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Derivations into various duals of Lau product of Banach algebras

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Abstract. For two Banach algebras A and B, an interesting product $A\times_\theta B$, called the θ-Lau product, was recently introduced and studied for some non-zero multiplicative linear functional θ on B. In this paper, by discussing general necessary and sufficient conditions for n-weak amenability of $A\times_\theta B$, we extend some results on the n-weak amenability of the unitization $A^\sharp$ of A, to the θ-Lau product $A\times_\theta B$. In particular, we improve several known results on n-weak amenability of $A\times_\theta B$ and answer some questions on this topic.

1. Introduction and some preliminaries

Let A and B be Banach algebras with $\sigma(B)\neq\emptyset$, and let $\theta\in\sigma(B)$, where $\sigma(B)$ is the set of all non-zero multiplicative linear functionals on B. The θ-Lau product $A\times_\theta B$ is a Banach algebra which is defined as the vector space $A\times B$ equipped with the algebra multiplication

$$(a_1,b_1)(a_2,b_2)=(a_1a_2+\theta(b_2)a_1+\theta(b_1)a_2,b_1b_2)\qquad(a_1,a_2\in A,\ b_1,b_2\in B),$$

and the norm $\|(a,b)\|=\|a\|+\|b\|$. This type of product was introduced by Lau [8] for a certain class of Banach algebras known as Lau algebras, and was extended by Sangani Monfared [9] for arbitrary Banach algebras. The unitization $A^\sharp$ of A can be regarded as the ι-Lau product $A\times_\iota\mathbb{C}$, where $\iota\in\sigma(\mathbb{C})$ is the identity map.

This product provides not only new examples of Banach algebras by themselves, but it can also serve as a source of (counter-) examples for various purposes in functional and harmonic analysis. From the homological algebra point

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of view, $A \times_\theta B$ is a strongly splitting Banach algebra extension of B by A, which means that A is a closed two-sided ideal of $A \times_\theta B$, and the quotient $(A \times_\theta B)/A$ is isometrically isomorphic to B. The Lau product of Banach algebras enjoys some properties that are not shared in general by arbitrary strongly splitting extensions. For instance, commutativity is not preserved by a generally strongly splitting extension. However, $A \times_\theta B$ is commutative if and only if both A and B are commutative.

Many basic properties of $A^\sharp$, some notions of amenability and some homological properties are extended to $A \times_\theta B$ by many authors; see, for example, [4], [7], [9], [10] and [11]. In particular, Ghaderi, Nasr-Isfahani and Nemati [4] extended some results on n-weak amenability of $A^\sharp$, obtained by Dales, Ghahramani and Gronbeak [3], to $A \times_\theta B$. They showed that if A and B are $(2n+1)$-weakly amenable, then $A \times_\theta B$ is $(2n+1)$-weakly amenable, [4, Theorem 4.1]. For a continuous derivation $D : A \times_\theta B \to (A \times_\theta B)^{(2n+1)}$ with $D(a, 0) = 0$, they claim that $D(0, b) \in B^{(2n+1)}$. By a careful look at their proof, we could only conclude that $(\iota^{(2n+1)} \circ D)(0, b) = 0$ on the subspace $\langle AA^{(2n)} \cup A^{(2n)}A \rangle$ of $A^{(2n)}$, where $\iota : A \to A \times_\theta B$ is the natural embedding. So, there appear to be some gaps in their proof. This result, under a suitable condition on A, was proved by Ebrahimi Vishki and Khoddami [12], but the question of whether there is an analogue to this result for the even case was left open. Moreover, in the case when A is unital, it was shown that $A \times_\theta B$ is n-weakly amenable if and only if both A and B are n-weakly amenable. It was also left as an open question whether this result holds for the case when A has a bounded approximate identity.

The n-weak amenability of the unitization $A^\sharp$ of a Banach algebra A was also studied by Zhang [13]. As a main result of [13, Section 3.2], he showed that if A is weakly amenable or has a bounded approximate identity, then for each $n \geq 0$, $A^\sharp$ is n-weakly amenable if and only if A is n-weakly amenable [13, Theorem 3.16].

In this paper, we discuss general necessary and sufficient conditions for $A \times_\theta B$ to be n-weakly amenable, for an integer $n \geq 0$. We extend some results about the n-weak amenability of $A^\sharp$, obtained by Zhang, to the θ-Lau product $A \times_\theta B$. In particular, we improve several results on n-weak amenability of $A \times_\theta B$, fix the gap in Theorem 4.1 of [4], and partially answer some questions on this topic.

2. $(2n + 1)$-weak amenability

We start this section with some preliminaries about n-weak amenability. Let A be a Banach algebra, and X a Banach A-bimodule. Then the dual space X^*

of X becomes a dual Banach A-bimodule with the module actions defined by

$$(fa)(x) = f(ax), \qquad (af)(x) = f(xa),$$

for all $a \in A, x \in X$ and $f \in X^*$. Similarly, the n-th dual $X^{(n)}$ of X is a Banach A-bimodule. In particular, $A^{(n)}$ is a Banach A-bimodule. A derivation from A into X is a linear mapping $D : A \to X$, satisfying

$$D(ab) = D(a)b + aD(b) \qquad (a, b \in A).$$

If $x \in X$, then $d_x : A \to X$ defined by $d_x(a) = ax - xa$ is a derivation. A derivation D is inner if there is an $x \in X$ such that $D = d_x$.

A Banach algebra A is called n-weakly amenable, for an integer $n \geq 0$, if every continuous derivation from A into $A^{(n)}$ is inner, where $A^{(0)} = A$. The algebra A is said to be weakly amenable if it is 1-weakly amenable. The concept of weak amenability was first introduced by Bade, Curtis and Dales in [1] for commutative Banach algebras, and was extended to the noncommutative case by Johnson [5]. Dales, Ghahramani and Grønbæk [3] initiated and intensively developed the study of n-weak amenability of Banach algebras.

Throughout the paper, n is assumed to be a non-negative integer, A and B are assumed to be Banach algebras, and θ an element of $\sigma(B)$. For brevity of notation, we usually identify an element of A with its canonical image in $A^{(2n)}$, as well as an element of A^* with its image in $A^{(2n+1)}$.

The Banach space $(A \times_\theta B)^{(2n+1)}$ can be identified with the Banach space $A^{(2n+1)} \times B^{(2n+1)}$ equipped with the maximum norm $\|(f, g)\| = \max\{\|f\|, \|g\|\}$ in the natural way. By induction, we find that the $(A \times_\theta B)$-bimodule actions on $(A \times_\theta B)^{(2n+1)}$ are formulated as follows:

$$(f, g)(a, b) = (fa + \theta(b)f, gb + f(a)\theta),$$
$$(a, b)(f, g) = (af + \theta(b)f, bg + f(a)\theta),$$

for $a \in A, b \in B, f \in A^{(2n+1)}$ and $g \in B^{(2n+1)}$.

To clarify the relation between $(2n+1)$-weak amenability of $A \times_\theta B$ and that of A and B, we begin with the following lemma which plays a key role in the sequel. This lemma was proved in [12, Proposition 2.1].

Lemma 2.1. *A mapping $D : A \times_\theta B \to (A \times_\theta B)^{(2n+1)}$ is a continuous derivation if and only if*

$$D(a, b) = (D_A(b) + T_A(a), D_B(b) + T_B(a)),$$

for all $a \in A$ and $b \in B$, where

(a) $D_B : B \to B^{(2n+1)}$ *is a continuous derivation.*

(b) $D_A : B \to A^{(2n+1)}$ *is a bounded linear operator such that* $D_A(b_1b_2) = \theta(b_1)D_A(b_2) + \theta(b_2)D_A(b_1)$ *for all* $b_1, b_2 \in B$, *and* $D_A(b)a = aD_A(b) = 0$ *for all* $a \in A$ *and* $b \in B$.

(c) $T_B : A \to B^{(2n+1)}$ *is a bounded linear operator such that* $bT_B(a) = T_B(a)b$ *and* $\theta(b)T_B(a) = bT_B(a) + D_A(b)(a)\theta$ *for all* $a \in A$ *and* $b \in B$.

(d) $T_A : A \to A^{(2n+1)}$ *is a continuous derivation such that* $(T_A(a_1)(a_2) + T_A(a_2)(a_1))\,\theta = T_B(a_1a_2)$ *for* $a_1, a_2 \in A$.

Moreover, $D = d_{(f,g)}$, *for some* $f \in A^{(2n+1)}$ *and* $g \in B^{(2n+1)}$, *if and only if* $D_B = d_g$, $T_A = d_f$, $D_A = 0$ *and* $T_B = 0$.

As a first result, we give general necessary and sufficient conditions for $A \times_\theta B$ to be $(2n+1)$-weakly amenable.

Theorem 2.2. *The* θ*-Lau product* $A \times_\theta B$ *is* $(2n+1)$*-weakly amenable if and only if*

(1) B *is* $(2n+1)$*-weakly amenable.*

(2) *The only bounded linear operator* $S : A \to B^{(2n+1)}$, *such that* $S(a_1a_2) = 0$ *for all* $a_1, a_2 \in A$ *and* $bS(a) = S(a)b = \theta(b)S(a)$ *for all* $b \in B$ *and* $a \in A$, *is zero.*

(3) *If* $T : A \to A^{(2n+1)}$ *is a continuous derivation such that there exists a bounded linear operator* $S : A \to B^{(2n+1)}$ *satisfying* $(T(a_1)(a_2) + T(a_2)(a_1))\,\theta = S(a_1a_2)$ *for all* $a_1, a_2 \in A$, *then* T *is inner.*

Before we prove this theorem, we need the following lemmas.

Lemma 2.3. *Condition* (2) *in Theorem 2.2 is equivalent to the density of* A^2 *in* A.

Proof. Let condition (2) in Theorem 2.2 hold, and let $f \in A^*$ be such that $f|_{A^2} = 0$. Define $S : A \to B^{(2n+1)}$ by $S(a) = f(a)\theta$ for all $a \in A$. Then S is a bounded linear operator such that $S(a_1a_2) = 0$ for all $a_1, a_2 \in A$, and $bS(a) = S(a)b = \theta(b)S(a)$ for all $b \in B$ and $a \in A$. So, $S = 0$. This shows that $f = 0$. Therefore, A^2 is dense in A. The converse is clear. □

Lemma 2.4. *Let* B *be a weakly amenable Banach algebra, and* X *be a Banach space. If* $D : B \to X$ *is a bounded linear operator such that* $D(b_1b_2) = \theta(b_1)D(b_2) + \theta(b_2)D(b_1)$ *for all* $b_1, b_2 \in B$, *then* $D = 0$.

Proof. Let $f \in X^*$. Then $f \circ D : B \to \mathbb{C}$ is a continuous point derivation at θ, so it is zero [3, Proposition 1.3]. This shows that $D = 0$. □

Now, we are ready to prove Theorem 2.2.

Proof. To prove the necessity, suppose that $A \times_\theta B$ is $(2n+1)$-weakly amenable. Let $D : B \to B^{(2n+1)}$ be a continuous derivation. Then $\overline{D} : A \times_\theta B \to (A \times_\theta B)^{(2n+1)}$ defined by $\overline{D}(a,b) = (0, D(b))$ is a continuous derivation, and hence it is inner. Lemma 2.1 implies that D is also inner. So B is $(2n+1)$-weakly amenable.

To prove (2), let $S : A \to B^{(2n+1)}$ be a bounded linear operator such that $S(a_1a_2) = 0$ for all $a_1, a_2 \in A$ and $bS(a) = S(a)b = \theta(b)S(a)$ for all $b \in B$ and $a \in A$. Define $\overline{D} : A \times_\theta B \to (A \times_\theta B)^{(2n+1)}$ by $\overline{D}(a,b) = (0, S(a))$. Then $\overline{D}$ is a continuous derivation, by Lemma 2.1. Thus $S = 0$ by the innerness of $\overline{D}$.

By a similar argument, we can prove (3). Indeed, suppose that $T : A \to A^{(2n+1)}$ is a continuous derivation, and $S : A \to B^{(2n+1)}$ is a bounded linear operator satisfying

$$(T(a)(c) + T(c)(a))\,\theta = S(ac),$$

for all $a, c \in A$. This, together with Lemma 2.3, implies that $bS(a) = S(a)b = \theta(b)S(a)$ for all $a \in A$ and $b \in B$. Define $\overline{D} : A \times_\theta B \to (A \times_\theta B)^{(2n+1)}$ by $\overline{D}(a,b) = (T(a), S(a))$. Then Lemma 2.1 implies that $\overline{D}$ is a continuous derivation, so it is inner. Therefore T is inner, as required. This completes the proof of necessity.

To prove the sufficiency, suppose that conditions (1)–(3) hold. Let $\overline{D} : A \times_\theta B \to (A \times_\theta B)^{(2n+1)}$ be a continuous derivation. Then

$$\overline{D}(a,b) = (D_A(b) + T_A(a), D_B(b) + T_B(a)), \quad (a \in A, b \in B),$$

in which the component mappings D_A, D_B, T_A and T_B are satisfying the conditions (a)–(d) of Lemma 2.1. By condition (1) and [3, Proposition 1.2], B is weakly amenable, and so Lemma 2.4 implies that $D_A = 0$. By conditions (1) and (3), D_B and T_A are inner derivations. Since $D_A = 0$, it follows that $bT_B(a) = T_B(a)b = \theta(b)T_B(a)$ for all $b \in B$ and $a \in A$. Moreover, since T_A is inner, $T_A(a_1)(a_2) + T_A(a_2)(a_1) = 0$, for all $a_1, a_2 \in A$. It follows that $T_B(a_1a_2) = 0$ for all $a_1, a_2 \in A$. From condition (2), $T_B = 0$. Therefore, $\overline{D}$ is inner, by Lemma 2.1. This proves that $A \times_\theta B$ is $(2n+1)$-weakly amenable, as claimed. □

As an immediate consequence of Theorem 2.2, we have the next result, which was proved in [12, Proposition 2.3]. Before, we recall that A is called $(2n+1)$-cyclicly weakly amenable if every continuous derivation $D : A \to A^{(2n+1)}$ for which $D(a_1)(a_2) + D(a_2)(a_1) = 0$, for all $a_1, a_2 \in A$, is inner. This result, for the case $n = 0$, was also proved in [9, Theorem 2.11].

Corollary 2.5. *If $A \times_\theta B$ is $(2n+1)$-weakly amenable, then B is $(2n+1)$-weakly amenable and A is $(2n+1)$-cyclicly weakly amenable.*

Proof. By condition (1) of Theorem 2.2, B is $(2n+1)$-weakly amenable. The $(2n+1)$-cyclic weak amenability of A follows from condition (3) of Theorem 2.2, if we take $S = 0$. □

Using Theorem 2.2, for $B = \mathbb{C}$ and $\theta = \iota$, we get the next result, which was already proved in [13, Proposition 3.9].

Corollary 2.6. *$A^\sharp$ is $(2n+1)$-weakly amenable if and only if*

(1) *$\langle A^2 \rangle$, the linear span of A^2, is dense in A.*

(2) *Every continuous derivation $D : A \to A^{(2n+1)}$, with the condition that there is a $T \in A^*$ such that $D(a_1)(a_2) + D(a_2)(a_1) = T(a_1 a_2)$ for all $a_1, a_2 \in A$, is inner.*

We know from [3, Proposition 1.3] that if A is $(2n+1)$-weakly amenable, then A^2 is dense in A. Thus, as a consequence of Lemma 2.3 and Theorem 2.2, we have the next result which extends the related results on $(2n+1)$-weak amenability of $A^\sharp$ [3, Proposition 1.4], and improves [12, Proposition 2.4]. This result has been proved in [4, Theorem 4.1], but the proof contains a gap that we fix here.

Proposition 2.7. *Let A and B be $(2n+1)$-weakly amenable. Then $A \times_\theta B$ is $(2n+1)$-weakly amenable.*

Using Theorem 2.2 and Proposition 2.7, with $B = A$, we have the following.

Corollary 2.8. *A is $(2n+1)$-weakly amenable if and only if $A \times_\theta A$ is $(2n+1)$-weakly amenable.*

It was shown in [13, Corollary 3.10 and 3.12] and [3, Proposition 1.4] that if A is commutative or weakly amenable, or has a bounded approximate identity, then $A^\sharp$ is $(2n+1)$-weakly amenable if and only if A is $(2n+1)$-weakly amenable. In the next result, which is a consequence of Theorem 2.2 and Proposition 2.7, we extend it to $A \times_\theta B$.

Theorem 2.9. *Suppose that one of the following statements holds:*

(i) *A has a bounded approximate identity.*

(ii) *A is weakly amenable.*

(iii) *A and B are commutative.*

Then $A \times_\theta B$ is $(2n+1)$-weakly amenable if and only if both A and B are $(2n+1)$-weakly amenable.

PROOF. By Proposition 2.7 and Theorem 2.2, in all three cases we have only to show that the $(2n+1)$-weak amenability of $A \times_\theta B$ implies that A is weakly amenable. So, assume that $A \times_\theta B$ is $(2n+1)$-weakly amenable. Assume that (i) holds and $D : A \to A^{(2n+1)}$ is a continuous derivation. Let $\{e_\alpha\}$ be a bounded approximate identity of A, and $E \in A^{**}$ be a weak* cluster point of $\{e_\alpha\}$. Define $S : A \to B^{(2n+1)}$ by $S(a) = D(a)(E)\theta$. Then

$$\begin{aligned} S(a_1a_2) &= (D(a_1)a_2 + a_1D(a_2))(E)\theta \\ &= (D(a_1)(a_2E) + D(a_2)(Ea_1))\theta = (D(a_1)(a_2) + D(a_2)(a_1))\theta, \end{aligned}$$

for all $a_1, a_2 \in A$. So, condition (3) of Theorem 2.2 implies that D is inner, as required.

Assume that (ii) holds and $D : A \to A^{(2n+1)}$ is a continuous derivation. Let $P : A^{(2n+1)} \to A^*$ be the projection with kernel $A^\perp$. Then $P \circ D : A \to A^*$ is an inner derivation. On the other hand, the continuous derivation $(I - P) \circ D : A \to A^\perp \subseteq A^{(2n+1)}$ satisfies condition (3) of Theorem 2.2, with $S = 0$. So, $(I - P) \circ D$ is inner. This shows that D is inner. So A is $(2n+1)$-weakly amenable.

Finally, assume that (iii) holds. Since $A \times_\theta B$ is commutative and A is a closed ideal of $A \times_\theta B$, it is enough to show that A^2 is dense in A, see [2, Theorem 2.8.69]. For this, we are assuming that $A \times_\theta B$ is $(2n+1)$-weakly amenable. Therefore, condition (2) of Theorem 2.2 is satisfied. By Lemma 2.3, we have that A^2 is dense in A, and this completes the proof. □

3. $(2n)$-weak amenability

In this section, we examine the conditions in which $A \times_\theta B$ is $(2n)$-weakly amenable. First, we recall that the Banach space $(A \times_\theta B)^{(2n)}$ can be also identified with the Banach space $A^{(2n)} \times B^{(2n)}$ equipped with the norm $\|(f, g)\| = \|f\| + \|g\|$ in the natural way. By induction, we find that the $(A \times_\theta B)$-bimodule actions on $(A \times_\theta B)^{(2n)}$ are formulated as follows:

$$\begin{aligned} (f, g)(a, b) &= (fa + g(\theta)a + \theta(b)f, gb)\,, \\ (a, b)(f, g) &= (af + g(\theta)a + \theta(b)f, bg)\,, \end{aligned}$$

for $a \in A, b \in B, f \in A^{(2n)}$ and $g \in B^{(2n)}$.

To clarify the relation between $(2n)$-weak amenability of $A \times_\theta B$ and that of A and B, we need the following lemma that was proved in [12, Proposition 2.2].

Lemma 3.1. *A mapping $D : A \times_\theta B \to (A \times_\theta B)^{(2n)}$ is a continuous derivation if and only if*

$$D(a, b) = (D_A(b) + T_A(a), D_B(b) + T_B(a)),$$

for all $a \in A$ and $b \in B$, where

(a) *$D_B : B \to B^{(2n)}$ is a continuous derivation.*

(b) *$D_A : B \to A^{(2n)}$ is a bounded linear operator such that $D_A(b_1b_2) = \theta(b_1)D_A(b_2) + \theta(b_2)D_A(b_1)$ for all $b_1, b_2 \in B$ and $D_A(b)a = aD_A(b) = -D_B(b)(\theta)a$ for all $a \in A$ and $b \in B$.*

(c) *$T_B : A \to B^{(2n)}$ is a bounded linear operator such that $T_B(a_1a_2) = 0$ for all $a_1, a_2 \in A$ and $bT_B(a) = T_B(a)b = \theta(b)T_B(a)$ for all $a \in A$ and $b \in B$.*

(d) *$T_A : A \to A^{(2n)}$ is a bounded linear operator such that $T_A(a_1a_2) = a_1T_A(a_2) + T_A(a_1)a_2 + T_B(a_2)(\theta)a_1 + T_B(a_1)(\theta)a_2$ for $a_1, a_2 \in A$.*

Moreover, $D = d_{(f,g)}$ for some $f \in A^{(2n)}$ and $g \in B^{(2n)}$ if and only if $D_B = d_g$, $T_A = d_f$, $D_A = 0$ and $T_B = 0$.

As a first result for $(2n)$-weak amenability of $A \times_\theta B$, we give the following characterization, which extends the related results in [12].

Theorem 3.2. *The θ-Lau product $A \times_\theta B$ is $(2n)$-weakly amenable if and only if*

(1) *A is $(2n)$-weakly amenable.*

(2) *If $T : B \to B^{(2n)}$ is a continuous derivation such that there is a bounded linear operator $D : B \to A^{(2n)}$ satisfying $D(b_1b_2) = \theta(b_1)D(b_2) + \theta(b_2)D(b_1)$ for all $b_1, b_2 \in B$, and $D(b)a = aD(b) = -T(b)(\theta)a$ for all $a \in A$ and $b \in B$, then T is inner.*

(3) *The only bounded linear operator $D : B \to A^{(2n)}$, such that $D(b_1b_2) = \theta(b_1)D(b_2) + \theta(b_2)D(b_1)$ for all $b_1, b_2 \in B$ and $aD(b) = D(b)a = 0$ for all $a \in A$ and $b \in B$, is zero.*

(4) *If $S : A \to B^{(2n)}$ is a bounded linear operator such that $S(a_1a_2) = 0$ for all $a_1, a_2 \in A$ and $bS(a) = S(a)b = \theta(b)S(a)$ for all $a \in A$ and $b \in B$, and there is a bounded linear operator $T : A \to A^{(2n)}$ satisfying $T(a_1a_2) = a_1T(a_2) + T(a_1)a_2 + S(a_1)(\theta)a_2 + S(a_2)(\theta)a_1$ for all $a_1, a_2 \in A$, then $S = 0$.*

Proof. To prove the necessity, suppose that $A \times_\theta B$ is $(2n)$-weakly amenable. Let $D : A \to A^{(2n)}$ be a continuous derivation. Then $\overline{D} : A \times_\theta B \to (A \times_\theta B)^{(2n)}$ defined by $\overline{D}(a, b) = (D(a), 0)$ is a continuous derivation, and hence it is inner. It follows from Lemma 3.1 that D is inner. Therefore, A is $(2n)$-weakly amenable.

To prove (2), let $T : B \to B^{(2n)}$ be a continuous derivation, and $D : B \to A^{(2n)}$ be a bounded linear operator satisfying (2). We define $\overline{D} : A \times_\theta B \to (A \times_\theta B)^{(2n)}$ by $\overline{D}(a,b) = (D(b), T(b))$. Then Lemma 3.1 implies that $\overline{D}$ is a continuous derivation, so it is inner. Hence, T is inner, as required.

Let $D : B \to A^{(2n)}$ be a bounded linear operator such that $D(b_1 b_2) = \theta(b_1)D(b_2) + \theta(b_2)D(b_1)$ for all $b_1, b_2 \in B$ and $aD(b) = D(b)a = 0$ for all $a \in A$ and $b \in B$. By Lemma 3.1, we conclude that $\overline{D} : A \times_\theta B \to (A \times_\theta B)^{(2n)}$, defined by $\overline{D}(a,b) = (D(b), 0)$, is a continuous derivation, and so it is inner. Hence, $D = 0$. This proves (3).

To prove (4), we use a similar argument. Indeed, if $S : A \to B^{(2n)}$ and $T : A \to A^{(2n)}$ are bounded linear operators satisfying (4), then Lemma 3.1 implies that $\overline{D} : A \times_\theta B \to (A \times_\theta B)^{(2n)}$, given by $\overline{D}(a,b) = (T(a), S(a))$, is a continuous derivation, and so it is inner. Hence, $S = 0$. This completes the proof of necessity.

For sufficiency, suppose that $\overline{D} : A \times_\theta B \to (A \times_\theta B)^{(2n)}$ is a continuous derivation. By Lemma 3.1, $\overline{D}$ is in the form

$$\overline{D}(a,b) = (D_A(b) + T_A(a), D_B(b) + T_B(a)), \quad (a \in A, b \in B),$$

in which the component mappings D_A, D_B, T_A and T_B satisfy the conditions (a)–(d) of Lemma 3.1. By condition (2), D_B is an inner derivation. Thus, there is $g \in B^{(2n)}$ such that $D_B = d_g$. Since $D_B(b)(\theta) = (bg - gb)(\theta) = g(\theta b - b\theta) = \theta(b)g(\theta - \theta) = 0$, from condition (3), we get $D_A = 0$. By condition (4), $T_B = 0$. This, together with condition (d) of Lemma 3.1, implies that T_A is a continuous derivation. From condition (1), it follows that there is $f \in A^{(2n+1)}$ such that $T_A = d_f$. Therefore, $\overline{D}$ is inner. This proves that $A \times_\theta B$ is $(2n)$-weakly amenable, as claimed. □

As an immediate consequence of Theorem 3.2, we have the next result, which has already been proved in [12, Proposition 2.5]; see also [3, Proposition 1.4]. Recall that B is called $\theta_{(2n)}$-null weakly amenable if every continuous derivation $D : B \to B^{(2n)}$ for which $D(b)(\theta) = 0$, for all $b \in B$, is inner.

Corollary 3.3. *If $A \times_\theta B$ is $(2n)$-weakly amenable, then A is $(2n)$-weakly amenable and B is $\theta_{(2n)}$-null weakly amenable.*

Proof. By condition (1) of Theorem 3.2, A is $(2n)$-weakly amenable. The $\theta_{(2n)}$-null weak amenability of B follows from condition (2) of Theorem 3.2, by taking $D = 0$. □

Proposition 3.4. *Suppose that A has a bounded approximate identity and $n \geq 1$. Then condition (2) in Theorem 3.2 is equivalent to $(2n)$-weak amenability of B. If A is unital, then the equivalence is also true for $n = 0$.*

Proof. If B is $(2n)$-weakly amenable, then it is trivial that condition (2) of Theorem 3.2 holds. For the converse, first let $n \geq 1$, and let $\{e_\alpha\}$ be a bounded approximate identity of A. Suppose that $T : B \to B^{(2n)}$ is a continuous derivation. Define $D : B \to A^{(2n)}$ by $D(b) = -T(b)(\theta)E$, where $E \in A^{**}$ is a weak* cluster point of $\{e_\alpha\}$. Then $aD(b) = D(b)a = -T(b)(\theta)a$ for all $a \in A$ and $b \in B$. Moreover,

$$\begin{aligned} D(b_1b_2) &= -T(b_1b_2)(\theta)E = (-T(b_1)b_2 - b_1T(b_2))(\theta)E \\ &= -\theta(b_2)T(b_1)(\theta)E - \theta(b_1)T(b_2)(\theta)E = \theta(b_2)D(b_1) + \theta(b_1)D(b_2), \end{aligned}$$

for all $b_1, b_2 \in B$. So, condition (2) of Theorem 3.2 implies that T is inner, as required.
Now, let $n = 0$ and $\mathbf{1}$ be the unit of A. If $T : B \to B$ is a continuous derivation, then $D : B \to A$, defined by $D(b) = -T(b)(\theta)\mathbf{1}$, satisfies condition (2) of Theorem 3.2, and so T is inner. Therefore, B is (0)-weakly amenable. □

Proposition 3.5. *Condition (3) of Theorem 3.2 holds if and only if $\langle AA^{(2n-1)} \cup A^{(2n-1)}A \rangle$ is dense in $A^{(2n-1)}$, or every continuous point derivation at θ is zero.*

Proof. It is clear that condition (3) of Theorem 3.2 holds if $\langle AA^{(2n-1)} \cup A^{(2n-1)}A \rangle$ is dense in $A^{(2n-1)}$. So, assume that every continuous point derivation at θ is zero, and $D : B \to A^{(2n)}$ is a bounded linear map satisfies condition (3) of Theorem 3.2. If $f \in A^{(2n+1)}$, then $f \circ D$ is a continuous point derivation at θ, so it is zero. This implies that $D = 0$.

For the converse, take a non-zero $f \in A^{(2n)}$ with $af = fa = 0$ for all $a \in A$, and let $d : B \to \mathbb{C}$ be a continuous point derivation at θ. Then $D : B \to A^{(2n)}$ defined by $D(b) = d(b)f$ satisfies condition (3) of Theorem 3.2, so it is zero. Thus $d = 0$, as required. □

Using Theorem 3.2, with $B = \mathbb{C}$ and $\theta = \iota$, we get the next result which extends [13, Proposition 3.13].

Corollary 3.6. *$A^\sharp$ is $(2n)$-weakly amenable if and only if*

(1) *A is $(2n)$-weakly amenable.*

(2) *Every $f \in A^*$, with the conditions that $f|_{A^2} = 0$, and for which there is a bounded linear operator $T : A \to A^{(2n)}$ such that $T(a_1a_2) = a_1T(a_2) + T(a_1)a_2 + f(a_1)a_2 + f(a_2)a_1$ for all $a_1, a_2 \in A$, is zero.*

We recall from [6] that B is called left (resp. right) θ-amenable if every continuous derivation from B into X^* is inner, for every Banach B-bimodule X with $b \cdot x = \theta(b)x$ (resp. $x \cdot b = \theta(b)x$); $(b \in B, x \in X)$. This notion of amenability is a generalization of the left amenability of a class of Banach algebras studied by Lau in [8], known as Lau algebras. Example of left (resp. right) θ-amenable Banach algebras include amenable Banach algebras and the Fourier algebra $A(G)$ for a locally compact group G.

In the next proposition, which extends the related results on $(2n)$-weak amenability of $A^\sharp$ [13, Proposition 3.13 and Corollary 3.14], we give an analogue to [12, Proposition 2.4] for the even case. This answers a question raised by Ebrahimi Vishki and Khoddami in [12].

Proposition 3.7. *Let A and B be $(2n)$-weakly amenable, and let $\langle A^2 \rangle$ be dense in A. Then $A \times_\theta B$ is $(2n)$-weakly amenable if one of the following statements holds:*

(i) *There is no non-zero continuous point derivation at θ.*

(ii) $\left\langle AA^{(2n-1)} \cup A^{(2n-1)}A \right\rangle$ *is dense in* $A^{(2n-1)}$.

(iii) *B is weakly amenable.*

(iv) *B is left (resp. right) θ-amenable.*

Proof. This follows from Theorem 3.2, Proposition 3.5 and the fact that if B is either weakly amenable or left (resp. right) θ-amenable, then there is no non-zero continuous point derivation at θ [3, Proposition 1.3] and [6, Remark 2.4]. □

For the converse of Proposition 3.7, we have the following.

Proposition 3.8. *Suppose that $A \times_\theta B$ is $(2n)$-weakly amenable and $n \geq 1$. Then A and B are $(2n)$-weakly amenable if one of the following statements holds:*

(i) *A has a bounded approximate identity.*

(ii) *B is (2)-weakly amenable.*

Proof. (i) It follows from Theorem 3.2 and Proposition 3.4. (ii) In view of Theorem 3.2, we have to show that B is $(2n)$-weakly amenable. To do this, let $T : B \to B^{(2n)}$ be a continuous derivation, and let $P : B^{(2n)} \to B^{**}$ be the projection with the kernel $B^{*\perp}$. Then $P \circ T : B \to B^{**}$ is an inner derivation. On the other hand, the continuous derivation $(I - P) \circ T : B \to B^{*\perp} \subseteq B^{(2n)}$ satisfies condition (2) of Theorem 3.2, with $D = 0$. So, $(I - P) \circ T$ is also inner. This shows that T is inner. So, B is $(2n)$-weakly amenable. □

From Propositions 3.7 and 3.8, we obtain also the following result which extends [12, Theorem 3.1].

Theorem 3.9. *Suppose that A has a bounded approximate identity, B is either weakly amenable or left (right) θ-amenable and $n \geq 1$. Then $A \times_\theta B$ is $(2n)$-weakly amenable if and only if both A and B are $(2n)$-weakly amenable.*

It was shown in [12, Theorem 3.1] that if A is unital, then the n-weak amenability of $A \times_\theta B$ is equivalent to the n-weak amenablility of both A and B. It was left as an open question for the case when A has a bounded approximate identity; see [12, Remark 3.1]. If we combine Theorems 2.9 and 3.9, we have the following theorem which partially answers this question.

Theorem 3.10. *Suppose that A has a bounded approximate identity and B is either weakly amenable or left (right) θ-amenable. Then $A \times_\theta B$ is n-weakly amenable, for $n \geq 1$, if and only if both A and B are n-weakly amenable. If A is unital, then the equivalence is also true for $n = 0$.*

As a consequence of the above theorem, with $A = \mathbb{C}$ and $\theta \in \sigma(B)$, we have the next result.

Corollary 3.11. *The θ-Lau product $\mathbb{C} \times_\theta B$ is n-weakly amenable if and only if B is n-weakly amenable.*

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