Publ. Math. Debrecen
90/3-4 (2017), 507–523
DOI: 10.5486/PMD.2017.7711

On generalized normal homogeneous Randers spaces

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Abstract. In this paper, we use the navigation method to study the geometric properties of generalized normal homogeneous Randers spaces. We first establish a relationship between generalized normal Randers spaces and generalized normal homogeneous Riemannian manifolds, which provides many non-Riemannian examples. We then give a complete classification of generalized normal Randers spaces with positive flag curvature.

1. Introduction

The purpose of this paper is to study the geometric properties of generalized normal homogeneous (or δ-homogeneous) Randers spaces. The concept of generalized normal homogeneous (or δ-homogeneous) spaces is first introduced by V. N. Berestovskii and C. P. Plaut in [5]. After that, generalized normal homogeneous Riemannian manifolds have been studied extensively in [7], [8], [9], and [10]. Recall that a metric space (M, ρ) is called generalized normal homogeneous (respectively, Clifford–Wolf homogeneous) if for any points $x, y \in M$, there exists an isometry f, called $\delta(x)$-translation (resp., Clifford–Wolf translation), of the space (M, ρ) onto itself, such that $f(x) = y$ and f has maximal displacement at the point x (respectively, equal displacements at all points), i.e., for every point $z \in M$, $\rho(z, f(z)) \leq \rho(x, f(x)) = \rho(x, y)$ (resp. $\rho(z, f(z)) = \rho(x, f(x))$). From this definition it is clear that any Clifford–Wolf homogeneous metric space

Mathematics Subject Classification: 22E46, 53C30.
Key words and phrases: generalized normal homogeneous space, Randers space, navigation data.
Supported by NSFC (no. 11271198, 11671212, 51535008) and SRFDP of China.
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is generalized normal homogeneous, and any generalized normal homogeneous metric space is homogeneous. A complete classification of connected simply connected indecomposable generalized normal homogeneous Riemannian manifolds with positive Euler characteristic is obtained in [10]. Moreover, a classification of generalized normal homogeneous Riemannian metrics on spheres is established in [7].

On the other hand, generalized normal homogeneous Finsler spaces are studied in [9]. Since for a coset space G/H there may exist many G-invariant Finsler metrics, it is natural to ask if $(G/H, h)$ is a generalized normal homogeneous Riemannian space, whether there exists a G-invariant non-Riemannian Finsler metric on G/H which makes $(G/H, F)$ a generalized normal homogeneous Finsler space. Conversely, if $(G/H, F)$ is a generalized normal homogeneous Finsler space, one may also ask whether there is a G-invariant Riemannian metric h on G/H such that $(G/H, h)$ is a generalized normal homogeneous Riemannian space.

Our first goal in this paper is to give an affirmative answer to the above problems in the Randers case, using the navigation method (see Theorem 3.8 below). The navigation method is a powerful tool in the study of Randers spaces. Bao and Robles [4] presented a very convenient way to describe Einstein–Randers metrics, as well as Randers spaces of constant curvature. More recently, Zhiguang Hu and the second author apply the navigation method to give a complete classification of homogeneous Randers spaces with isotropic S-curvature and positive flag curvature. Combining their results with the classification of generalized normal homogeneous Riemannian metrics on spheres [7], we can give a complete classification of positively curved generalized normal homogeneous Randers spaces (up to isometries).

Here is a brief description of the individual sections. In Section 2, we present some preliminaries on Randers spaces. In Section 3, we study the general geometric properties of generalized normal homogeneous Randers spaces. We prove that a connected homogeneous Randers space $(G/H, F)$ with navigation data (h, w) is generalized normal homogeneous if and only if $(G/H, h)$ is a generalized normal homogeneous Riemannian manifold, and w is a Killing vector field of constant length with $h(w, w) < 1$. This establishes a relationship between generalized normal homogeneous Randers spaces and generalized normal homogeneous Riemannian manifolds. Finally, in Section 4, we give a complete classification of positive curved generalized normal homogeneous Randers spaces. The list consists of S^{2n+1}, S^{4n+3} and $N_{(1,1)}$, with suitable metrics.

2. Preliminaries

A Randers metric is built from a Riemannian metric and a 1-form: $F = \alpha + \beta$, where α is a Riemannian metric and β is a 1-form whose length with respect to α is everywhere less than 1. There is another presentation of such metrics, by the so-called navigation data (see [3]):

$$F(x, y) = \frac{\sqrt{h(y, w)^2 + \lambda h(y, y)}}{\lambda} - \frac{h(y, w)}{\lambda}, \tag{1}$$

where h is a Riemannian metric, w is a vector field on M with $h(w, w) < 1$ and $\lambda = 1 - h(w, w)$. The pair (h, w) is called the navigation data of the Randers metric F. This version of Randers metric is convenient when handling some problems concerning flag curvature and Ricci scalar (see, for example, [4]).

In a local coordinate system, the transformation law between the defining form and the navigation data can be described as the following (see [4]). If

$$F = \alpha + \beta = \sqrt{a_{ij} y^i y^j} + b_i y^i, \tag{2}$$

then the navigation data has the form

$$h_{ij} = (1 - \|\beta\|^2)(a_{ij} - b_i b_j), \qquad w^i = -\frac{a^{ij} b_j}{1 - \|\beta\|_\alpha^2}. \tag{3}$$

Conversely, the defining form can also be expressed by the navigation data by the formula:

$$a_{ij} = \frac{h_{ij}}{\lambda} + \frac{w_i}{\lambda}\frac{w_j}{\lambda}, \qquad b_i = \frac{-w_i}{\lambda}, \tag{4}$$

here $w_i = h_{ij} w^j$ and $\lambda = 1 - w^i w_i = 1 - h(w, w)$.

A connected Randers space (M, F) is called homogeneous if its full group of isometries acts transitively on M. In this case, the manifold M can be written as G/H, where G is the unity component of the full group of isometries, and H is the isotropy subgroup of G at a fixed point. Since H is compact, the coset space G/H is reductive in the sense of [18], namely, there is a decomposition of the Lie algebra

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, \quad \text{(direct sum of subspaces)} \tag{5}$$

where $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{h} = \mathrm{Lie}(H)$, and $\mathfrak{m}$ is a linear subspace of $\mathfrak{g}$ satisfying $\mathrm{Ad}(h)(\mathfrak{m}) \subset \mathfrak{m}$, $\forall h \in H$. In this case, it is easily seen that both the underlying Riemannian metric α and the 1-form β are invariant under G. Further, in the navigation data (h, w), both h and w are also G-invariant. This reduces the study of homogeneous Randers spaces to the study of invariant Randers metrics on reductive homogeneous manifolds.

Invariant Randers metrics on reductive homogeneous manifolds have been studied by DENG and HOU (see [12] and [11]). Let G/H be a reductive homogeneous space with decomposition (5). Then one can identify $\mathfrak{m}$ with the tangent space $T_o(G/H)$ of G/H at the origin o through the mapping $X \to \frac{d}{dt}|_{t=0}(\exp(tX)H)$. Under this identification, the G-invariant Riemannian metric on G/H is in one-to-one correspondence with the H-invariant inner product on m. Fix a G-invariant Riemannian metric α on G/H, and let $\langle , \rangle$ be the corresponding inner product on $\mathfrak{m}$. Then there is a one-to-one correspondence between the G-invariant 1-form on G/H and the H-invariant vector in $\mathfrak{m}$. In fact, each G-invariant 1-form β corresponds to an H-invariant vector u through

$$\beta(x) = \langle u, x \rangle, \quad x \in \mathfrak{m}.$$

Furthermore, the length of the form β is equal to the length of the vector u.

3. General properties of generalized normal homogeneous Finsler spaces

In this section, we give the definition of generalized normal homogeneous Finsler spaces and study the fundamental properties of them.

Definition 3.1. Let (M, F) be a connected Finsler space, and $x \in M$. An isometry $f : M \to M$ is called a $\delta(x)$-translation at x (resp., a Clifford–Wolf translation) if x is a point of maximal displacement of f, i.e., if for every $y \in M$ we have $d(y, f(y)) \leq d(x, f(x))$ (resp., for any $y \in M$, we have $d(x, f(x)) = d(y, f(y))$), where d is the distance function of (M, F).

Definition 3.2. A Finsler space (M, F) is called generalized normal homogeneous (resp. CW-homogeneous) if for every $x, y \in M$, there exists a $\delta(x)$-translation (resp. Clifford–Wolf translation) of (M, F) sending x to y.

Remark 3.3. Clearly, any Clifford–Wolf translation is a $\delta(x)$-translation for all $x \in M$, hence any CW-homogeneous Finsler space is generalized normal homogeneous. Moreover, since a generalized normal homogeneous or CW-homogeneous space must be homogeneous in the usual sense, we can write such a space as $M = G/H$, where $G = I_0(M, F)$. Sometimes, we will also call a generalized normal homogeneous Finsler space $(G/H, F)$ as a (G)-δ-homogeneous Finsler space or a G-generalized normal homogeneous Finsler space.

The following theorem gives a characterization of generalized normal homogeneous Finsler spaces in terms of Killing vector fields.

Theorem 3.4. *A connected homogeneous Finsler space* $(M = G/H, F)$ *is generalized normal homogeneous if and only if, for any point* $x \in M$ *and any nonzero tangent vector* $y \in T_xM$*, there is a nonzero vector of* $\mathfrak{g} = \mathrm{Lie}(G)$ *which defines a Killing vector field* Y *of* (M, F) *such that* $Y(x) = y$*, where the function* $F(Y(\cdot))$ *reaches a maximal value.*

The proof is similar to the Riemannian case, using the following result on Killing vector fields. We will omit the details.

Lemma 3.5 ([14])**.** *Let* X *be a Killing vector field on a Finsler space* (M, F)*, and* $U \subset M$ *an open subset such that* X *is nowhere zero on* U*. Then on* U *we have* $\nabla^X_X X = -\frac{1}{2}\widetilde{\nabla}^X |X|^2$*, where* $\widetilde{\nabla}^{(X)} f = g_{ij}(X) f_{x^j} \frac{\partial}{\partial x^i}$ *is the gradient field of* f *for the Riemannian metric* $g_X = g_{ij}(X)$ *on* V*.*

Definition 3.6. Let $(G/H, F)$ be a generalized normal homogeneous Finsler space. A vector $X \in \mathfrak{g}$ is called δ-vector if $F(X|_{\mathfrak{m}}) \geq F((\mathrm{Ad}(a)X)|_{\mathfrak{m}})$, for any $a \in G$.

Using the above definition, we can give another characterization of generalized normal homogeneous Finsler spaces.

Proposition 3.7. *A homogeneous Finsler space* $(G/H, F)$ *with connected Lie group* G *is generalized normal homogeneous if and only if for any* $X \in \mathfrak{m}$*, there exists* $Y \in \mathfrak{h}$ *such that* $X + Y$ *is a* δ*-vector.*

Proof. Identify $T_H(G/H)$ with $\mathfrak{m}$. Given $X \in \mathfrak{m}$, denote by $\widetilde{X}$ the Killing vector field generated by $\overline{X}$ such that $\widetilde{X}_H = \overline{X}|_{\mathfrak{m}} = X$. Then by Theorem 3.4, there exists $\overline{X} \in \mathfrak{g} = \mathrm{Lie}(G)$, and $F(\widetilde{X}(\cdot))$ reaches its maximum at H. For convenience, we still denote by F the $\mathrm{Ad}(H)$-invariant norm on $\mathfrak{m}$ induced by the G-invariant Randers metric F. Then we have

$$F(X) = F(\overline{X}_{\mathfrak{m}}) = F(\widetilde{X}_H) \geq F(\widetilde{X}_{aH}) = F((L_{a^{-1}})_*\widetilde{X}_{aH}) = F(\mathrm{Ad}(a^{-1})\overline{X}|_{\mathfrak{m}}),$$

for any $a \in G$. Thus $\overline{X}$ is a δ vector, and $Y = \overline{X} - X \in \mathfrak{h}$.

Conversely, since G/H is homogeneous, for any $U \in T_{gH}G/H$, $(L_{g^{-1}})_*U \in \mathfrak{m}$, denoting $X = (L_{g^{-1}})_*U \in \mathfrak{m}$, there exists $Y \in \mathfrak{h}$ such that $X + Y$ is a δ vector, that is,

$$F(X) \geq F(\mathrm{Ad}(a)(X + Y)|_{\mathfrak{m}}).$$

Let $\overline{X} = X + Y$, and $\widetilde{U}$ be the Killing vector generated by $\mathrm{Ad}(g)\overline{X}$. Then we have

$$\begin{aligned}\widetilde{U}_{gH} &= \frac{d}{dt}|_{t=0}(\exp(\mathrm{Ad}(g)\overline{X})gH) = \frac{d}{dt}|_{t=0}(g\cdot g^{-1}\exp(t\mathrm{Ad}(g)\overline{X})gH)\\ &= (L_g)_*(\mathrm{Ad}(g^{-1})\mathrm{Ad}(g)\overline{X})|_m = (L_g)_*(\overline{X})|_{\mathfrak{m}} = (L_g)_*X = U,\end{aligned}$$

and

$$\begin{aligned}F(\widetilde{U}_{gH}) = F(X) &\geq F(\mathrm{Ad}(a^{-1})\mathrm{Ad}(g)\overline{X}|_{\mathfrak{m}})\\ &= F((L_a)_*(\mathrm{Ad}(a^{-1})\mathrm{Ad}(g)\overline{X})|_m) = F(\widetilde{U}_{aH}).\end{aligned}$$

By Theorem 3.4, $(G/H, F)$ is generalized normal homogeneous. □

Now, let $(M = G/H, F)$ be a homogeneous Randers space with navigation data (h, w), and suppose the Lie group G has a smooth effective action on M. Then it is easily seen that F is invariant under the action of G if and only if both h and w are invariant under the action of G. Using the navigation method, we can give a nice relationship between $(G/H, F)$ and $(G/H, h)$.

Theorem 3.8. *A connected homogeneous Randers space $(M = G/H, F)$ with navigation data (h, w) is generalized normal homogeneous if and only if $(M = G/H, h)$ is a generalized normal homogeneous Riemannian manifold, and w is a Killing vector field of constant length with $h(w, w) < 1$.*

Proof. We fist prove the "only if" part. Let (h, w) be the navigation data of F, and $\varphi_{t;w}$ the flow generated by the vector field w. By Theorem 3.4, for any $y \in T_xM$, there is a Killing vector field X, such that $X_x = y$ and $F(X(\cdot))$ reaches its maximum at x. Note that the Killing vector field of $(G/H, F)$ is also a Killing vector field of $(G/H, h)$, and we have $\mathcal{L}_X w = [X, w] = 0$. Since $(M = G/H, F)$ is generalized normal homogeneous, both F and w are G-invariant. Let $\psi_{s;X}$ denote the flow generated by X. Then we have

$$\varphi_{t;w} \circ \psi_{s;X} = \psi_{s;X} \circ \varphi_{t;w}. \tag{6}$$

Let $t \geq 0$, and suppose $\varphi_{t;w}(x) = x'$. Then we have $X(\varphi_{t;w}(x)) = (\varphi_{t;w})_* X_x$. Denoting $y' = (\varphi_{t;w})_* y$, we get

$$F(y) = F(X_x) \geq F((\varphi_{t;w})_* y) = F(y'). \tag{7}$$

Similarly, for x' and y', there is a Killing vector field X' such that $X'_{x'} = y'$, and $F(X'(\cdot))$ reaches its maximum at x'. Since $\varphi_{t;w}$ is a diffeomorphism of M, it induces a linear isomorphism $(\varphi_{t;w})_* : T_xM \to T_{x'}M$, with $\varphi_{-t;w}(y') = y$. Let $\psi'_{s;X'}$ be the flow generated by X'. Then we have $\mathcal{L}_{X'}w = [X', w] = 0$ and

$$\varphi_{t;w} \circ \psi'_{s;X'} = \psi'_{s;X'} \circ \varphi_{t;w}. \tag{8}$$

Therefore, we have

$$F(y') = F(X'_{x'}) \geq F(X'(\varphi_{-t;w}(x'))) = F((\varphi_{-t;w})_* X'_{x'}) = F(y). \tag{9}$$

Thus $F(y) = F((\varphi_{t;w})_*)y = F(y)$. From this we conclude that w is a Killing vector field. Since w is G-invariant, w is a Killing vector of constant length with respect to F, as well as to h.

Now, we prove that $(M = G/H, h)$ is a generalized normal homogeneous Riemannian manifold. Given any $x \in M$ and $y \in T_xM$, we need to find a Killing vector field X, such that $X_x = y$, and $h(X, X)$ reaches its maximum at x. Since $(M = G/H, F)$ is generalized normal homogeneous, there exists a Killing vector field X' of F such that $F(X'(\cdot))$ reaches its maximum 1 at x. Obviously, $X' - w$ is also a Killing vector field of $(M = G/H, F)$, as well as of $(M = G/H, h)$. By the navigation method, $h(X' - w, X' - w)$ reaches its maximum 1 at x due to the fact that

$$F(X) > 1 \Leftrightarrow h(X - w, X - w) > 1. \tag{10}$$

Now let $y' = y - w$. Then there exists a Killing vector field X' such that $X'_x = y'$, and $F(X'(\cdot))$ reaches its maximum 1 at x. Consequently, $X = X' + w$ is a Killing vector field which meets all the requirements.

Next, we proof the "if" part. Let $(M = G/H, h)$ be a generalized normal homogeneous Riemannian space. Since G/H is a reductive homogeneous space with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, H is the isotropy subgroup fixing one point. If (h, w) is the navigation data of $(M = G/H, F)$, then w and F are both G-invariant. Identify the tangent space $T_{eH}G/H$ of G/H at the origin eH with $\mathfrak{m}$. For convenience, we use the same symbol w to denote the $\mathrm{Ad}(H)$-invariant vector in $\mathfrak{m}$, and let $\langle , \rangle_h$ be the $\mathrm{Ad}(H)$-invariant inner product induced by h. Then $(G/H, h)$ is generalized normal homogeneous if and only if for any $X \in \mathfrak{m}$, there is $Y \in \mathfrak{h}$ such that for any $a \in G$,

$$\langle X, X \rangle_h \ \geq \langle (Ad(a)(X + Y))_{\mathfrak{m}}, (Ad(a)(X + Y))_{\mathfrak{m}} \rangle_h \tag{11}$$

Assume $\langle X, X \rangle_h = 1$. Then by the navigation principle, we have

$$\langle X, X \rangle_h = 1 \Leftrightarrow F(X + w|_{\mathfrak{m}}) = 1. \tag{12}$$

Hence,

$$\langle(\mathrm{Ad}(a)(X+Y))_{\mathfrak{m}},(\mathrm{Ad}(a)(X+Y))_{\mathfrak{m}}\rangle_h \le 1 \Leftrightarrow F(\mathrm{Ad}(a)(X+Y+w)_{\mathfrak{m}}) \le 1. \quad (13)$$

Here we still denote by F the $\mathrm{Ad}(H)$-invariant Minkowski norm induced by F. Suppose $\overline{X} = X + w$. Then for any $\overline{X} \in \mathfrak{m}$, there is $Y \in \mathfrak{h}$ such that for any $a \in G$, we have

$$F(\overline{X}) \ge F(\mathrm{Ad}(a)(\overline{X}+Y)_{\mathfrak{m}}). \quad (14)$$

Then $\overline{X} + Y$ is a δ vector. By Proposition 3.7, $(M = G/H, F)$ is generalized normal homogeneous. □

This result generalizes [15, Theorem 1.1].

Remark 3.9. By [15, Theorem 1.1], a homogeneous Randers space $(G/H, F = \alpha+\beta)$ with navigation data (h, w) is CW-homogeneous if and only if $(G/H, h)$ is CW-homogeneous, and w is a Killing vector field of constant length with respect to h. Here we point out that $(G/H, \alpha)$ can be non-CW-homogeneous. In fact, consider a three-dimensional compact simple Lie group SU(2). Let X_1, X_2, X_3 be an orthonormal basis of $\mathfrak{su}(2)$, with respect to the inner product defined by the negative of its Killing form. Then for any $X, Y \in \mathfrak{su}(2)$, we can write $X = x_1X_1 + x_2X_2 + x_3X_3$, and $Y = y_1X_1 + y_2X_2 + y_3X_3$. Now, let h be the canonical bi-invariant inner product. Then we have $h(X,Y) = x_1y_1+x_2y_2+x_3y_3$, and $w = sX_1$ $(s < 1)$. Thus the Randers metric defined by (h, w) is CW-homogeneous. While by the expression of Randers metric 1 and 2, we can see that the left-invariant metric α may not be bi-invariant, since any two bi-invariant inner products on a compact simple Lie group can only differ by a constant multiplication. Thus $(\mathrm{SU}(2), \alpha)$ is not CW-homogeneous. Moreover, by [7, Proposition 7], $(\mathrm{SU}(2), \alpha)$ is not SU(2)-generalized normal homogeneous, since every left-invariant generalized normal homogeneous metric on SU(2) must be bi-invariant.

Corollary 3.10. *A generalized normal homogeneous Randers space $(G/H, F)$ is either compact or isometric to the direct product of a Euclidean space and a compact generalized normal homogeneous Randers space.*

Proof. Let $(M = G/H, F)$ be a generalized normal homogeneous Randers space with navigation data (h, w). Then by Theorem 3.8, $(G/H, h)$ is a generalized normal homogeneous Riemannian manifold, and w is a Killing vector field of constant length. According to [8, Theorem 4], $(G/H, h)$ is either compact or isometric to the direct product of a Euclidean space and a compact generalized normal homogeneous Riemannian manifold. If $(G/H, h)$ is compact, we are done.

Assume that $G/H = E_1 \times G_2/H_2$, where E_1 is a Euclidean space, and G_2/H_2 is a compact generalized normal homogeneous Riemannian manifold. Let $h = h_1 + h_2$, and w_i be the Killing vector field of constant length of h_i $(i = 1, 2)$, respectively. Then the Randers metric on E_1 with the navigation data (h_1, w_1), and the one on the coset space G_2/H_2 with the navigation data (h_2, w_2) are both generalized normal homogeneous Randers spaces. This completes the proof of the corollary. □

Corollary 3.11. *Let $(G/H, F)$ be a generalized normal homogeneous Randers space. Then $(G/H, F)$ is a g. o. Randers space. In particular, it has vanishing S-curvature.*

Proof. We fix $o = eH$ as the basic point. Since the Randers space is homogeneous, it suffices to prove that all the geodesics through o are of the form $\gamma(t) = \exp(tX)o$, where $X \in \mathfrak{g}$. By Theorem 3.4, for any $y \in T_oG/H$, there exists a Killing vector field X such that $X_o = y$, and $F(X(\cdot))$ reaches its maximum. Moreover, from the proof of Theorem 3.4, one can see that $\gamma(t) = \exp(tX)o$ is a geodesic.

On the other hand, the above two assertions are also true with respect to the navigation data (h, w). By Theorem 3.8, $(G/H, h)$ is a generalized normal homogeneous Riemannian manifold, hence $(G/H, h)$ is a Riemannian g. o. space. Now by [23, Theorem 6.7], $(G/H, F)$ is also a g. o. Randers space. □

For the study of Flag curvature of generalized normal homogeneous Randers spaces, we need a result of Huang–Mo. In [16], Huang-Mo studied the property of the change of flag curvature of a Finsler metric under the influence of a vector field. They found that the flag curvature will decrease under a navigation. In particular, they proved the following:

Proposition 3.12 ([16]). *Let (M, F) be a Finsler manifold, and let $\widetilde{u}$ be a homothetic vector field with dilation σ, and $F(x, \widetilde{u}) < 1$. Let $\widetilde{F}$ be the Finsler metric produced by navigation problem by F and $\widetilde{u}$, then the flag curvature of $\widetilde{F}$ (resp. F), denoted by $\widetilde{K}(y, v)$ (resp. $K(y, v)$) satisfies*

$$\widetilde{K}(y, v) = K(\widetilde{y}, v) - \sigma^2,$$

where $\widetilde{y} = y - F(x, y)\widetilde{u}$.

A special case is when the homothetic vector field is a Killing vector field. In this case, if F has positive flag curvature, then $\widetilde{F}$ also has positive flag curvature. Now, we prove

Proposition 3.13. *Every generalized normal homogeneous Randers space has nonnegative flag curvature.*

Proof. Every generalized normal homogeneous Randers space can be expressed as $(G/H, F)$ with navigation data (h, w). By Theorem 3.8, $(G/H, h)$ is a generalized normal homogeneous Riemannian manifold, and w is a Killing vector field of constant length with respect to α. It is proved in [5] that $(G/H, h)$ has nonnegative sectional curvature. Then by Proposition 3.12, $(G/H, F)$ also has nonnegative flag curvature. □

4. Generalized normal homogeneous Randers spaces with positive flag curvature

In this section, we give a complete classification of positively curved generalized normal homogeneous Randers spaces. Let G be a compact Lie group, H a closed subgroup, and $\mathrm{Lie}(G) = \mathfrak{g}$, $\mathrm{Lie}(H) = \mathfrak{h}$. Fix a bi-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$, and an orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ with respect to $\langle\cdot,\cdot\rangle$. Then the restriction of $\langle\cdot,\cdot\rangle$ to $\mathfrak{m}$ induces a normal homogeneous Riemannian metric on G/H. It follows from [7, Theorem 2] that any normal homogeneous Riemannian manifold is also generalized normal homogeneous. If there exists a one dimensional subspace $\mathfrak{m}_0$ in $\mathfrak{m}$ such that the action of $\mathrm{Ad}(H)$ on $\mathfrak{m}_0$ is trivial, then G/H admits non-negative flag curvature generalized normal homogeneous Randers metrics.

Homogeneous Randers spaces with positively flag curvature and vanishing S-curvature have been classified by Deng and Hu in [17] (see Table 1). Since every generalized normal homogeneous Randers space has vanishing S-curvature, all possible positively curved generalized normal homogeneous Randers spaces must be one of the spaces in their list. Hence, we just need to find out which positively curved Randers spaces with vanishing S-curvature are generalized normal homogeneous. By the navigation method, this is equivalent to finding all compact generalized homogeneous Riemannian manifolds with positive curvature and nonzero $\mathrm{Ad}(H)$-invariant vectors.

Homogeneous Randers spaces with positive flag curvature and vanishing S-curvature are listed as follows:

(I). $\mathrm{SU}(n+1)/\mathrm{SU}(n)$, (II). $\mathrm{Sp}(n+1)/\mathrm{Sp}(n)$,
(III). $N_{1,1}$, (IV). $N_{k,l}, \gcd(k,l) = 1, kl(k+l) \neq 0$.

We first consider the cases (III) and (IV). These are the Aloff–Wallach spaces, which are defined as $N_{k,l} = \mathrm{SU}(3)/S^1_{k,l}$, where the embedding of U(1) into SU(3) is as the following:

$$S^1_{k,l} : e^{2\pi\sqrt{-1}\theta} \to \mathrm{diag}(e^{2k\pi\sqrt{-1}\theta}, e^{2l\pi\sqrt{-1}\theta}, e^{2m\pi\sqrt{-1}\theta}),$$

here k, l, m are integers with greatest common divisor 1, and $k + l + m = 0, k \geq l \geq 0$.

If $(k, l) \neq (1, 1)$ and $k \neq l$, then the isotropy representation has a decomposition as

$$\mathfrak{m} = V_0 \oplus V_{2k+l} \oplus V_{2l+k} \oplus V_{k-l},$$

where

$$V_{2k+l} = \left\{ \begin{pmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ -\overline{z} & 0 & 0 \end{pmatrix} \middle| z \in \mathbb{C} \right\},$$

$$V_{2l+k} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & z \\ 0 & -\overline{z} & 0 \end{pmatrix} \middle| z \in \mathbb{C} \right\},$$

$$V_{k-l} = \left\{ \begin{pmatrix} 0 & z & 0 \\ -\overline{z} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| z \in \mathbb{C} \right\},$$

and

$$V_0 = \left\{ \sqrt{-1} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -(a+b) \end{pmatrix} \middle| ak + bl + (a+b)(k+l) = 0, a, b \in \mathbb{R} \right\}.$$

The Lie algebra of $S^1_{k,l}$ is $\mathbb{R}h_{k,l}$, where $h_{k,l} = \sqrt{-1}\mathrm{diag}\{k, l, -(k+l)\}$. Set

$$V_1 = V_0 \oplus V_{k-l}, \qquad V_2 = V_{2k+l} \oplus V_{2l+k},$$

and define

$$q_0(X, Y) = -\mathrm{Re}(\mathrm{tr}(XY)), \quad X, Y \in \mathfrak{su}(3).$$

Then q_0 is an Ad(SU(3))-invariant inner product on $\mathfrak{su}(3)$, hence it defines a bi-invariant Riemannian metric on SU(3).

For $X, Y \in \mathfrak{m}$, set $X = X_1 + X_2$, $Y = Y_1 + Y_2$, $X_i, Y_i \in V_i$, and define

$$q_t(X, Y) = (1 + t)q_0(X_1, Y_1) + q_0(X_2, Y_2) = q_0(X, Y) + tq_0(X_1, Y_1). \qquad (15)$$

It is shown in [1] that if $-1 < t < 0$, then q_t defines a SU(3)-invariant Riemannian metric on $\mathrm{SU}(3)/S^1_{k,l}$ with positive curvature. Now, we prove

Proposition 4.1. *The space* $N_{k,l} = \mathrm{SU}(3)/S^1_{k,l}$, *where* $kl(k+l) \neq 0$, $\gcd(k,l) = 1$, *and* $(k,l) \neq (1,1)$, *with the* $\mathrm{SU}(3)$*-invariant metric* q_t *is not a generalized normal homogeneous Riemannian space.*

Proof. Up to conjugacy, every nontrivial circle in SU(3) is of the form

$$S^1_{k,l} : e^{2\pi\sqrt{-1}\theta} \to \mathrm{diag}(e^{2k\pi\sqrt{-1}\theta}, e^{2l\pi\sqrt{-1}\theta}, e^{2m\pi\sqrt{-1}\theta}),$$

where $|k| + |l| \neq 0$, $k, l \in \mathbf{Z}$. Let

$$K = \left\{ \begin{pmatrix} g & 0 \\ 0 & \det g^{-1} \end{pmatrix} \middle| g \in \mathrm{U}(2) \right\}.$$

Then we have $S^1_{k,l} \subset K$. Let $\mathfrak{g}_1$ be the Lie algebra of G_1. Then a direct computation shows that $V_1 = h^{\perp}_{k,l} \cap \mathfrak{g}_1$, and $V_2 = \mathfrak{g}_1^{\perp}$ (here "$\perp$" is with respect to q_0). Thus, $(\mathrm{SU}(3), K)$ is a symmetric pair corresponding to $\mathbb{C}P^2$. The Lie algebra of K is $\mathfrak{k} = h_{k,l} \oplus V_1$ and $S^1_{k,l} \subset K$ are both closed subgroups of SU(3). With respect to the Ad(SU(3))-invariant inner product q_0, we have an orthogonal decomposition

$$\mathfrak{su}(3) = \mathfrak{k} \oplus V_2 = h_{k,l} \oplus V_1 \oplus V_2.$$

Then $q_t = (1+t)q_0|_{V_1} + q_0|_{V_2}$ is a SU(3)-invariant Riemannian metric on $\mathrm{SU}(3)/S^1_{k,l}$. If $\mathrm{SU}(3)/S^1_{k,l}$ is generalized normal homogeneous, then it must be a g. o. space with respect to q_t, that is, for any $X_1 + X_2 \in V_1 \oplus V_2$, there exists $Z \in h_{k,l}$ such that $W = X_1 + X_2 + Z$ is a geodesic vector. By [8, Proposition 16], we have

$$[Z, X_1] = 0, \ [X_2, X_1] = \frac{1}{t}[X_2, Z].$$

However, a direct computation shows that $[h_{k,l}, V_{k-l}] = V_{k-l} \neq 0$ for $k \neq l$, which is a contradiction. □

Wilking proved in [21] that the homogeneous Riemannian manifold $N_{1,1}$ is normal and can be written as $\mathrm{SU}(3) \times \mathrm{SO}(3)/\mathrm{U}^*(2)$, hence $N_{1,1} = \mathrm{SU}(3) \times \mathrm{SO}(3)/\mathrm{U}^*(2)$ is generalized normal homogeneous.

Let q_t be as above. Then q_t defines a SU(3)-invariant Riemannian metric on $N_{(1,1)}$, and this metric has positive curvature if and only if $-1 < t < 0$. Thus, for any $X \in \mathfrak{m}_0 = V_0 \oplus V''', q_t(X,X) < 1$, we obtain a positively flag curvature generalized normal homogeneous Randers space with navigation data (q_t, X), and it is non-Riemannian if and only if $X \neq 0$.

Now, we consider Case (I). Let $\mathfrak{g} = \mathfrak{su}(n+1)$, $\mathfrak{h} = \mathfrak{su}(n)$, and define $\mathfrak{m}$ to be the direct sum as

$$\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1,$$

where

$$\mathfrak{m}_0 = \mathbb{R}X_0,\ X_0 = \sqrt{-1}\begin{pmatrix} -\frac{1}{n}E & 0 \\ 0 & 1 \end{pmatrix}, \tag{16}$$

and

$$\mathfrak{m}_1 = \left\{ \begin{pmatrix} 0 & \alpha \\ -\overline{\alpha}' & 0 \end{pmatrix} \Big| \alpha' = (x_1, \ldots, x_n) \in \mathbb{C}^n \right\}.$$

Then we have a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Any $\mathrm{SU}(n+1)$-invariant Riemannian metric on $\mathrm{SU}(n+1)/\mathrm{SU}(n)$ can be induced by an inner product

$$(\cdot,\cdot)_t = \frac{2nt}{n+1}\langle\cdot,\cdot\rangle|_{\mathfrak{m}_0} + \langle\cdot,\cdot\rangle|_{\mathfrak{m}_1} \tag{17}$$

on $\mathfrak{m}$, where $t > 0$, and $\langle\cdot,\cdot\rangle$ is the $\mathrm{Ad}(\mathrm{SU}(n+1))$-invariant inner product defined by

$$\langle X, Y\rangle = -\frac{1}{2}Retr(XY), \quad X, Y \in \mathfrak{su}(n+1).$$

It is shown in [20] that the homogeneous metric defined by (17) has positive curvature if and only if $0 < \frac{2n}{n+1}t < \frac{8n}{3(n+1)}$. Hence, by Table 2, the $\mathrm{SU}(n+1)$-invariant generalized normal homogeneous Riemannian metrics $(\cdot,\cdot)_t$ must have positive curvature. Thus, for any $cX \in \mathfrak{m}_0$ with $|c| < \frac{1}{\sqrt{t}}, \frac{n+1}{2n} \leq t \leq 1$, we obtain a positively curved generalized normal homogeneous Randers metrics on $\mathrm{SU}(n+1)/\mathrm{SU}(n)$, and it is non-Riemannian if and only if $c \neq 0$.

Finally, we consider case (II). We can take the subspace $\mathfrak{m}$ of $\mathfrak{sp}(n+1)$ to be

$$\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1,$$

where

$$\mathfrak{m}_0 = \mathbb{R}X_1 \oplus \mathbb{R}X_2 \oplus \mathbb{R}X_3 \tag{18}$$

is the subspace of H-fixed vectors in $\mathfrak{m}$, and $X_i, i = 1, 2, 3$ denote the elements of $\mathbb{H}^{(n+1)\times(n+1)}$ with the only non-zero element at the $(n+1, n+1)$-entry which is equal to $\sqrt{2}I, \sqrt{2}J$ and $\sqrt{2}K$, respectively. Here I, J, K denote the standard imaginary units in $\mathbb{H}$, and

$$\mathfrak{m}_1 = \left\{ \begin{pmatrix} 0 & \alpha \\ -\overline{\alpha}' & 0 \end{pmatrix} \Big| \alpha' = (x_1, \ldots, x_n) \in \mathbb{H}^n \right\}.$$

Then up to a positive multiple, any $\mathrm{Sp}(n+1)$-invariant Riemannian metric on $S^{(4n+3)} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$ can be written as

$$g_{t_1,t_2,t_3}(\cdot,\cdot) = t_1\langle\cdot,\cdot\rangle|_{\mathbb{R}X_1} + t_2\langle\cdot,\cdot\rangle|_{\mathbb{R}X_2} + t_3\langle\cdot,\cdot\rangle|_{\mathbb{R}X_3} + \langle\cdot,\cdot\rangle|_{\mathfrak{m}_1}. \tag{19}$$

Now, we prove

Proposition 4.2. *If the metrics $g_{t,t,t}$ is $\mathrm{Sp}(n+1)$-generalized normal homogeneous, then it has positive curvature. Moreover, if the metric $g_{s,t,t}$ is $\mathrm{Sp}(n+1)\times \mathrm{U}(1)$-generalized normal homogeneous, then it has positive curvature.*

Proof. The condition for such a metric to have positive curvature can be stated as follows. Let

$$V_i = (t_j^2 + t_k^2 - 3t_i^2 + 2t_it_j + 2t_it_k - 2t_jt_k)/t_i \quad \text{and} \quad H_i = 4 - 3t_i, \tag{20}$$

with (i,j,k) a cyclic permutation of $(1,2,3)$. Then it is shown in [20] that the homogeneous metrics g_{t_1,t_2,t_3} have positive sectional curvature if and only if

$$V_i > 0,\ H_i > 0 \quad \text{and} \quad 3|t_jt_k - t_j - t_k + t_i| < t_jt_k + \sqrt{H_iV_i}, \tag{21}$$

with (i,j,k) a cyclic permutation of $(1,2,3)$. It is also pointed out in [20] that the set (t_1,t_2,t_3) satisfying the above condition forms a non-empty slice.

By Table 2, the metrics $g_{t,t,t}$ (i.e., $t = t_1 = t_2 = t_3$) is $\mathrm{Sp}(n+1)$-generalized normal homogeneous if and only if $t \in [\frac{1}{2}, 1]$, and the metrics $g_{s,t,t}$ (i.e., $t_1 = s, t_2 = t_3 = t$) is $\mathrm{Sp}(n+1)\times\mathrm{U}(1)$-generalized normal homogeneous if and only if $t \in [\frac{1}{2}, 1]$ and $s \in (0, t]$.

In the case $t = t_1 = t_2 = t_3$, (21) is equivalent to

$$\begin{cases} V_1 = V_2 = V_3 = t > 0, \\ H_1 = H_2 = H_3 = 4 - 3t > 0, \\ 3|t^2 - t| < t^2 + \sqrt{t(4-3t)}. \end{cases}$$

Solving the above inequalities system, we get $0 < t < \frac{4}{3}$. This implies that if $(S^{(4n+3)}, g_{t,t,t})$ is $\mathrm{Sp}(n+1)$-generalized normal homogeneous, then it has positive curvature.

In the case $s = t_1 \neq t_2 = t_3 = t$, (21) is equivalent to

$$\begin{cases} H_1 = 4 - 3s > 0, \\ H_2 = H_3 = 4 - 3t > 0, \\ V_1 = 4t - 3s > 0, \\ V_2 = V_3 = \frac{s^2}{t} > 0, \\ 3|t^2 - 2t + s| < t^2 + \sqrt{(4-3s)(4t-3s)}, \\ 3|st - s| < st + \sqrt{(4-3t)\frac{s^2}{t}}. \end{cases} \tag{22}$$

Since $s > 0$, the last inequality is reduced to

$$3|t-1| < t + \sqrt{\frac{(4-3t)}{t}}.$$

The solution to the above inequality is $0 < t < \frac{4}{3}$. Now, we show that the set

$$P = \left\{ (s,t) | 0 < s \leq t, t \in \left[\frac{1}{2}, 1 \right] \right\}$$

satisfies the inequality system (22). We just need to verify that the set P satisfies the fifth inequality. Observe that if $(s,t) \in P$, then $t^2 - 2t + s < 0$, and for $\frac{1}{2} < t < 1$, we always have

$$-(3t^2 - 2t + s) < t^2 + (4t - 3s).$$

Since $t \leq 1$, we have $4 - 3s \geq 4t - 3s$. On the other hand, if $\frac{1}{2} < t \leq 1$, $0 < s \leq t$, then we have

$$3|t^2 - 2t + s| < t^2 + (4t - 3s) < t^2 + \sqrt{(4-3s)(4t-3s)}.$$

If $t = \frac{1}{2}$, it is easily seen that any $s \in [0, \frac{1}{2}]$ satisfies the fifth inequality. From this, we conclude that if $(S^{(4n+3)}, g_{s,t,t})$ is $\mathrm{Sp}(n+1) \times \mathrm{U}(1)$-generalized normal homogeneous, then it has positive curvature. □

Therefore, any $\mathrm{Sp}(n+1)$-generalized normal homogeneous Randers metric on S^{4n+3} with positive flag curvature must be a $\mathrm{Sp}(n+1)$-generalized normal homogeneous Riemannian metric $g_{t,t,t}$ in (19), under the influence of a Killing vector field generated by $X = x_1X_1 + x_2X_2 + x_3X_3$ in $\mathfrak{m}_0$, where x_1, x_2, x_3 can be any real numbers satisfying $|x_1|^2 + |x_2|^2 + |x_3|^2 < \frac{1}{t}$. Moreover, any $\mathrm{Sp}(n+1) \times \mathrm{U}(1)$-generalized normal homogeneous Randers metric on S^{4n+3} with positive flag curvature must be a $\mathrm{Sp}(n+1) \times \mathrm{U}(1)$-generalized normal homogeneous Riemannian metric $g_{s,t,t}$ in (19), under the influence of a Killing vector generated by $X = x_1X_1$, where $x_1 \in \mathbb{R} \in \mathfrak{m}_0$ satisfies $|x_1|^2 < \frac{1}{s}$. On the other hand, [17, Theorem 5.1] implies that two connected simply connected generalized normal Randers spaces (M, F_1), (M, F_2) with navigation data (h, w_1),(h, w_2) are isometry if and only if w_1 and w_2 have the same length with respect to h. This gives a complete classification of generalized normal homogeneous Randers spaces up to isometry. The classification results are summarized in Table 3.

Acknowledgements. We are deeply grateful to the referee for a careful reading of the manuscript and valuable suggestions. The first author would like to thank Ming Xu for valuable comments and suggestions.

coset spaces	isotropy representations	Ad(H)-fixed vectors
(I) $\mathrm{SU}(n+1)/\mathrm{SU}(n)$	$\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_1$	$\mathfrak{m}_0$
(II) $\mathrm{Sp}(n+1)/\mathrm{Sp}(n)$	$\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_1$	$\mathfrak{m}_0$
(III) $N_{1,1} = \mathrm{SU}(3)\times\mathrm{SO}(3)/\mathrm{U}^*(2)$	$\mathfrak{m} = V_0 \oplus V' \oplus V'' \oplus V'''$	$V_0 \oplus V'''$
(IV) $N_{k,l} = \mathrm{SU}(3)/S^1_{k,l}$,	$\mathfrak{m} = V_0 \oplus V_{2k+l} \oplus V_{2l+k} \oplus V_{k-l}$	V_0
$\gcd(k,l) = 1, kl(k+l) \neq 0$		

Table 1. Homogeneous Randers spaces with $S = 0$ and $K > 0$.

G	H	$(,)\|_{m_0}$	$(,)\|_{m_1}$	$(,)\|_{m_2}$
$\mathrm{SU}(n+1)$	$\mathrm{SU}(n)$	$trg_{\mathrm{can}}, \frac{n+1}{2n} \leq t \leq 1$	rg_{can}	
$\mathrm{Sp}(n+1)$	$\mathrm{Sp}(n)$	$trg_{\mathrm{can}}, \frac{1}{2} \leq t \leq 1$	rg_{can}	
$\mathrm{Sp}(n+1)S^1$	$\mathrm{Sp}(n)S^1$	$srg_{\mathrm{can}}, s \leq t$	rg_{can}	$trg_{\mathrm{can}}, \frac{1}{2} \leq t \leq 1$

Table 2. Generalized normal homogeneous metrics on spheres.

coset spaces	navigation data	the conditions
(I) $\mathrm{SU}(n+1)/\mathrm{SU}(n)$	$((\cdot,\cdot)_t, X)$	$X = cX_0 \in \mathfrak{m}_0, \|c\| < \frac{1}{\sqrt{t}}$, $\frac{n+1}{2n} \leq t \leq 1$
(II) $\mathrm{Sp}(n+1)/\mathrm{Sp}(n)$	$(g_{t,t,t}, X)$	$X = x_1X_1 + x_2X_2 + x_3X_3$, $\|x_1\|^2 + \|x_2\|^2 + \|x_3\|^2 < \frac{1}{t}$, $\frac{1}{2} \leq t \leq 1$
(III) $\mathrm{Sp}(n+1)\mathrm{U}(1)/\mathrm{Sp}(n)\mathrm{U}(1)$	$(g_{s,t,t}, X)$	$X = x_1X_1$, $\|x_1\|^2 < \frac{1}{s}$
(IV) $\mathrm{SU}(3)\times\mathrm{SO}(3)/\mathrm{U}^*(2)$	(q_t, X)	$q_t(X,X) < 1$

Table 3. Generalized normal homogeneous Randers spaces with positive flag curvature.

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(Received May 16, 2016; revised August 7, 2016)