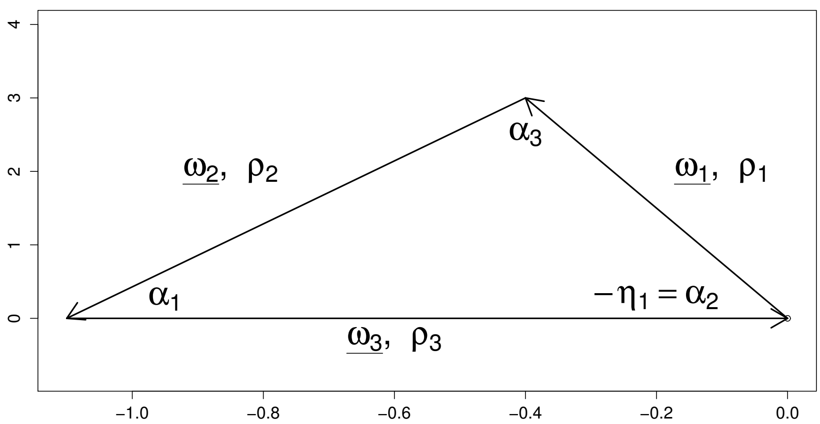

Figure 2. Triangle

Proof. Formulae for integrals of Bessel functions require care and attention, see, for instance, [Vil68, p. 224], and the Addition Theorem [Kor02, p. 27]. The assumptions imply that a triangle according to (ρ_1, ρ_2, ρ_3) can be formed, see Figure 2. Formally, the following relations are valid; $\rho_1^2 = \rho_2^2 + \rho_3^2 - 2\rho_2\rho_3\cos\alpha_1$,

$\rho_2 - \rho_3 \cos\alpha_1 = \rho_1 \cos\alpha_3$, $\rho_3 \sin\alpha_1 = \rho_1 \sin\alpha_3$, [EMOT81, p. 54], equivalently, $\sqrt{4\rho_2^2\rho_3^2 - (\rho^2 - \rho_2^2 - \rho_3^2)^2} = 2\rho_2\rho_3 \sin\alpha_1$, $\alpha_1 \in (0,\pi)$. Let us start with the Graf's Addition Theorem:

$$e^{i\ell_1\alpha_2} J_{\ell_1}(\rho_1) = \sum_{m=-\infty}^{\infty} J_m(\rho_2) J_{m+\ell_1}(\rho_3) e^{im\alpha_1}.$$

The system $e^{im\alpha}$ is orthogonal on the $[0, 2\pi]$, but the angle α_1 is changing on interval $[0,\pi]$, in this case, we have the integral

$$\int_0^\pi e^{i(m-\ell_2)\alpha} d\alpha = \begin{cases} \pi & \text{if} \quad m = \ell_2, \\ \frac{i}{m-\ell_2}\left(1 - (-1)^{m-\ell_2}\right) & \text{if} \quad m \neq \ell_2, \end{cases}$$

hence

$$\int_0^\pi e^{i\ell_1\alpha_2} J_{\ell_1}(\rho_1) e^{-i\ell_2\alpha_1} d\alpha_1 = \int_0^\pi \sum_{m=-\infty}^{\infty} J_m(\rho_2) J_{m+\ell_1}(\rho_3) e^{i(m-\ell_2)\alpha_1} d\alpha_1$$
$$= \pi J_{\ell_2}(\rho_2) J_{\ell_1+\ell_2}(\rho_3) + 2i \sum_{k=-\infty}^{\infty} \frac{1}{2k+1} J_{2k+1+\ell_2}(\rho_2) J_{2k+1+\ell_1+\ell_2}(\rho_3).$$

The real part of the above equality provides

$$\int_0^\pi \cos(\ell_1\alpha_2 - \ell_2\alpha_1) J_{\ell_1}(\lambda\rho_1) d\alpha_1 = \pi J_{\ell_2}(\lambda\rho_2) J_{\ell_1+\ell_2}(\lambda\rho_3). \tag{A.1}$$

Now integrate over $\lambda d\lambda$, and applying the formula (B.1) we get

$$\begin{aligned}
&\int_0^\infty J_{\ell_1}(\rho_1\lambda) J_{\ell_2}(\rho_2\lambda) J_{\ell_1+\ell_2}(\rho_3\lambda) \lambda d\lambda \\
&= \int_0^\infty J_{\ell_1}(\rho_1\lambda) \frac{1}{\pi} \int_0^\pi \cos(\ell_1\alpha_2 - \ell_2\gamma) J_{\ell_1}(\rho\lambda) d\gamma\lambda d\lambda \\
&= \frac{1}{\pi} \int_{|\rho_2-\rho_3|}^{\rho_2+\rho_3} \int_0^\infty J_{\ell_1}(\rho_1\lambda) J_{\ell_1}(\rho\lambda) \lambda d\lambda \frac{\cos(\ell_1\alpha_2 - \ell_2\gamma)\rho d\rho}{\rho_2\rho_3 \sin\gamma} \\
&= \frac{1}{\pi} \int_{|\rho_2-\rho_3|}^{\rho_2+\rho_3} \frac{\cos(\ell_1\alpha_2 - \ell_2\gamma)}{\rho_2\rho_3 \sin\gamma} \frac{\delta(\rho_1 - \rho)}{\rho_1} \rho d\rho = \frac{\cos(\ell_1\alpha_2 - \ell_2\alpha_1)}{\pi\rho_2\rho_3 \sin\alpha_1}.
\end{aligned}$$

The integral is zero if the inequality $|\rho_2 - \rho_3| \leq \rho_1 \leq \rho_2 + \rho_3$ is not satisfied [Vil68, p. 224]. □

We consider a quadrilateral according to the wave numbers $(\alpha_1, \rho_2, \rho_3, \rho_4)$ defined by two triangles (ρ_1, ρ_2, κ) and (κ, ρ_3, ρ_4), where $\kappa = |\underline{\kappa}|$ is the diagonal and $\rho_j = |\underline{\omega}_j|$, see Figure 3. In other words, $(\underline{\omega}_1, \underline{\omega}_2, \underline{\kappa})$ and $(\underline{\omega}_3, \underline{\omega}_4, -\underline{\kappa})$ are triangulars, and their sides (ρ_1, ρ_2, κ) and (κ, ρ_3, ρ_4) fulfil the triangle relation, i.e. the assumption

$$\max\left(|\rho_2 - \rho_1|, |\rho_4 - \rho_3|\right) < \kappa < \min\left(\rho_1 + \rho_2, \rho_3 + \rho_4\right),$$

fulfils, see Figure 3.

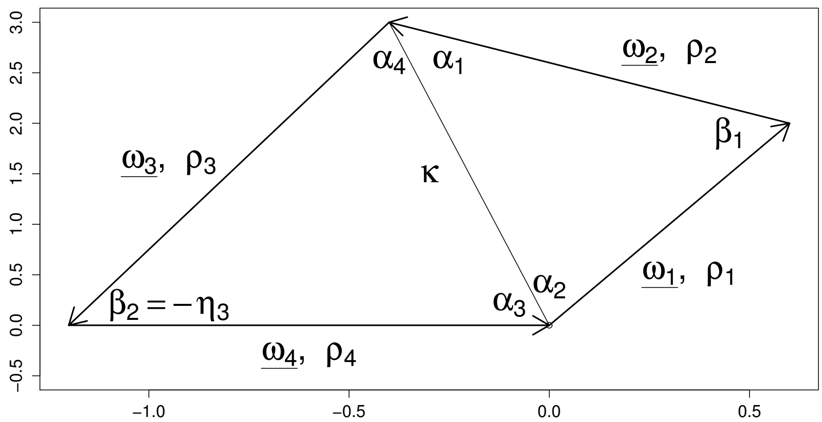

Figure 3. Quadrilateral

Lemma 2. *Assume* $\kappa^2 = \rho_3^2 + \rho_4^2 - 2\rho_3\rho_4 \cos\beta_2$, $\beta_2 \in (0, \pi)$ *and* (ρ_1, ρ_2, κ) *defines a triangle, see Figure 3, then*

$$\int_0^\infty J_{\ell_1}(\rho_1\lambda) J_{\ell_2}(\rho_2\lambda) J_{\ell_3}(\rho_3\lambda) J_{\ell_1+\ell_2+\ell_3}(\rho_4\lambda) \lambda d\lambda$$
$$= \frac{1}{\pi^2}\int_0^\pi \cos\left((\ell_1+\ell_2)\alpha_3 - \ell_3\beta_2\right) \frac{\cos\left(\ell_1\alpha_2 - \ell_2\alpha_1\right)}{\rho_2\kappa\sin\alpha_1} \delta\left(\triangle|\rho_1, \rho_2, \kappa\right) d\beta_2,$$

where the notations correspond to Figure 3, and $\delta\left(\triangle|\rho_1, \rho_2, \kappa\right)$ *is zero if* (ρ_1, ρ_2, κ) *does not form a triangle, otherwise it is* 1.

Proof. The equation (A.1) and Lemma 1 give

$$J_{\ell_3}(\rho_3\lambda) J_{\ell_1+\ell_2+\ell_3}(\rho_4\lambda) = \frac{1}{\pi}\int_{|\rho_4-\rho_3|}^{\rho_3+\rho_4} J_{\ell_1+\ell_2}(\kappa\lambda) \frac{\cos\left((\ell_1+\ell_2)\alpha_3 - \ell_3\beta_2\right)\kappa d\kappa}{\rho_3\rho_4\sin\beta_2},$$

$$\int_0^\infty J_{\ell_1}(\rho_1\lambda) J_{\ell_2}(\rho_2\lambda) J_{\ell_1+\ell_2}(\kappa\lambda)\lambda d\lambda = \frac{\cos(\ell_1\alpha_2 - \ell_2\alpha_1)}{\pi\rho_1\kappa\sin\alpha_2}\delta(\triangle|\rho_1,\rho_2,\kappa),$$

hence

$$\begin{aligned}
&\int_0^\infty J_{\ell_1}(\rho_1\lambda) J_{\ell_2}(\rho_2\lambda) J_{\ell_3}(\rho_3\lambda) J_{\ell_1+\ell_2+\ell_3}(\rho_4\lambda)\lambda d\lambda \\
&= \frac{1}{\pi}\int_{|\rho_4-\rho_3|}^{\rho_4+\rho_3} \frac{\cos((\ell_1+\ell_2)\alpha_3 - \ell_3\beta_2)}{\rho_3\rho_4\sin\beta_2}\int_0^\infty J_{\ell_1}(\rho_1\lambda) J_{\ell_2}(\rho_2\lambda) J_{\ell_1+\ell_2}(\kappa\lambda)\lambda d\lambda\kappa d\kappa \\
&= \frac{1}{\pi^2}\int_{|\rho_4-\rho_3|}^{\rho_4+\rho_3} \frac{\cos((\ell_1+\ell_2)\alpha_3 - \ell_3\beta_2)}{\rho_3\rho_4\sin\beta_2}\frac{\cos(\ell_1\alpha_2 - \ell_2\alpha_1)}{\rho_1\kappa\sin\alpha_2}\delta(\triangle|\rho_1,\rho_2,\kappa)\kappa d\kappa \\
&= \frac{1}{\pi^2}\int_0^\pi \cos((\ell_1+\ell_2)\alpha_3 - \ell_3\beta_2)\frac{\cos(\ell_1\alpha_2 - \ell_2\alpha_1)}{\rho_2\kappa\sin\alpha_1}\delta(\triangle|\rho_1,\rho_2,\kappa)d\beta_2,
\end{aligned}$$

where $\sqrt{(2\rho_3\rho_4)^2 - (\kappa^2-\rho_3^2-\rho_4^2)^2} = 2\rho_3\rho_4\sin\beta_2$, $\kappa d\kappa = \rho_3\rho_4\sin(\beta_2)d\beta_2$, $\rho_1\sin\beta_1 = \kappa\sin\alpha_1$, see Figure 3. Note that if we are given wave numbers $(\rho_1,\rho_2,\rho_3,\rho_4)$ and if κ changes, then not only β_2 will change, but all the angles as well. □

For further generalization of Lemma 1, we consider multilaterals on the plane. A multilateral of order 5, say, has 5 vertices and 2 diagonals, see Figure 4. Under the motion of a rigid body, the angles, the lengths of the sides, and the diagonals are invariant. The multilateral will be well defined if the length of the sides and diagonals are given, one may replace the diagonals by the angle opposite them. For instance, the $\kappa_2 = |\underline{\kappa}_2|$ and angle $\beta_{2,2}$ are equivalent in determining the triangle together with sides $\rho_4 = |\underline{\omega}_4|$ and $\rho_5 = |\underline{\omega}_5|$.

Theorem 3. *Let $p \geq 4$ and consider a multilateral of order p, then*

$$\begin{aligned}
&\int_0^\infty J_{\Sigma_1^{p-1}\ell_k}(\rho_p\lambda)\prod_{k=1}^{p-1} J_{\ell_k}(\rho_k\lambda)\lambda d\lambda \\
&\quad = \frac{1}{\pi^{p-2}}\int_0^\pi\cdots\int_0^\pi \frac{\cos(\ell_1\alpha_2 - \ell_2\alpha_1)}{\rho_2\kappa_1\sin(\alpha_1)} \\
&\quad\times\prod_{k=2}^{p-2}\cos\left(\alpha_{k+1}\textstyle\sum_{j=1}^k \ell_j - \ell_{k+1}\beta_{k-1,2}\right)\delta(\triangle|\rho_{k+2},\kappa_k,\kappa_{k+1})d\beta_{k-1,2},
\end{aligned}$$

where each angle α_k is opposite to ρ_k, and angles $\beta_{k,1}$, $\beta_{k,2}$ are opposite to diagonal κ_k on the right and on the left, respectively, see Figure 4 for notations.

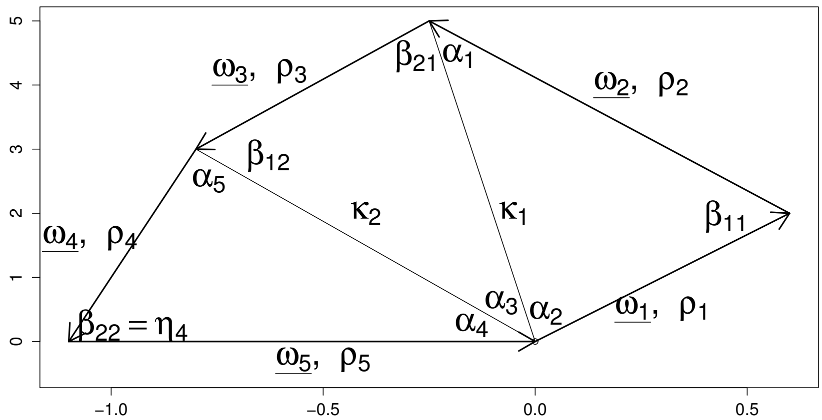

Figure 4. Multilateral

Proof. A multilateral can be split up into $p-2$ triangles, see Figure 4. We show that from $p=4$ follows $p=5$, such that the pattern of general induction shows up. By the Addition Theorem, we have

$$J_{\ell_4}(\rho_4\lambda)\,J_{\ell_1+\ell_2+\ell_3+\ell_4}(\rho_5\lambda) = \frac{1}{\pi}\int_0^{\pi} J_{\ell_1+\ell_2+\ell_3}(\kappa_2\lambda)\cos\left((\ell_1+\ell_2+\ell_3)\,\alpha_4-\ell_4\beta_{2,2}\right)d\beta_{2,2},$$

and the result of Lemma 2 leads us to the formula

$$\int_0^{\infty} J_{\ell_1}(\rho_1\lambda)\,J_{\ell_2}(\rho_2\lambda)\,J_{\ell_3}(\rho_3\lambda)\,J_{\ell_1+\ell_2+\ell_3}(\kappa_2\lambda)\,\lambda d\lambda = \frac{1}{\pi^2}\int_0^{\pi}\cos\left((\ell_1+\ell_2)\,\alpha_3-\ell_3\beta_{1,2}\right)\cos\left(\ell_1\alpha_2-\ell_2\alpha_1\right)\frac{d\beta_{1,2}}{\rho_2\kappa_1\sin\alpha_1},$$

hence we obtain

$$\begin{aligned}
&\int_0^{\infty} J_{\ell_1}(\rho_1\lambda)\,J_{\ell_2}(\rho_2\lambda)\,J_{\ell_3}(\rho_3\lambda)\,J_{\ell_4}(\rho_4\lambda)\,J_{\ell_1+\ell_2+\ell_3+\ell_4}(\rho_5\lambda)\,\lambda d\lambda\\
&\quad=\frac{1}{\pi^3}\int_0^{\infty} J_{\ell_1}(\rho_1\lambda)\,J_{\ell_2}(\rho_2\lambda)\,J_{\ell_3}(\rho_3\lambda)\int_0^{\pi} J_{\ell_1+\ell_2+\ell_3}(\kappa_2\lambda)\\
&\qquad\times\cos\left((\ell_1+\ell_2+\ell_3)\,\alpha_4-\ell_4\beta_{2,2}\right)d\beta_{2,2}\lambda d\lambda\\
&\quad=\frac{1}{\pi^3}\int_0^{\pi}\int_0^{\pi}\cos\left(\ell_1\alpha_2-\ell_2\alpha_1\right)\cos\left((\ell_1+\ell_2)\,\alpha_3-\ell_3\beta_{1,2}\right)\\
&\qquad\times\frac{\cos\left((\ell_1+\ell_2+\ell_3)\,\alpha_4-\ell_4\beta_{2,2}\right)d\beta_{2,2}d\beta_{1,2}}{\rho_2\kappa_1\sin\alpha_1}.
\end{aligned}$$

□

Appendix B. Dirac-function in polar coordinates

The covariance function of a homogeneous random field has a symmetric spectral representation where the Dirac 'function' is involved. Let $\delta(\cdot)$ denote the Dirac 'function', more precisely, $\delta(\cdot)$ is a distribution putting all the mass at zero, for instance, the integral of Bessel functions provides Dirac function

$$\int_0^\infty J_\ell(\rho r)\, J_\ell(\kappa r)\, r dr = \frac{\delta(\rho-\kappa)}{\rho}, \tag{B.1}$$

see [AW01, Section 11, p. 691]. We shall apply the Jacobi–Anger expansion on the plane

$$e^{i\rho r\cos(\varphi-\eta)} = \sum_{\ell=-\infty}^{\infty} i^\ell J_\ell(\rho r)\, e^{i\ell(\varphi-\eta)}. \tag{B.2}$$

In order to understand the influence of the Dirac 'function' in polar coordinates, we express it by the integral through the Jacobi–Anger expansion (B.2), and obtain

$$\delta\left(\Sigma_1^p \rho_k \underline{\widehat{\omega}}_k\right) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{i\left(\underline{\lambda}\cdot\Sigma_1^p \underline{\omega}_k\right)} d\underline{\lambda}, \tag{B.3}$$

where the sum of vectors is invariant under permutation:

$$\begin{aligned}
&\delta\left(\Sigma_1^p \rho_k \underline{\widehat{\omega}}_k\right)\\
&= \frac{1}{(2\pi)^2}\int_0^\infty\int_0^{2\pi}\prod_{k=1}^{p}\sum_{m_k=-\infty}^{\infty} i^{m_k} J_{m_k}(\rho_k\lambda)\, e^{im_k(\eta_k-\xi)}\lambda d\lambda d\xi\\
&= \frac{1}{(2\pi)^2}\int_0^\infty\int_0^{2\pi}\sum_{m_{1:p}=-\infty}^{\infty} i^{\Sigma_1^p m_k} e^{i\Sigma_1^p m_k(\eta_k-\xi)}\prod_{k=1}^{p} J_{m_k}(\rho_k\lambda)\,\lambda d\lambda d\xi\\
&= \frac{\delta_{\Sigma_1^p m_k}}{2\pi}\int_0^\infty\sum_{m_{1:p-1}=-\infty}^{\infty} e^{i\Sigma_1^{p-1} m_k\left(\eta_k-n_p\right)} J_{-\Sigma_1^{p-1} m_k}(\rho_p\lambda)\prod_{k=1}^{p-1} J_{m_k}(\rho_k\lambda)\,\lambda d\lambda\\
&= \frac{\delta_{\Sigma_1^p m_k}}{2\pi}\int_0^\infty\sum_{m_{1:p-1}=-\infty}^{\infty} (-1)^{\Sigma_1^{p-1} m_k} e^{i\Sigma_1^{p-1} m_k\left(\eta_k-n_p\right)} J_{\Sigma_1^{p-1} m_k}(\rho_p\lambda)\prod_{k=1}^{p-1} J_{m_k}(\rho_k\lambda)\,\lambda d\lambda\\
&= \frac{\delta_{\Sigma_1^p m_k}}{2\pi}\int_0^\infty\sum_{m_{1:p-1}=-\infty}^{\infty} e^{i\Sigma_1^{p-1} m_k\left(\eta_k-n_p-\pi\right)} J_{\Sigma_1^{p-1} m_k}(\rho_p\lambda)\prod_{k=1}^{p-1} J_{m_k}(\rho_k\lambda)\,\lambda d\lambda
\end{aligned}$$

since $\Sigma_1^p m_k = 0$, $m_p = -\Sigma_1^{p-1} m_k$. We can apply here Theorem 3 for a clear expression. Some particular cases are as follows.

(1) If $\boldsymbol{p=2}$,

$$\delta\left(\Sigma_1^2\rho_k\underline{\widehat{\omega}}_k\right) = \frac{1}{2\pi}\int_0^\infty \sum_{m=-\infty}^{\infty} (-1)^m e^{im(\eta_1-\eta_2)} J_m(\rho_1\lambda) J_m(\rho_2\lambda)\lambda d\lambda$$
$$= \frac{1}{2\pi}\frac{\delta(\rho_1-\rho_2)}{\rho_1}\sum_{m=-\infty}^{\infty} e^{im(\eta_1-\eta_2-\pi)} = \frac{\delta(\rho_1-\rho_2)}{\rho_2}\delta(\eta_1-\eta_2-\pi),$$

hence the integral is taken according to the subspace $\rho_1=\rho_2$ and $\eta_1=\eta_2+\pi$, it corresponds to $\underline{\omega}_1=-\underline{\omega}_2$, this subspace is the one what is expected.

(2) For $\boldsymbol{p=3}$, we apply Lemma 1,

$$\begin{aligned}
&\delta\left(\Sigma_1^3\rho_k\underline{\widehat{\omega}}_k\right)\\
&= \frac{1}{2\pi}\int_0^\infty \sum_{m_{1:2}=-\infty}^{\infty} e^{i\Sigma_1^2 m_k(\eta_k-\eta_3-\pi)} J_{m_1+m_2}(\rho_3\lambda)\prod_{k=1}^{2} J_{m_k}(\rho_k\lambda)\lambda d\lambda\\
&= \frac{\delta(\triangle|\rho_1,\rho_2,\rho_3)}{2\pi^2\rho_2\rho_3\sin\alpha_1}\sum_{m_{1:2}=-\infty}^{\infty} e^{i\Sigma_1^2 m_k(\eta_k-\eta_3-\pi)}\cos(m_2\alpha_1-m_1\alpha_2)\\
&= \frac{\delta(\triangle|\rho_1,\rho_2,\rho_3)}{(2\pi)^2\rho_2\rho_3\sin\alpha_1}\left(\sum_{m_1=-\infty}^{\infty} e^{im_1(\eta_1-\eta_3-\pi-\alpha_2)}\sum_{m_2=-\infty}^{\infty} e^{im_2(\eta_2-\eta_3-\pi+\alpha_1)}\right.\\
&\quad\left.+\sum_{m_1=-\infty}^{\infty} e^{im_1(\eta_1-\eta_3-\pi+\alpha_2)}\sum_{m_2=-\infty}^{\infty} e^{im_2(\eta_2-\eta_3-\pi-\alpha_1)}\right)\\
&= \frac{\delta(\triangle|\rho_1,\rho_2,\rho_3)}{\rho_2\rho_3\sin\alpha_1}\left(\delta(\eta_1-\eta_3-\pi-\alpha_2)\,\delta(\eta_2-\eta_3-\pi+\alpha_1)\right.\\
&\quad\left.+\delta(\eta_1-\eta_3-\pi+\alpha_2)\,\delta(\eta_2-\eta_3-\pi-\alpha_1)\right),
\end{aligned} \tag{B.4}$$

where the notations of Figure 2 are used. Here the Dirac 'function' is concentrated on the subspace when (ρ_1,ρ_2,ρ_3) forms a triangle, this triangle defines angles α_1, α_2, α_3, see Figure 2. Once α_1, α_2, α_3 are given, there are two possible choices for angles $\eta_1-\eta_3$, $\eta_2-\eta_3$, such that η_3 varies from 0 to 2π. Actually, we plotted the case when $\eta_3=0$, see Figure 2. One can also check that the set $\Sigma_1^3\rho_k\underline{\widehat{\omega}}_k=0$ will not change if we put $m_2=-m_1-m_3$ in (B.4) instead of $m_3=-m_1-m_2$, although it may be counted when the principal domain of the bispectrum is of interest.

(3) Similarly, for $\boldsymbol{p}=\mathbf{4}$, we have

$$\begin{aligned}
&\delta\left(\Sigma_1^4\rho_k\underline{\widehat{\omega}}_k\right)\\
&\quad=\frac{1}{2\pi}\int_0^\infty\sum_{m_{1:3}=-\infty}^{\infty}e^{i\Sigma_1^3 m_k(\eta_k-\eta_4-\pi)}J_{m_1+m_2+m_3}\left(\rho_4\lambda\right)\prod_{k=1}^{3}J_{m_k}\left(\rho_k\lambda\right)\lambda d\lambda\\
&\quad=\frac{1}{2\pi^3}\sum_{m_{1:3}=-\infty}^{\infty}e^{i\Sigma_1^3 m_k(\eta_k-\eta_4-\pi)}\int_0^\pi\cos\left(m_1\alpha_2-m_2\alpha_1\right)\\
&\quad\quad\times\cos\left(\left(m_1+m_2\right)\alpha_3-m_3\beta_2\right)\frac{\delta\left(\triangle|\rho_1,\rho_2,\kappa\right)}{\rho_2\kappa\sin\alpha_1}d\beta_2, \qquad \text{(B.5)}
\end{aligned}$$

see Lemma 2 and Figure 3 for this case. Since $\rho_k\underline{\widehat{\omega}}_k$ are given, one can expect some more precise expression. Indeed,

$$\begin{aligned}
&\frac{1}{2\pi^3}\sum_{m_{1:3}=-\infty}^{\infty}e^{i\Sigma_1^3 m_k(\eta_k-\eta_4-\pi)}\cos\left(m_1\alpha_2-m_2\alpha_1\right)\cos\left(\left(m_1+m_2\right)\alpha_3-m_3\beta_2\right)\\
&\quad=\delta\left(\eta_1-\eta_4-\pi+\alpha_2+\alpha_3\right)\delta\left(\eta_2-\eta_4-\pi-\alpha_1+\alpha_3\right)\delta\left(\eta_3-\eta_4-\pi-\beta_2\right)\\
&\quad\quad+\delta\left(\eta_1-\eta_4-\pi+\alpha_2-\alpha_3\right)\delta\left(\eta_2-\eta_4-\pi-\alpha_1-\alpha_3\right)\delta\left(\eta_3-\eta_4-\pi+\beta_2\right)\\
&\quad\quad+\delta\left(\eta_1-\eta_4-\pi-\alpha_2+\alpha_3\right)\delta\left(\eta_2-\eta_4-\pi+\alpha_1+\alpha_3\right)\delta\left(\eta_3-\eta_4-\pi-\beta_2\right)\\
&\quad\quad+\delta\left(\eta_1-\eta_4-\pi-\alpha_2-\alpha_3\right)\delta\left(\eta_2-\eta_4-\pi+\alpha_1-\alpha_3\right)\delta\left(\eta_3-\eta_4-\pi+\beta_2\right). \qquad \text{(B.6)}
\end{aligned}$$

Now, for a given $\alpha_1,\rho_2,\rho_3,\rho_4$, the diagonal κ and β_2 are equivalent, $\kappa\left(\beta_2\right)=\sqrt{\rho_3^2+\rho_4^2-2\rho_3\rho_4\cos\beta_2}$, say, and let β_2 be the subject of changes. Hence α_3 is determined, together with α_1 and α_2, see Figure 3. It follows that $\eta_3-\eta_4=\pi\pm\beta_2$, then with each choice of $\eta_3-\eta_4$ we have two possibilities for $\eta_1-\eta_4$ and $\eta_2-\eta_4$. These later angles $\eta_1-\eta_4$ and $\eta_2-\eta_4$ are determined by α_1, α_2 and α_3.

Appendix C. Cumulants of spectral measures $Z_\ell\left(\rho d\rho\right)$

We generalize the joint cumulant stochastic spectral measures

$$\begin{aligned}
&\operatorname{Cum}\left(Z_0\left(\rho_1 d\rho_1\right),Z_\ell\left(\rho_2 d\rho_2\right),Z_{-\ell}\left(\rho_3 d\rho_3\right)\right)\\
&\quad=2\left(-1\right)^\ell\delta\left(\triangle|\rho_1,\rho_2,\rho_3\right)\frac{\cos\left(\ell\arccos\left(R\right)\right)}{\rho_2\rho_3\sqrt{1-R^2}}S_3\left(\rho_1,\rho_2,\rho_3\right)\prod_{k=1}^{3}\rho_k d\rho_k,
\end{aligned}$$

where $R=\left(\rho_2^2+\rho_3^2-\rho_1^2\right)/\left(2\rho_2\rho_3\right)=\cos\alpha_1$ and $\delta\left(\triangle|\rho_1,\rho_2,\rho_3\right)=1$, if ρ_1,ρ_2,ρ_3 constitute a triangle, and 0 otherwise, see [Ter14] in order to get the formula for trispectrum and higher order spectra. $\delta(\triangle|\rho_1,\rho_2,\rho_3)$ implies that the wave numbers ρ_1, ρ_2, and ρ_3 should satisfy the triangle relation.

Consider the fourth-order cumulant

$$\begin{aligned}&\operatorname{Cum}\left(Z_0(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2), Z_{\ell_3}(\rho_3 d\rho_3), Z_{-(\ell_2+\ell_3)}(\rho_4 d\rho_4)\right)\\ &\quad = \iiiint_0^{2\pi} e^{-i\ell_2(\eta_2-\eta_4)-i\ell_3(\eta_3-\eta_4)}\delta\left(\Sigma_1^4 \rho_k \underline{\widehat{\omega}}_k\right) S_4(\alpha_1,\rho_{2:4},\beta_2)\prod_{k=1}^4 d\eta_k \prod_{k=1}^4 \rho_k d\rho_k,\end{aligned}$$

replace the Dirac-function by (B.3), (B.5), and use the orthogonality of the 'spherical harmonics',

$$\iiiint_0^{2\pi} e^{-i\ell_2(\eta_2-\eta_4)-i\ell_3(\eta_3-\eta_4)} e^{i\Sigma_1^3 m_k(\eta_k-\eta_4)}\prod_{k=1}^4 d\eta_k = \delta_{m_1}\delta_{m_2-\ell_2}\delta_{m_3-\ell_3}(2\pi)^4,$$

and a particular case of Lemma 2

$$\begin{aligned}&\int_0^\infty J_0(\rho_1\lambda)\,J_{\ell_2}(\rho_2\lambda)\,J_{\ell_3}(\rho_3\lambda)\,J_{\ell_2+\ell_3}(\rho_4\lambda)\,\lambda d\lambda\\ &\quad = \frac{1}{\pi^2}\int_0^\pi \cos\left((\ell_1+\ell_2)\alpha_3-\ell_3\beta_2\right)\frac{\cos(\ell_1\alpha_2-\ell_2\alpha_1)}{\pi\rho_2\kappa\sin\alpha_1}\delta(\triangle|\rho_1,\rho_2,\kappa)\,d\beta_2,\end{aligned}$$

see Figure 3 for notations. The result is

$$\begin{aligned}&\operatorname{Cum}\left(Z_0(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2), Z_{\ell_3}(\rho_3 d\rho_3), Z_{-(\ell_2+\ell_3)}(\rho_4 d\rho_4)\right)\\ &\quad = 4(-1)^{\ell_2+\ell_3}\int_0^\pi \frac{\delta(\triangle|\rho_1,\rho_2,\kappa)}{\rho_1\kappa\sin\alpha_2}\cos(\ell_2\alpha_1)\cos(\ell_2\alpha_3-\ell_3\beta_2)\\ &\qquad \times S_4(\alpha_1,\rho_{2:4},\beta_2)\,d\beta_2\prod_{k=1}^4\rho_k d\rho_k.\end{aligned}$$

We obtain the cumulant similarly for general p, it follows from a particular case of Theorem 3, when $\ell_1 = 0$, see Figure 4.

Lemma 3.

$$\begin{aligned}&\operatorname{Cum}\left(Z_0(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2), Z_{\ell_3}(\rho_3 d\rho_3),\ldots, Z_{-(\ell_2+\ell_3\cdots+\ell_{p-1})}(\rho_p d\rho_p)\right)\\ &\quad = (-1)^{\ell_2+\ell_3\cdots+\ell_{p-1}}\,2^{p-2}\int_0^\pi\cdots\int_0^\pi S_p(\alpha_1,\rho_{2:p},\beta_{1:p-3,2})\\ &\quad \times L(\ell_{2:p-1},\alpha_1,\beta_{1:p-3,1})\prod_{k=2}^{p-2}d\beta_{k-1,2}\prod_{m=1}^p\rho_m d\rho_m,\end{aligned}$$

where

$$L\left(\ell_{2:p-1},\alpha_{1:p-1},\beta_{1:p-1,2}\right) = \frac{\cos\left(\ell_2\alpha_1\right)}{\rho_2\kappa_1\sin\left(\alpha_1\right)}\prod_{k=2}^{p-2}\cos\left(\alpha_{k+1}\sum\nolimits_{j=2}^{k}\ell_j-\ell_{k+1}\beta_{k-1,2}\right)\delta\left(\triangle|\rho_{k+2},\kappa_k,\kappa_{k+1}\right).$$

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GYÖRGY TERDIK
FACULTY OF INFORMATICS
UNIVERSITY OF DEBRECEN
H-4002 DEBRECEN
P. O. BOX 400
HUNGARY

E-mail: terdik.gyorgy@inf.unideb.hu

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