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Contact structures on Lie algebroids

By CRISTIAN IDA (†) (Braşov) and PAUL POPESCU (Craiova)

Abstract. In this paper, we generalize the main notions from the geometry of (almost) contact manifolds in the category of Lie algebroids. Also, using the framework of generalized geometry, we obtain an (almost) contact Riemannian Lie algebroid structure on a vertical Liouville distribution over the big-tangent manifold of a Riemannian manifold.

1. Introduction

The importance of contact and symplectic geometry is without question. Contact manifolds can be viewed as an odd-dimensional counterpart of symplectic manifolds. Both contact and symplectic geometry are motivated by the mathematical formalism of classical mechanics, where one can consider either the even-dimensional phase space of a mechanical system or the odd-dimensional extended phase space that includes the time variable. For more about contact geometry, the reader can consult the outstanding works [5], [6], [8].

On the other hand, in the last decades, the Lie algebroids have occupied an important place in the context of some different categories in differential geometry and mathematical physics and represent an active domain of research. The Lie algebroids ([29]) are generalizations of Lie algebras and integrable distributions. In fact, a Lie algebroid is an anchored vector bundle with a Lie bracket on module of sections and many geometrical notions which involve the tangent bundle have been generalized to the context of Lie algebroids. In the category of almost complex geometry, the notion of almost complex Lie algebroids over

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almost complex manifolds was introduced in [11] as a natural extension of the notion of an almost complex manifold to that of an almost complex Lie algebroid. More generally, in [2], [14], [24], [38], the notion of almost complex Lie algebroids over a smooth manifold is considered, as well as some problems concerning the geometry of almost complex Lie algebroids over smooth manifolds are studied in relation with corresponding notions from the geometry of almost complex manifolds. Taking into account the major role of (almost) complex geometry in the study of (almost) contact geometry, a natural generalization of (almost) contact geometry of manifolds to that of (almost) contact Lie algebroids can be of some interest. We notice that for the particular class of Lie algebroids defined by the tangent bundle along the leaves of a foliation of odd dimension, the contact structures are introduced and studied in some recent papers [12], [36], under the name of foliated contact structures. In general, the notion of contact Lie algebroids appears in some very recent talks (see [32], [33], [34]), where this notion is used in order to obtain Jacobi manifolds on spheres of linear Poisson manifolds with a bundle metric. Also, the Albert cosymplectic and contact reduction theorems are extended in the Lie algebroid framework, and this reduction theory can represent a rich source in obtaining some new examples of cosymplectic or contact Lie algebroids (see [32]). The study of symplectic Lie algebroids and their reductions can be found, for instance, in [25].

Our aim in this paper is to generalize some basic facts from the (almost) contact geometry on odd-dimensional manifolds (see [5], [6], [8], [35]), in the framework of Lie algebroids of odd rank, and to present new examples of contact Lie algebroids. This generalization is possible mainly using the differential calculus on Lie algebroids: exterior derivative, interior product, Lie derivative (see, for instance, [30]), but also using the connections theory on Lie algebroids (see [15]), and the technique of Riemannian geometry on Lie algebroids (see [7]).

The paper is organized as follows. In the second section, we present the almost contact and the almost contact Riemannian structures on Lie algebroids of odd rank, and we give the main properties of these structures in relation with similar properties from the case of almost contact manifolds. In the third section, we present the normal almost contact structures on Lie algebroids, we study these structures, and using the definition of the direct product of two Lie algebroids (see [29]), we characterize the direct product of two Lie algebroids endowed with some additional (almost Hermitian and almost contact Riemannian) structures. In the fourth section, we give the basic definitions and results about contact structures on Lie algebroids in relation with similar notions from contact manifolds theory, we present some examples (see [33], [34]) and a bijective corespondence between contact Riemannian structures and almost contact Riemannian structures on Lie algebroids, as well as give some characterizations of contact Riemannian Lie algebroids. Also, the notions of K-contact, Sasakian and Kenmotsu Lie algebroids are introduced, and some of their properties are studied as in the manifolds case. In the last section, using the framework of generalized geometry and starting from the geometry of big-tangent manifold introduced and intensively studied in [45], we obtain an (almost) contact Riemannian structure on the vertical Liouville distribution over the big-tangent manifold of a paracompact manifold M which admits a Riemannian metric g. More exactly, we construct a vertical framed Riemannian $f(3,1)$-structure on the vertical bundle over the big-tangent manifold of a Riemannian manifold (M,g), and when we restrict this structure to a vertical Liouville distribution which is integrable (so it is a Lie algebroid), we obtain an (almost) contact Riemannian structure on this Lie algebroid.

Other problems and some future works can be addressed as, for instance: the study of deformations of Sasakian structures on Lie algebroids, the study of curvature problems on contact Riemannian Lie algebroids, K-contact, Sasaki and Kenmotsu Lie algebroids, as well as the study of F_E-sectional curvature and a Schur type theorem on Sasakian Lie algebroids. Also, taking into account the recent harmonic theory on Riemannian Lie algebroids (see [3]), a harmonic and C-harmonic theory for differential forms on Sasakian Lie algebroids can be investigated, since every almost contact Lie algebroid will be invariantly oriented (see Corollary 2.1). Another important problem is that of the integrability of Jacobi structures, being closely related to that of Poisson structures and giving rise to contact groupoids. The progress in this direction is described at large in the recent paper [13], but we do not consider here relations with contact groupoids, which would involve some more problems, beyond the scope of our work.

The main notions introduced here are natural generalizations from the category of manifolds to that of Lie algebroids, and most of the proofs are similar to the ones given for the case of (almost) contact manifolds (see, for instance, [5], [6], [35]). For this reason, they are omitted here.

2. Almost contact Lie algebroids

In this section, we define the almost contact and the almost contact Riemannian structures on Lie algebroids, and some properties of these structures are analyzed by analogy with the almost contact manifolds case (see [5], [6], [35]).

Let $p : E \to M$ be a vector bundle of *rank* m over a smooth n-dimensional manifold M, and $\Gamma(E)$ the $C^\infty(M)$-module of sections of E. A *Lie algebroid structure* on E is given by a triplet $(E, \rho_E, [\cdot\,,\cdot]_E)$, where $[\cdot\,,\cdot]_E$ is a Lie bracket on $\Gamma(E)$ and $\rho_E : E \to TM$ is called the *anchor map*, such that if we also denote by $\rho_E : \Gamma(E) \to \mathcal{X}(M)$ the homomorphism of $C^\infty(M)$-modules induced by the anchor map, then we have

$$[s_1, fs_2]_E = f[s_1, s_2]_E + \rho_E(s_1)(f)s_2, \qquad \forall s_1, s_2 \in \Gamma(E),\ \forall f \in C^\infty(M). \quad (2.1)$$

Remark 2.1. If $(E, \rho_E, [\cdot\,,\cdot]_E)$ is a Lie algebroid over M, then the anchor map $\rho_E : \Gamma(E) \to \mathcal{X}(M)$ is a homomorphism between the Lie algebras $(\Gamma(E), [\cdot\,,\cdot]_E)$ and $(\mathcal{X}(M), [\cdot\,,\cdot])$.

The exterior derivative on Lie algebroids is defined by

$$\begin{aligned}(d_E\omega)(s_0, \ldots, s_p) &= \sum_{i=0}^{p}(-1)^i \rho_E(s_i)(\omega(s_0, \ldots, \widehat{s_i}, \ldots, s_p)) \\ &\quad + \sum_{i<j=1}^{p}(-1)^{i+j}\omega([s_i, s_j]_E, s_0, \ldots, \widehat{s_i}, \ldots, \widehat{s_j}, \ldots, s_p), \quad (2.2)\end{aligned}$$

for $\omega \in \Omega^p(E)$ and $s_0, \ldots, s_p \in \Gamma(E)$, where $\Omega^p(E)$ is the set of p-forms on E. For more details about Lie algebroids and all calculus on Lie algebroids (interior product, Lie derivative, etc.), we refer, for instance, to [15], [21], [27], [29], [30] and [37].

Let $(E, \rho_E, [\cdot\,,\cdot]_E)$ be a Lie algebroid of $\operatorname{rank} E = 2m + 1$ over a smooth n-dimensional manifold M. If there are a section $\xi \in \Gamma(E)$, 1-form $\eta \in \Gamma(E^*)$ and a $(1,1)$-tensor $F_E \in \Gamma(E \otimes E^*)$ such that

$$F_E^2 = -I_E + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad (2.3)$$

where I_E denotes the Kronecker tensor on E, then we say that (F_E, ξ, η) is an *almost contact structure* on the Lie algebroid $(E, \rho_E, [\cdot\,,\cdot]_E)$, or $(E, \rho_E, [\cdot\,,\cdot]_E, F_E, \xi, \eta)$ is an *almost contact Lie algebroid*. The 1-section ξ is called *Reeb section* or *fundamental section*. Obviously, the set $\Gamma_\xi(E) = \{f\xi | f \in \mathcal{F}(M)\}$ has a module structure over $\mathcal{F}(M)$ and a Lie algebra structure, called the *Lie algebra of Reeb sections*.

Let $D_x = \{s_x \in E_x | \imath_{s_x}\eta_x = 0\} \subseteq E_x$ for $x \in M$. Then, the distribution $D = \cup_{x\in M} D_x$ is a vector subbundle of E of rank $2m$ called the *contact subbundle* of $(E, \rho_E, [\cdot\,,\cdot]_E, F_E, \xi, \eta)$. We notice that $D = \ker \eta = \operatorname{im} F_E$.

Remark 2.2. The above definition of the almost contact structure from (2.3) does not depend on the anchor ρ_E and the bracket $[\cdot\,,\cdot]_E$, hence it can be considered for a general vector bundle $E \to M$ of odd rank which will be referred to as an *almost contact bundle*.

Now, let us briefly present some basic properties of almost contact structures on Lie algebroids (or general vector bundles, when the notions do not depend on the anchor or bracket).

Proposition 2.1. *If (F_E, ξ, η) is an almost contact structure on the vector bundle E, then:*
(i) $F_E(\xi) = 0$; (ii) $F_E^3 = -F_E$; (iii) $\eta \circ F_E = 0$; (iv) $\operatorname{rank} F_E = 2m$.

Proof. Follows in a similar manner as for almost contact manifolds (see [5] and [6]). □

Also, the following theorem holds.

Theorem 2.1. *Let E be a vector bundle with an almost contact structure (F_E, ξ, η). There exists on E a fiber-wise Riemannian metric (or simply Riemannian metric) g_E with the property*

$$g_E\left(F_E(s_1), F_E(s_2)\right) = g_E(s_1, s_2) - \eta(s_1)\eta(s_2), \tag{2.4}$$

for any $s_1, s_2 \in \Gamma(E)$.

Proof. We recall that a Riemannian metric in the vector bundle $p : E \to M$ is a mapping g_E that assigns to every $x \in M$ a scalar product $g_E(x)$ in the local fiber E_x such that, for every local sections $s_1, s_2 \in \Gamma(E)$, the function $x \mapsto g_E(x)(s_1, s_2)$ is smooth. Since E is paracompact, there exists a Riemannian metric g_E^{**} on E, and then, we define g_E by

$$g_E(s_1, s_2) = \frac{1}{2}\left[g_E^*(F_E(s_1), F_E(s_2)) + g_E^*(s_1, s_2) + \eta(s_1)\eta(s_2)\right], \tag{2.5}$$

where g_E^* has the expression $g_E^*(s_1, s_2) = g_E^{**}\left(F_E^2(s_1), F_E^2(s_2)\right) + \eta(s_1)\eta(s_2)$. Then, it is easy to check that g_E given by (2.5) is a Riemannian metric on E and satisfies the condition (2.4). □

The vector bundle E with the almost contact structure (F_E, ξ, η) and the Riemannian metric g_E satisfying the condition (2.4) is called an *almost contact Riemannian vector bundle* or an *almost contact Riemannian Lie algebroid* (when this is the case), and (F_E, ξ, η, g_E) is an *almost contact Riemannian structure* on E. Sometimes, we say that g_E is a metric *compatible* with the almost contact structure (F_E, ξ, η).

In a similar manner as in the case of almost contact manifolds (see [5], [6]), some elementary but useful properties of such metrics are specified in the following:

Proposition 2.2. *If g_E is a metric compatible with the almost contact structure (F_E, ξ, η) on the vector bundle E of rank $2m+1$, then:*

(i) $\eta(s) = g_E(s, \xi)$ *for all* $s \in \Gamma(E)$*;*

(ii) *on the domain U of each local chart from M there exists an orthonormal basis of local sections of E over U,* $\{s_1, \ldots, s_n, F_E(s_1), \ldots, F_E(s_n), \xi\}$*;*

(iii) $F_E + \eta \otimes \xi$ *and* $-F_E + \eta \otimes \xi$ *are orthogonal transformations with respect to metric g_E;*

(iv) $g_E(F_E(s_1), s_2) = -g_E(s_1, F_E(s_2))$ *for every* $s_1, s_2 \in \Gamma(E)$.

The local basis $\{s_1, \ldots, s_m, s_{1^*} = F_E(s_1), \ldots, s_{m^*} = F_E(s_m), \xi\}$ of sections of E, obtained above and denoted sometimes by $\{s_a, s_{a^*}, \xi\}$, $a = 1, \ldots, m$, is called a F_E*–basis* for the almost contact Riemaniann vector bundle (E, F_E, ξ, η, g_E) (or Lie algebroid, when this is the case). The existence of metrics compatible with an almost contact structure (F_E, ξ, η) on E allows us to state the following characterization of almost contact bundles (or Lie algebroids) by means of the structure group of the vector bundle E.

Theorem 2.2. *The structure group of the almost contact vector bundle (or Lie algebroid) E of rank $2m+1$ reduces to $U(m) \times 1$. Conversely, if the structure group of the vector bundle E reduces to $U(m) \times 1$, then E has an almost contact structure.*

Proof. The proof follows in the same manner as for the almost contact manifolds case (see, for instance, [5], [35]). However, we briefly present here its generalization to the general vector bundles case. Let g_E be a metric on E, compatible with the almost contact structure (F_E, ξ, η), and consider two open neighborhoods U, V on M trivializing E with $U \cap V \neq \emptyset$. Also, we denote by $\mathcal{B}_U = \{s_a, s_{a^*}, \xi\}$ and $\mathcal{B}_V = \{s'_a, s'_{a^*}, \xi\}$ the corresponding F_E-bases from Proposition 2.2 (ii). The matrix (F_E) of F_E with respect to these bases is

$$(F_E) = \begin{pmatrix} 0 & -I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For $x \in U \cap V$ and $s_x \in E_x$, we denote by (s_x^U), (s_x^V) the column matrices of components of the section s_x with respect to $\mathcal{B}_U$ and $\mathcal{B}_V$, respectively. Then $(s_x^V) = P \cdot (s_x^U)$, where

$$P = \begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and $A, B, C, D \in \mathcal{M}_{m\times m}(\mathbb{R})$. But P is orthogonal and commutes with the matrix (F_E) (see Proposition 2.2 (ii)), thus we have $D = A$, $C = -B$, and this proves that $P \in U(m) \times 1$.

Conversely, if the structure group of the vector bundle E reduces to $U(m)\times 1$, then there exists a covering $\{U_\alpha\}_{\alpha\in I}$ of M, for which we can choose the orthonormal local bases of sections of E with the property that on the intersection $U_\alpha \cap U_\beta \neq \emptyset$ these are transformed by the action of the group $U(m) \times 1$. With respect to such bases, we can define the endomorphism $F_E|_\alpha : \Gamma(E|_{U_\alpha}) \to \Gamma(E|_{U_\alpha})$ by the matrix (F_E). But (F_E) commutes with $U(m) \times 1$, hence $\{F_E|_\alpha\}_{\alpha\in I}$ determine a global endomorphism $F_E : \Gamma(E) \to \Gamma(E)$. In a similar way, the sections $\xi \in \Gamma(E)$ and $\eta \in \Omega^1(E)$ are globally defined by the matrices of their components with respect to each open set U_α, namely,

$$\xi : (0, \dots, 0, 1)^t, \quad \eta : (0, \dots, 0, 1).$$

Finally, the fact that (F_E, ξ, η) is an almost contact structure on E is straightforward. □

Also, we notice that the determinants of the matrices from the proof of Theorem 2.2 are positive, which yields

Corollary 2.1. *Any almost contact bundle (or Lie algebroid) is orientable.*

Let us define $\Omega_E(s_1, s_2) = g_E(s_1, F_E(s_2))$ for all $s_1, s_2 \in \Gamma(E)$. Then, from Proposition 2.2 (iv) it follows that Ω_E is a 2-form on E. It is called the *fundamental* 2*-form* or the *Sasaki* 2*-form* of the almost contact Riemannian vector bundle (or Lie algebroid) (E, F_E, ξ, η, g_E). Moreover, it is easy to see that Ω_E has the following obvious properties:

$$\Omega_E(s_1, F_E(s_2)) = -\Omega_E(F_E(s_1), s_2) \text{ and } \Omega_E(F_E(s_1), F_E(s_2)) = \Omega_E(s_1, s_2). \quad (2.6)$$

If $\{e^a, e^{a^*}, \eta\}$ is the dual basis of the F_E-basis from Proposition 2.2, then the fundamental 2-form Ω_E is locally given by

$$\Omega_E = -2 \sum_{a=1}^{m} e^a \wedge e^{a^*}.$$

We remark that $\operatorname{rank}\Omega_E = 2m$, and then $\eta \wedge \Omega_E^m$ (where Ω_E^m is the exterior product of m copies of Ω_E) vanish nowhere on M. The converse of this result is also true, namely, we have

Theorem 2.3. *Let E be a vector bundle over M of $\operatorname{rank} E = 2m+1$ and $\eta \in \Omega^1(E)$. If there exists $\Omega_E \in \Omega^2(E)$ such that $\eta \wedge \Omega_E^m \neq 0$ at each point of M, then E has an almost contact structure.*

Proof. Follows as in the case of almost contact manifolds (see [5], [35]). □

Moreover, in the case of Lie algebroids, we have

Theorem 2.4. *Let $(E, \rho_E, [\cdot\,,\cdot]_E)$ be a Lie algebroid of $\operatorname{rank} E = 2m+1$ and $\eta \in \Omega^1(E)$. If $\eta \wedge (d_E\eta)^m \neq 0$ on M, then the Lie algebroid $(E, \rho_E, [\cdot\,,\cdot]_E)$ has an almost contact Riemannian structure (F_E, ξ, η, g_E) whose fundamental form is $d_E\eta$, and the Reeb section ξ is completely determined by the conditions $\eta(\xi) = 1$ and $\imath_\xi(d_E\eta) = 0$.*

3. Normal almost contact structures on Lie algebroids

In this section, we define normal almost contact structures on Lie algebroids and characterize these structures. Also, the direct product between an almost Hermitian Lie algebroid and an almost contact Riemannian Lie algebroid or the direct product of two almost contact Riemannian Lie algebroids are investigated.

We recall that for a general tensor $A \in \Gamma(E \otimes E^*)$ of type $(1,1)$ on E, the Nijenhuis tensor of A is a tensor $N_A \in \Gamma(\otimes^2 E^* \otimes E)$ given by

$$N_A(s_1, s_2) = [A(s_1), A(s_2)]_E - A([A(s_1), s_2]_E) - A([s_1, A(s_2)]_E) + A^2([s_1, s_2]_E).$$

As usual, we say that an almost contact structure (F_E, ξ, η) on a Lie algebroid $(E, \rho_E, [\cdot\,,\cdot]_E)$ of rank $2m+1$ is *normal* if

$$N_E^{(1)} \equiv N_{F_E} + 2d_E\eta \otimes \xi = 0. \tag{3.1}$$

Other useful tensors on E are the following:

$$\begin{aligned} N_E^{(2)}(s_1, s_2) &\equiv \left(\mathcal{L}_{F_E(s_1)}\eta\right)(s_2) - \left(\mathcal{L}_{F_E(s_2)}\eta\right)(s_1), \\ N_E^{(3)}(s) &\equiv \frac{1}{2}\left(\mathcal{L}_\xi F_E\right)(s), \quad N_E^{(4)}(s) \equiv \left(\mathcal{L}_\xi \eta\right)(s). \end{aligned} \tag{3.2}$$

Using the differential calculus on Lie algebroids (exterior differential and Lie derivative), we can easily prove that if the almost contact structure (F_E, ξ, η) is normal, then $N_E^{(2)} = N_E^{(3)} = N_E^{(4)} = 0$.

Replacing in the definition of the Nijenhuis tensor N_{F_E} the brackets by their expressions (since the Levi–Civita connection ∇ on Riemannian Lie algebroids is torsionless, see [7]), similarly to the almost contact Riemannian manifolds (see [39]), we obtain

Proposition 3.1. *An almost contact Riemannian structure (F_E, ξ, η, g_E) on a Lie algebroid $(E, \rho_E, [\cdot\,,\cdot]_E)$ is normal if and only if one of the following conditions is satisfied:*

$$F_E\left(\nabla_{s_1} F_E\right) s_2 - \left(\nabla_{F_E(s_1)} F_E\right) s_2 - \left[\left(\nabla_{s_1}\eta\right) s_2\right]\xi = 0, \tag{3.3}$$

$$\left(\nabla_{s_1} F_E\right) s_2 - \left(\nabla_{F_E(s_1)} F_E\right) F_E(s_2) + \eta(s_2)\nabla_{F_E(s_1)}\xi = 0, \tag{3.4}$$

for every $s_1, s_2 \in \Gamma(E)$.

Since the eigenvalues of $F_E|_D$ are i and $-i$, we deduce that the complexified $D_{\mathbb{C}} = D \otimes_{\mathbb{R}} \mathbb{C}$ of D has the decomposition

$$D_{\mathbb{C}} = D^{1,0} \oplus D^{0,1}, \tag{3.5}$$

where $D^{1,0}$ and $D^{0,1}$ are the eigensubbundles corresponding to i and $-i$, respectively. A simple argument shows that

$$D^{1,0} = \{s - iF_E(s) | s \in \Gamma(D)\}, \quad D^{0,1} = \{s + iF_E(s) | s \in \Gamma(D)\},$$

and extending to $E_{\mathbb{C}}$ the metric g_E by

$$g_E^c(s_1 + is_2, s) = g_E(s_1, s) + ig_E(s_2, s), \quad g_E^c(s, s_1 + is_2) = g_E(s, s_1) - ig_E(s, s_2),$$

we obtain a Hermitian metric g_E^c on $E_{\mathbb{C}}$. From Proposition 2.2 (iv), we deduce that with respect to this metric, the decomposition (3.5) is orthogonal, and, consequently, the following orthogonal decomposition of the complexified vector bundle $E_{\mathbb{C}}$ is associated:

$$E_{\mathbb{C}} = D_{\mathbb{C}} \oplus \langle\xi\rangle_{\mathbb{C}} = D^{1,0} \oplus D^{0,1} \oplus \langle\xi\rangle_{\mathbb{C}}, \tag{3.6}$$

where $\langle\xi\rangle_{\mathbb{C}} = \langle\xi\rangle \otimes_{\mathbb{R}} \mathbb{C}$.

On the other hand, $(E_{\mathbb{C}}, g_E^c)$ is a Hermitian vector bundle over M, and the natural extension ∇^c of the Levi–Civita connection ∇ from E is a Hermitian connection in this bundle (see [24]). Moreover, $(D_{\mathbb{C}}, g_E^c|_{D_{\mathbb{C}}})$ is a Hermitian subbundle of $(E_{\mathbb{C}}, g_E^c)$, with the Hermitian connection $\nabla^{D_{\mathbb{C}}}$ induced by the following decomposition

$$\nabla^c s = \nabla^{D_{\mathbb{C}}} s + A^{D_{\mathbb{C}}} s, \tag{3.7}$$

where $s \in \Gamma(D_{\mathbb{C}})$, $\nabla^{D_{\mathbb{C}}} s \in L(E_{\mathbb{C}}, D_{\mathbb{C}})$ and $A^{D_{\mathbb{C}}} s \in L(E_{\mathbb{C}}, \langle\xi\rangle_{\mathbb{C}})$. A simple calculation shows that

$$A_s^{D_{\mathbb{C}}} s' = -\Omega_E(s, s')\xi, \quad \nabla^{D_{\mathbb{C}}} F_E|_{D_{\mathbb{C}}} = 0,$$

hence $\nabla^{D_{\mathbb{C}}}$ is an almost complex connection ([24]), in the complex bundle $D_{\mathbb{C}}$.

Let $g_E^{1,0}$ be the restriction of the metric $g_E^c|_{D_{\mathbb{C}}}$ to $D^{1,0}$. Then, following the same argument as above, we deduce that $(D^{1,0}, g_E^{1,0})$ is a Hermitian subbundle of $(D_{\mathbb{C}}, g_E^c|_{D_{\mathbb{C}}})$, with Hermitian connection $\nabla^{1,0}$ induced by the following decomposition

$$\nabla^{D_{\mathbb{C}}} s = \nabla^{1,0} s + A^{1,0} s, \tag{3.8}$$

where $s \in \Gamma(D^{1,0})$, $\nabla^{1,0} s \in L(D_{\mathbb{C}}, D^{1,0})$ and $A^{1,0} s \in L(D_{\mathbb{C}}, D^{0,1})$.

The direct product of two given Lie algebroids $(E_1, \rho_{E_1}, [\cdot\,,\cdot]_{E_1})$ over M_1 and $(E_2, \rho_{E_2}, [\cdot\,,\cdot]_{E_2})$ over M_2 is defined in [29, p. 155] as a Lie algebroid structure on $E_1 \times E_2 \to M_1 \times M_2$. The general sections of $E_1 \times E_2$ are of the form $s = \sum(f_i \otimes s_i^1) \oplus \sum(g_j \otimes s_j^2)$, where $f_i, g_j \in C^\infty(M_1 \times M_2)$, $s_i^1 \in \Gamma(E_1)$, $s_j^2 \in \Gamma(E_2)$, the anchor map is defined by

$$\rho_E\left(\sum(f_i \otimes s_i^1) \oplus \sum(g_j \otimes s_j^2)\right) = \sum(f_i \otimes \rho_{E_1}(s_i^1)) \oplus \sum(g_j \otimes \rho_{E_2}(s_j^2)),$$

and the Lie bracket on $E = E_1 \times E_2$ is:

$$\begin{aligned}
&[s, s']_E \\
&\quad = \left(\sum f_i f_k' \otimes [s_i^1, s_k'^1]_{E_1} + \sum f_i \rho_{E_1}(s_i^1)(f_k') \otimes s_k'^1 - \sum f_k' \rho_{E_1}(s_k'^1)(f_i) \otimes s_i^1\right) \\
&\quad\quad \oplus \left(\sum g_j g_l' \otimes [s_j^2, s_l'^2]_{E_2} + \sum g_j \rho_{E_2}(s_j^2)(g_l') \otimes s_l'^2 - \sum g_l' \rho_{E_2}(s_l'^2)(g_j) \otimes, s_j^2\right),
\end{aligned}$$

for every $s = \sum(f_i \otimes s_i^1) \oplus \sum(g_j \otimes s_j^2)$ and $s' = \sum(f_k' \otimes s_k'^1) \oplus \sum(g_l' \otimes s_l'^2)$ in $\Gamma(E)$.

Now, by direct verification and using a simple calculation, we can prove the following two results concerning the direct product of Lie algebroids.

Proposition 3.2. *Let us consider two Lie algebroids, $(E_1, \rho_{E_1}, [\cdot\,,\cdot]_{E_1})$ over M_1 of rank $2m_1$ equipped with an almost Hermitian structure (J_{E_1}, g_{E_1}),* [24], *and* $(E_2, \rho_{E_2}, [\cdot\,,\cdot]_{E_2})$ *over* M_2 *of rank* $2m_2+1$ *equipped with an almost contact Riemannian structure* $(F_{E_2}, \xi_2, \eta_2, g_{E_2})$*. Then, the tensors* F_E*,* ξ*,* η*,* g_E*, given by*

$$F_E\left(\sum(f_i\otimes s_i^1)\oplus\sum(g_j\otimes s_j^2)\right)=\sum(f_i\otimes J_{E_1}(s_i^1))\oplus\sum(g_j\otimes F_{E_2}(s_j^2)),$$
$$\eta\left(\sum(f_i\otimes s_i^1)\oplus\sum(g_j\otimes s_j^2)\right)=\sum(g_j\otimes\eta_2(s_j^2)),\quad \xi=0\oplus\xi_2,$$

and

$$g_E\left(\left(\sum(f_i\otimes s_i^1)\oplus\sum(g_j\otimes s_j^2)\right),\sum(f'_k\otimes s'^1_k)\oplus\sum(g'_l\otimes s'^2_l)\right)$$
$$=\sum f_if'_k\otimes g_{E_1}(s_i^1,s'^1_k)\oplus\sum g_jg'_l\otimes g_{E_2}(s_j^2,s'^2_l),$$

define an almost contact Riemannian structure on the direct product Lie algebroid $E=E_1\times E_2$*.*

Proposition 3.3. *Let us consider two Lie algebroids,* $(E_1, \rho_{E_1}, [\cdot\,,\cdot]_{E_1})$ *over* M_1 *of rank* $2m_1+1$ *equipped with an almost contact Riemannian structure* $(F_{E_1}, \xi_1, \eta_1, g_{E_1})$*, and* $(E_2, \rho_{E_2}, [\cdot\,,\cdot]_{E_2})$ *over* M_2 *of rank* $2m_2+1$ *equipped with an almost contact Riemannian structure* $(F_{E_2}, \xi_2, \eta_2, g_{E_2})$*. Then, the tensor* F_E *given by*

$$F_E\left(\sum(f_i\otimes s_i^1)\oplus\sum(g_j\otimes s_j^2)\right)$$
$$=\sum(f_i\otimes F_{E_1}(s_i^1)-g_j\otimes\eta_2(s_j^2)\xi_1)\oplus\sum(g_j\otimes F_{E_2}(s_j^2)+f_i\otimes\eta_1(s_i^1)\xi_2),$$

defines an almost Hermitian structure on the direct product Lie algebroid $E=E_1\times E_2$*, with the metric* g_E *from Proposition 3.2. This structure is Hermitian (that is* $N_{F_E}=0$*) if and only if both almost contact Riemannian structures are normal.*

Remark 3.1. Let (E, F_E, ξ, η) be an almost contact Lie algebroid of rank $2m+1$ over a smooth manifold M, and L be a line Lie algebroid over M such that $\Gamma(L)=\mathrm{span}\{s_L\}$. Then, if we consider the Lie algebroid $\widetilde{E}$ given by direct product $\widetilde{E}=E\times L$, we remark that the map

$$J_{\widetilde{E}}:\Gamma(\widetilde{E})\to\Gamma(\widetilde{E}),\quad J_{\widetilde{E}}(s\oplus fs_L)=(F_E(s)-f\xi)\oplus\eta(s)s_L$$

for every $f\in C^\infty(M)$, $s\in\Gamma(E)$ is linear, and $J^2_{\widetilde{E}}=-I_{\widetilde{E}}$, that is $(\widetilde{E}, J_{\widetilde{E}})$ is an almost complex Lie algebroid of rank $2m+2$. Also, as usual, we can prove that the almost contact structure (F_E, ξ, η) on E is normal if $J_{\widetilde{E}}$ is integrable.

The following formula is useful for the calculation of the covariant derivative of F_E depending on the tensors $N_E^{(1)}$ and $N_E^{(2)}$, in the case of arbitrary almost contact Riemannian structures on Lie algebroids.

Proposition 3.4. *Let (F_E, ξ, η, g_E) be an almost contact Riemannian structure on the Lie algebroid $(E, \rho_E, [\cdot\,,\cdot]_E)$ of rank $2m+1$ over a smooth manifold M. If ∇ is the Levi–Civita connection of the metric g_E, then*

$$\begin{aligned} 2g_E\left((\nabla_{s_1}F_E)s_2, s_3\right) &= 3d_E\Omega_E(s_1, F_E(s_2), F_E(s_3)) - 3d_E\Omega_E(s_1, s_2, s_3) \\ &\quad + g_E(N_E^{(1)}(s_2, s_3), F_E(s_1)) + N_E^{(2)}(s_2, s_3)\eta(s_1) \\ &\quad + 2d_E\eta(F_E(s_2), s_1)\eta(s_3) - 2d_E\eta(F_E(s_3), s_1)\eta(s_2) \end{aligned}$$

for every $s_1, s_2, s_3 \in \Gamma(E)$.

Proof. Follows by direct calculus. □

4. Contact structures on Lie algebroids

In this section, we give the basic definitions and results about contact structures on Lie algebroids in relation with similar notions from contact manifolds theory, we present some examples from [33], [34] and a bijective corespondence between contact Riemannian structures and almost contact Riemannian structures on Lie algebroids, as well as give some characterizations of contact Riemannian Lie algebroids. Also, the notions of K-contact, Sasakian and Kenmotsu Lie algebroids are introduced, and some of their properties are analyzed.

4.1. Contact Lie algebroids. Firstly, we recall that a contact structure on an odd-dimensional manifold M^{2n+1} is defined by a maximally non-integrable distribution of rank $2n$, $\mathcal{D}^{2n} \subset TM$, called *contact distribution*. Equivalently, we have that the curvature form of the distribution $\mathcal{D}^{2n}$ is non-degenerate. Moreover, if there exists a 1-form $\eta \in \Omega^1(M)$ such that $\ker \eta = \mathcal{D}^{2n}$, then the contact structure is called *cooriented*. Also, we notice that the contact structure on foliated manifolds was recently introduced (see, for instance, [12], [36]) as a triple $(M^{2n+1+m}, \mathcal{F}^{2n+1}, \mathcal{D}^{2n})$, where M is a smooth manifold of dimension $2n+1+m$, $\mathcal{F}$ is a foliation of codimension m ($\dim \mathcal{F} = 2n+1$), and $\mathcal{D} \subset T\mathcal{F}$ is a distribution of dimension $2n$ (of the tangent bundle along the leaves) that is contact on each leaf of $\mathcal{F}$. This generalizes the contact fiber bundles construction from [28]. A standard example of foliated contact structure is the *space of foliated oriented* *contact elements* on the cotangent spheric bundle $S(T^*\mathcal{F})$ of the leafwise cotangent bundle of $\mathcal{F}$ (see [36]), which can be also obtained directly by pullback of the natural foliated contact structure on the projectivised cotangent bundle $P(T^*\mathcal{F})$ of $\mathcal{F}$ via the double-cover $S(T^*\mathcal{F}) \to T^*\mathcal{F} \to P(T^*\mathcal{F})$ (see [12]).

These notions concerning foliated contact structures can serve as elementary examples of our next general considerations, because it is well known that for a given regular foliated manifold $(M, \mathcal{F})$, the tangent bundle along the leaves $T\mathcal{F}$ has a natural structure of a Lie algebroid, where the anchor is the inclusion $i : T\mathcal{F} \to TM$ and the bracket is the usual Lie bracket of vectors fields tangent to the leaves. Hence, the study of contact structures in the context of general Lie algebroids is natural and it can be of some interest.

Let us continue with some basic definitions and results about contact Lie algebroids in relation with similar notions from contact manifolds/foliations theory.

Let $(E, \rho_E, [\cdot,\cdot]_E)$ be a Lie algebroid of rank $2m+1$ over a smooth manifold M. If a 1-form η on E, satisfying the condition from Theorem 2.4 is given, namely, if $\eta \wedge (d_E\eta)^m \neq 0$ everywhere on M, then we say that η defines a *contact structure* on E, or that (E, η) is a *contact Lie algebroid*, and η is called the *contact form* of E. We remark that if $f \in C^\infty(M)$ nowhere vanishes on M, then $f\eta$ also is a contact form on E. Moreover, η and $f\eta$ determine the same contact subbundle D, hence the authentic invariant of this change of contact forms is the contact subbundle. For this reason, it is more natural to define a contact structure by a subbundle D of rank $2m$ of E with the property that there exists a 1-form $\eta \in \Omega^1(E)$ so that $D = \cup_{x\in M} D_x$, where $\ker \eta_x = D_x$ and $\eta \wedge (d_E\eta)^m$ nowhere vanishes on M. Alternatively, a *contact structure on E* is given by a pair (θ_E, Ω_E), where $\theta_E \in \Omega^1(E)$ is a 1-form on E and $\Omega_E \in \Omega^2(E)$ is a 2-form on E such that $\Omega_E = d_E\theta_E$ and $(\theta_E \wedge \Omega_E \wedge \cdots {}^m \cdots \wedge \Omega_E)(x) \neq 0$, for every $x \in M$. The Reeb section $R \in \Gamma(E)$ is defined by $\imath_R\theta_E = 1$ and $\imath_R\Omega_E = 0$.

Example 4.1 ([33]). For a Lie algebroid $(E, [\cdot,\cdot]_E, \rho_E)$ of rank m over M, we can consider the prolongation of E over its dual bundle $p^* : E^* \to M$ (see [21], [27]), which is a vector bundle $(\mathcal{T}^E E^*, p_1^*, E^*)$, where $\mathcal{T}^E E^* = \cup_{u^*\in E^*} \mathcal{T}^E_{u^*} E^*$ with

$$\mathcal{T}^E_{u^*} E^* = \{(u_x, V_{u^*}) \in E_x \times T_{u^*}E^* | \rho_E(u_x) = (p^*)_*(V_{u^*}), \quad p^*(u^*) = x \in M\},$$

and the projection $p_1^* : \mathcal{T}^E E^* \to E^*$ given by $p_1^*(u_x, V_{u^*}) = u^*$. A section $\widetilde{s} \in \Gamma(\mathcal{T}^E E^*)$ is called projectable if and only if there exist $s \in \Gamma(E)$ and $V \in \mathcal{X}(E^*)$ such that $(p^*)_*(V) = \rho_E(s)$ and $\widetilde{s} = ((s(p^*(u^*)), V(u^*))$. We notice that $\mathcal{T}^E E^*$ has a Lie algebroid structure of rank $2m$ over E^* with anchor $\rho_{\mathcal{T}^E E^*} : \mathcal{T}^E E^* \to TE^*$ given by $\rho_{\mathcal{T}^E E^*}(u, V) = V$ and the Lie bracket uniquely determined by

$$[(s_1, V_1), (s_2, V_2)]_{\mathcal{T}^E E^*} = ([s_1, s_2]_E, [V_1, V_2]), \quad s_1, s_2 \in \Gamma(E), V_1, V_2 \in \mathcal{X}(E^*).$$

The Liouville section $\lambda_E \in \Gamma((\mathcal{T}^E E^*)^*)$ is given by $\lambda_E(u^*)(u,V) = u^*(u)$, $u^* \in E^*$, $(u,V) \in \mathcal{T}^E E^*$, and the canonical symplectic section $\omega_E \in \Omega^2(\mathcal{T}^E E^*)$ is given by $\omega_E = -d_{\mathcal{T}^E E^*}\lambda_E$, thus $(\mathcal{T}^E E^*, \omega_E)$ is a symplectic Lie algebroid.

Now, we suppose that we have a bundle metric g_E on E and consider the associated spherical bundle $p_{S(E^*)} : S(E^*) \to M$ of rank $m-1$, where $S(E^*) = \{u^* \in E^* | g_E^*(u^*, u^*) = 1\}$.

Similarly as above, we can consider the prolongation $\mathcal{T}^E S(E^*)$ of E over the spherical bundle $S(E^*)$, and for the following diagram

$$\begin{array}{ccc} \mathcal{T}^E S(E^*) & \xrightarrow{\mathcal{T}_E i} & \mathcal{T}^E E^* \\ {\scriptstyle \tau_{\mathcal{T}^E S(E^*)}}\downarrow & & \downarrow{\scriptstyle \tau_{\mathcal{T}^E E^*} = p_1^*} \\ S(E^*) & \xrightarrow{i} & E^* \end{array}$$

we have $d_{\mathcal{T}^E S(E^*)}((\mathcal{T}_E i)^*\varphi) = (\mathcal{T}_E i)^*(d_{\mathcal{T}^E E^*}\varphi), \varphi \in \Omega(\mathcal{T}^E E^*)$, i.e. $\mathcal{T}^E S(E^*) \to S(E^*)$ is a Lie subalgebroid of $\mathcal{T}^E E^* \to E^*$. Now, for $\eta_E = -(\mathcal{T}_E i)^*(\lambda_E) \in \Omega^1(\mathcal{T}^E S(E^*))$ we have $\eta_E \wedge (d_{\mathcal{T}^E S(E^*)}\eta_E)^{m-1} \neq 0$, that is, $(\mathcal{T}^E S(E^*), \eta_E)$ is a contact Lie algebroid.

Remark 4.1. More generally, if $(E, [\cdot\,,\cdot]_E, \rho_E)$ is an exact symplectic Lie algebroid over M of rank $2m$ with exact symplectic section $\Omega = -d_E\lambda$ and $F \to N$ is a Lie subalgebroid of rank $2m-1$ of E, then, according to [33], [34], $(F, \eta = i_F^*(\lambda))$ is a contact Lie algebroid, where $i_F : F \to E$ is the natural inclusion.

The above definition is that of the so-called cooriented contact stucture and η such that $\ker \eta = D$ is called a coorientation of the contact structure (E, D). However, as in the case of smooth manifolds (see, for instance, [13]), we can talk about general contact structures on Lie algebroids (not necessarily cooriented) and their brackets as follows.

Definition 4.1. A *contact structure* on a Lie algebroid $(E, [\cdot\,,\cdot]_E, \rho_E)$ of rank $2m+1$ is a subbundle D of rank $2m$ of E which is maximally non-integrable, that is, the curvature $\mathrm{Curv}(D) : D \times D \to L$ is non-degenerate, where L is the quotient line bundle $L := E/D$ and $\mathrm{Curv}(D)$ is given at the level of sections by $\mathrm{Curv}(D)(s_1, s_2) = [s_1, s_2]_E \bmod D$. The pair (E, D) is called a *contact Lie algebroid.*

Definition 4.2. A Reeb section of the contact Lie algebroid (E, D) is every section $\xi \in \Gamma(E)$ such that $[\xi, \Gamma(D)]_E \subset \Gamma(D)$, and we denote by $\Gamma_{\mathrm{Reeb}}(E, D)$ the set of Reeb sections.

Proposition 4.1. *The set of Reeb sections of a contact Lie algebroid* (E, D) *is a Lie subalgebra of the Lie algebra* $\Gamma(E)$ *of all sections of* E *and* $\Gamma(E) = \Gamma_{\mathrm{Reeb}}(E, D) \oplus \Gamma(D)$.

PROOF. Follows as in the contact manifolds case (see [13]). □

Also, it is useful to consider the dual point of view on contact structures on Lie algebroids, that is, to view D as the kernel of a 1-form on E with values in L ($\theta_E \in \Omega^1(E, L)$ and viewed as the canonical projection from E to L). Now, the curvature of D can be rewritten as $\mathrm{Curv}(D)(s_1, s_2) = \theta_E([s_1, s_2]_E)$, and we say that θ_E is of *contact type*. The case when L is the trivial line bundle gives rise to the above cooriented case. The previous proposition yields

Corollary 4.1. *The* 1*-form* θ_E *with values in* L *restricts to a vector space isomorphism*

$$\theta_E|_{\Gamma_{\mathrm{Reeb}}(E,D)} : \Gamma_{\mathrm{Reeb}}(E, D) \overset{\cong}{\to} \Gamma(L). \tag{4.1}$$

Thus, the Lie algebra structure of $\Gamma_{\mathrm{Reeb}}(E, D)$ (from Proposition 4.1) can be transferred to a Lie algebra structure on $\Gamma(L)$ and denote the corresponding bracket by $\{\cdot, \cdot\}_L$.

Definition 4.3. The bracket $\{\cdot, \cdot\}_L$ on $\Gamma(L)$ is called the *Reeb bracket* associated to the contact Lie algebroid (E, D) (which is a Kirillov-type bracket [26]).

Also, we notice that similarly as in the contact manifolds case (see [13, Lemma 2.5]), Proposition 4.1 can be reformulated in the form:

Proposition 4.2. *The map* $\Gamma(E) \cong \Gamma(L) \oplus \Gamma(\mathrm{Hom}(D, L))$, *given by* $s \mapsto (\theta_E(s), \theta_E([\cdot, s]_E))$, *is an isomorphism of vector spaces, and the induced* $C^\infty(M)$*-module structure on the right hand side is given by* $f \cdot (s, \phi) = (fs, \phi + d_E f \otimes s)$, *for every* $s \in \Gamma(L)$ *and* $\phi \in \Gamma(\mathrm{Hom}(D, L))$.

The surjectivity of (4.1) implies that for every section $s \in \Gamma(L)$, there exists a unique section $\xi_s \in \Gamma(E)$ such that $\theta_E(\xi_s) = s$ and $\theta_E([\xi_s, t]) = 0$ for every section $t \in \Gamma(D)$. In this case, ξ_s is called the *Reeb section associated to* s, and the Reeb bracket $\{\cdot, \cdot\}_L$ has the following characteristic property: $[\xi_{s_1}, \xi_{s_2}]_E = \xi_{\{s_1, s_2\}_L}$, for every $s_1, s_2 \in \Gamma(L)$. Moreover, applying θ_E, we get the explicit formula for the Reeb bracket in terms of the 1-form θ_E, namely,

$$\{s_1, s_2\}_L = \theta_E([\xi_{s_1}, \xi_{s_2}]_E), s_1, s_2 \in \Gamma(L). \tag{4.2}$$

Proposition 4.2 implies that, for $f \in C^\infty(M)$ and $s \in \Gamma(L)$, we have

$$\xi_{fs} = f\xi_s + \beta(d_E f \otimes s), \tag{4.3}$$

where $\beta : \mathrm{Hom}(D, L) \to D$ is the isomorphism induced by $\mathrm{Curv}(D)$, that is

$$\mathrm{Hom}(D, L) \ni \mathrm{Curv}(D)(t, \cdot) \mapsto t \in \Gamma(D).$$

Also, we notice that the inverse of the isomorphism defined in Proposition 4.2 sends (s, ϕ) to $\xi_s - \beta(\phi)$.

Example 4.2. When L is the trivial line bundle, the Reeb section associated to the constant function 1 is the standard Reeb section ξ associated to the contact form η, and it is uniquely determined by $\imath_\xi \eta = 1$ and $\imath_\xi(d_E \eta) = 0$. The other Reeb section corresponding to an arbitrary smooth function $f \in C^\infty(M)$ is $\xi_f = f\xi + \beta(d_E f)$. In this case, $\beta : D^* \to D$ is the isomorphism induced by $d_E \eta$. Finally, we notice that the Reeb bracket becomes a Jacobi bracket on $C^\infty(M)$ as follows:

$$\{f, g\}_L = \Lambda(d_E f, d_E g) + \rho_E(\xi)(f)g - f\rho_E(\xi)(g), \tag{4.4}$$

where the bisection $\Lambda \in \Gamma(\bigwedge^2 E)$ is defined by using β, that is $\Lambda(d_E f, d_E g) = d_E \eta(\beta(d_E f), \beta(d_E g))$.

Remark 4.2. In some recent papers (see [10], [18]) are introduced contact structures on principal $\mathbb{R}^\times := \mathbb{R} - \{0\}$-bundles (using a new language about contact structures). More exactly, for a given $\mathbb{R}^\times$-action $h : \mathbb{R}^\times \times P \to P$ on a vector bundle $P \to M$, a *contact structure* is referred to as a triple (P, h, ω), where ω is a 1-homogeneous symplectic form on P, that is $(h_t)^*\omega = t\omega (t \neq 0)$. Using a similar language, the construction from [33] (recalled in Example 4.1) can be formulated in the non-coorientable case as follows. Let $(E, [\cdot, \cdot]_E, \rho_E)$ be a Lie algebroid of rank m over M, $p^* : E^* \to M$ the dual vector bundle of E, and $h : \mathbb{R}^\times \times E^* \to E^*$ be the multiplicative $\mathbb{R}^\times$-action on E^* (then the projective bundle of E^* is $P(E^*) := E^*/\mathbb{R}^\times \to M$, rank $P(E^*) = m - 1$). As usual (for tangent and cotangent lifts of a $\mathbb{R}^\times$-action on manifolds or supermanifolds [18]), there is a natural lift of h to a $\mathbb{R}^\times$-action on $\mathcal{T}^E E^*$ denoted by $\mathcal{T}^E h : \mathbb{R}^\times \times \mathcal{T}^E E^* \to \mathcal{T}^E E^*$ given by $(\mathcal{T}^E h)_t = \mathcal{T}^E(h_t)$, which is a compatible action, that is, $(\mathcal{T}^E h)_t$ are Lie algebroid automorphisms (see [31]). Then, there is a natural Lie algebroid (over quotient spaces) $P(\mathcal{T}^E E^*) \to P(E^*)$ which is isomorphic with the prolongation $\mathcal{T}^E P(E^*)$ of E over the projective bundle $P(E^*) \to M$. Now, since the canonical symplectic section $\omega_E \in \Omega^2(\mathcal{T}^E E^*)$ is linear, thus homogeneous, the triple $(\mathcal{T}^E E^*, \mathcal{T}^E h, \omega_E)$ is a contact structure. In a traditional language, it corresponds to a contact structure on the Lie algebroid $\mathcal{T}^E P(E^*) \to P(E^*)$, let us say a maximally non-integrable subbundle $\mathcal{D}^E P(E^*) \subset \mathcal{T}^E P(E^*)$ of rank $2m-2$, and then, the contact structure $\mathcal{D}^E P(E^*)$ is pulled back to a contact structure on the Lie algebroid $\mathcal{T}^E S(E^*)$ through the double-cover $S(E^*) \to E^* \to P(E^*)$.

4.2. Contact Riemannian Lie algebroids. In what follows, we consider only the coorientable case. When an almost contact Riemannian structure defined in Theorem 2.4 is fixed on the contact Lie algebroid (E, η), then we say that (E, F_E, ξ, η, g_E) is a *contact Riemannian Lie algebroid*.

Remark 4.3. From the definition of the fundamental form and from Theorem 2.4, it results that for a given contact Riemannian structure, the endomorphism F_E is uniquely determined by the 1-form η and by the metric g_E.

For the contact Riemannian Lie algebroid (E, F_E, ξ, η, g_E), we consider the contact subbundle D. Taking into account Theorem 2.4, the restriction to D of the 2-form $d_E\eta$ is non-degenerate, and then we can state the following:

Proposition 4.3. *The contact subbundle D of a contact Riemannian Lie algebroid has a symplectic vector bundle structure with the symplectic 2-form $d_E\eta|_D$.*

Denote by $\mathcal{J}(D)$ the set of almost complex structures on D, compatible with $d_E\eta$, that is, the structures $\mathcal{J} : D \to D$ with the properties

$$\mathcal{J}^2 = -I_D, \quad d_E\eta(\mathcal{J}(s_1), \mathcal{J}(s_2)) = d_E\eta(s_1, s_2), \quad d_E\eta(\mathcal{J}(s), s) \geq 0 \tag{4.5}$$

for every $s, s_1, s_2 \in \Gamma(D)$. This means that we consider on D only almost complex structures compatible with its symplectic bundle structure. We remark that if (F_E, ξ, η, g_E) is the almost contact Riemannian structure associated to the contact Riemannian structure defined in Theorem 2.4 on the Lie algebroid E, then $F_E|_D \in \mathcal{J}(D)$.

For each $\mathcal{J} \in \mathcal{J}(D)$, the map $g_{\mathcal{J}}$, defined by

$$g_{\mathcal{J}}(s_1, s_2) = d_E\eta(\mathcal{J}(s_1), s_2), \quad s_1, s_2 \in \Gamma(D), \tag{4.6}$$

is a Hermitian metric on D, that is, it satisfies the condition

$$g_{\mathcal{J}}(\mathcal{J}(s_1), \mathcal{J}(s_2)) = g_{\mathcal{J}}(s_1, s_2), \quad s_1, s_2 \in \Gamma(D). \tag{4.7}$$

Moreover, if we denote by $\mathcal{G}(D)$ the set of all Riemannian metrics on D, satisfying the equality (4.7), it is easy to see that the map $\mathcal{J} \in \mathcal{J}(D) \mapsto g_{\mathcal{J}} \in \mathcal{G}(D)$ is bijective. Since η nowhere vanishes on M, we denote by ξ a section of E such that $\eta(\xi) = 1$ and extend $\mathcal{J}$ to an endomorphism F_E of $\Gamma(E)$ by setting $F_E|_D = \mathcal{J}$, $F_E(\xi) = 0$. Consider the decompositions $s_1 = s_1^D + a\xi$, $s_2 = s_2^D + b\xi$, where s_1^D, s_2^D are the D components of the sections s_1 and s_2, respectively. Similarly, we extend $g_{\mathcal{J}}$ to a metric on E by

$$g_E(s_1, s_2) = g_{\mathcal{J}}(s_1^D, s_2^D) + ab \tag{4.8}$$

for every $s_1, s_2 \in \Gamma(E)$. Taking into account (4.6), we can prove that $d_E\eta(s_1, s_2) = g_E(s_1, F_E(s_2))$, hence the contact structure on E is a Riemannian one. Moreover, (F_E, ξ, η, g_E) is an almost contact Riemannian structure on E, and then the set of almost contact Riemannian structures on E is in bijective correspondence with the set of almost complex structures of Hermitian type $(\mathcal{J}, g_{\mathcal{J}})$ defined on the contact subbundle D.

Using the notion of a Killing section on Riemannian Lie algebroids (introduced recently in [9]) and the classical calculus on Lie algebroids, similar arguments used in the study of contact Riemannian manifolds (see [5], [35]) yield

Proposition 4.4. *Let E be a contact Riemannian Lie algebroid, and let (F_E, ξ, η, g_E) be the associated almost contact Riemannian structure. Then*

(i) $N_E^{(2)} = 0$, $N_E^{(4)} = 0$;

(ii) $N_E^{(3)} = 0$ *if and only if* ξ *is a Killing section, i.e.* $\mathcal{L}_\xi g_E = 0$;

(iii) $\nabla_\xi F_E = 0$.

A more suitable form of the results from Proposition 4.4 is the following:

Proposition 4.5. *Let E be a contact Riemannian Lie algebroid, and let (F_E, ξ, η, g_E) be the associated almost contact Riemannian structure. Then,*

$$\mathcal{L}_\xi \eta = 0, \quad \mathcal{L}_\xi(d_E\eta) = 0, \quad \left(\mathcal{L}_{F_E(s_1)}\eta\right)(s_2) = \left(\mathcal{L}_{F_E(s_2)}\eta\right)(s_1)$$

for every $s_1, s_2 \in \Gamma(E)$.

Another useful result in relation with corresponding notions from contact Riemannian manifolds is

Proposition 4.6. *On a contact Riemannian Lie algebroid the following formulas hold:*

(i) $g_E(N_E^{(3)}(s_1), s_2) = g_E(s_1, N_E^{(3)}(s_2))$;

(ii) $\nabla_s \xi = -F_E(s) - F_E(N_E^{(3)}(s))$;

(iii) $F_E \circ N_E^{(3)} = -N_E^{(3)} \circ F_E$;

(iv) $\operatorname{trace} N_E^{(3)} = 0$, $\operatorname{trace}(N_E^{(3)} \circ F_E) = 0$, $N_E^{(3)}(\xi) = 0$, $\eta(N_E^{(3)}(s)) = 0$;

(v) $(\nabla_{s_1} F_E)(s_2) + (\nabla_{F_E(s_1)} F_E) F_E(s_2)$
$= 2g_E(s_1, s_2)\xi - \eta(s_2)\Big(s_1 + N_E^{(3)}(s_1) + \eta(s_1)\xi\Big)$.

Now, by putting into other words Theorem 2.4, we can assert that if η defines a contact structure on the Lie algebroid E, then there exists an almost contact Riemannian structure (F_E, ξ, η, g_E) with $\Omega_E = d_E\eta$ as fundamental form. Then, it is natural to ask what kind of relation can exist between the form $\eta\wedge(d_E\eta)^m$ and the volume form $dV_{g_E} = \sqrt{\det g_E}e^1 \wedge \cdots \wedge e^{2m+1}$ of the Riemannian metric g_E on E. More exactly, following step by step the proof from the case of contact Riemannian manifolds (see [5], [6], [35]), we have the following:

Theorem 4.1. *Let E be a contact Riemannian Lie algebroid of rank $2m+1$ with contact 1-form η. The volume form with respect to the metric g_E of E is given by*

$$dV_{g_E} = \frac{1}{2^m m!}\eta \wedge (d_E\eta)^m. \tag{4.9}$$

A morphism $\mu : (E_1, \eta_1) \to (E_2, \eta_2)$ between two contact Lie algebroids over the same manifold M is called a *contact morphism* if there is $f \in C^\infty(M)$ nowhere zero on M and such that

$$\mu^*\eta_2 = f\eta_1. \tag{4.10}$$

If $f \equiv 1$, the morphism μ is called a *strict contact morphism*. Also, we easily obtain

Proposition 4.7. *The morphism $\mu : (E_1, \eta_1) \to (E_2, \eta_2)$ between two contact Lie algebroids over the same manifold M is a contact morphism if and only if $\mu(D_1) \subseteq D_2$.*

4.3. K-contact, Sasakian and Kenmotsu Lie algebroids. A contact Riemannian Lie algebroid with the property that its Reeb section ξ is a Killing section is called a *K-contact Lie algebroid*. From Propositions 4.4 (ii) and 4.6 (ii) easily follows

Proposition 4.8. *A contact Riemannian Lie algebroid E is K-contact if and only if*

$$\nabla_s\xi = -F_E(s) \tag{4.11}$$

for every $s \in \Gamma(E)$.

From the formula (4.11), it results the following

Proposition 4.9. *On a K-contact Lie algebroid E the following equalities hold:*

$$(\nabla_{s_1}\eta)s_2 = g_E(\nabla_{s_1}\xi, s_2) = \Omega_E(s_1, s_2), \quad (\nabla_s F_E)\xi = -s + \eta(s)\xi \tag{4.12}$$

for every $s, s_1, s_2 \in \Gamma(E)$.

The contact Riemannian Lie algebroid E is called a *Sasakian Lie algebroid* if the associated almost contact Riemannian structure (F_E, ξ, η, g_E) is normal. Otherwise, the almost contact Riemannian structure (F_E, ξ, η, g_E) is a *Sasakian structure* if $d_E\eta = \Omega_E$ and $N_E^{(1)} = 0$.

From (3.1) and Proposition 4.4 (ii) easily follows

Theorem 4.2. *Every Sasakian Lie algebroid is K-contact.*

A characterization of Sasakian Lie algebroids by the Levi–Civita connection ∇ of g_E can be obtained as in the manifolds case (see [5], [35]), that is

Theorem 4.3. *The almost contact Riemannian structure (F_E, ξ, η, g_E) on E is Sasakian if and only if*

$$(\nabla_{s_1} F_E)s_2 = g_E(s_1, s_2)\xi - \eta(s_2)s_1 \tag{4.13}$$

for every sections $s_1, s_2 \in \Gamma(E)$.

Choosing an F_E-basis $\{e_a\} = \{s_a, s_{a^*}, \xi\}$ on $\Gamma(E)$, from (4.11) it follows that

$$(\nabla_{e_a}\eta)e_b = g_E(\nabla_{e_a}\xi, e_b) = -g_E(F_E(e_a), e_b) = 0. \tag{4.14}$$

Now, using the $\star$-Hodge operator on invariantly oriented Lie algebroids (see [3]), the exterior coderivative on Lie algebroids can be expressed as

$$d_E^*\varphi = -\sum_{a=1}^{2m+1} \imath_{e_a}(\nabla_{e_b}\varphi), \quad \varphi \in \Omega^\bullet(E). \tag{4.15}$$

Thus, from (4.14) and (4.15) we deduce $d_E^*\eta = 0$, hence we can state the following:

Proposition 4.10. *The contact form of a K-contact Lie algebroid is coclosed.*

Remark 4.4. Assuming that the elements of the basis $\{e_a\}$ are eigensections of the operator $N_E^{(3)}$, by a similar argument, it follows that Proposition 4.10 is valid for every contact Riemannian Lie algebroid.

Proposition 4.11. *Every K-contact Lie algebroid of rank 3 is Sasakian.*

Proof. Denote by $\{e, F_E(e), \xi\}$ a F_E-basis of $\Gamma(E)$. Then we have

$$g_E((\nabla_s F_E)e, e) = 0, \quad g_E((\nabla_s F_E)e, F_E(e)) = 0, \quad g_E((\nabla_s F_E)e, \xi) = g_E(s, e).$$

We deduce $(\nabla_s F_E)e = g_E(s, e)\xi$ for every $s \in \Gamma(E)$, and then (4.13) is satisfied for $s_2 = e$. Similarly, one can verify (4.13) for $s_2 = F_E(e)$ and $s_2 = \xi$, hence, by Theorem 4.3, the K-contact Lie algebroid of rank 3 is Sasakian. $\square$

A Lie algebroid $(E, \rho_E, [\cdot\,,\cdot]_E)$ of rank $E = 2m+1$ endowed with an almost contact Riemannian structure (F_E, ξ, η, g_E) is called an *almost Kenmotsu Lie algebroid* if the following conditions are satisfied:

$$d_E\eta = 0, \quad d_E\Omega_E = 2\eta \wedge \Omega_E. \tag{4.16}$$

With the name *Kenmotsu Lie algebroid* we refer to every normal almost Kenmotsu Lie algebroid.

Theorem 4.4. *A Lie algebroid $(E, \rho_E, [\cdot\,,\cdot]_E)$ of* rank $E = 2m+1$ *endowed with an almost contact Riemannian structure (F_E, ξ, η, g_E) is a Kenmotsu Lie algebroid if and only if*

$$(\nabla_{s_1}F_E)s_2 = -\eta(s_2)F_E(s_1) - g_E(s_1, F_E(s_2))\xi. \tag{4.17}$$

Proof. Follows as in the case of Kenmotsu manifolds (see [35]). □

Also, by straightforward calculation it follows

Proposition 4.12. *On a Kenmotsu Lie algebroid the following equalities hold:*

$$(\nabla_{s_1}\eta)(s_2) = g_E(s_1, s_2) - \eta(s_1)\eta(s_2),$$
$$\mathcal{L}_\xi g_E = 2(g_E - \eta \otimes \eta), \quad \mathcal{L}_\xi F_E = 0, \quad \mathcal{L}_\xi \eta = 0.$$

From Proposition 4.12, it follows that the Reeb section ξ of a Kenmotsu Lie algebroid cannot be Killing, hence such a Lie algebroid cannot be Sasakian, and, more generally, it cannot be K-contact.

5. An almost contact Lie algebroid structure on the vertical Liouville distribution on the big-tangent manifold

The following definition generalizes the notion of framed $f(3,1)$-structure from manifolds to Lie algebroids, and it will be important for our next considerations.

Definition 5.1. A framed $f(3,1)$-structure of corank s on a Lie algebroid $(E, \rho_E, [\cdot\,,\cdot]_E)$ of rank $(2n+s)$ is a natural generalization of an almost contact structure on E, and it is a triplet $(f, (\xi_a), (\omega^a))$, $a = 1, \ldots, s$, where $f \in \Gamma(E \otimes E^*)$ is a tensor of type $(1,1)$, (ξ_a) are sections of E, and (ω^a) are 1-forms on E such that

$$\omega^a(\xi_b) = \delta^a_b, \quad f(\xi_a) = 0, \quad \omega^a \circ f = 0, \quad f^2 = -I_E + \sum_a \omega^a \otimes \xi_a. \tag{5.1}$$

The name of $f(3,1)$-structure was suggested by the identity $f^3+f=0$. For an account of such kind of structures on manifolds, we refer, for instance, to [17], [48].

In this section, we introduce a natural framed $f(3,1)$-structure of corank 2 on the Lie algebroid defined by the vertical bundle over the big-tangent manifold of a Riemannian manifold (M,g). When we restrict it to an integrable vertical Liouville distribution over the big-tangent manifold, which has a natural structure of Lie algebroid, we obtain an almost contact structure.

5.1. Vertical framed f-structures on the big-tangent manifold. The aim of this subsection is to construct some framed $f(3,1)$-structures on the vertical bundle $V=V_1\oplus V_2$ over the big-tangent manifold $\mathcal{T}M$ when (M,g) is a Riemannian manifold.

Let M be an n-dimensional smooth manifold, and let us consider $\pi:TM\to M$ its tangent bundle, $\pi^*:T^*M\to M$ its cotangent bundle and $\tau\equiv\pi\oplus\pi^*:TM\oplus T^*M\to M$ its big-tangent bundle defined as the Whitney sum of the tangent and the cotangent bundles of M. The total space of the big-tangent bundle, called *big-tangent manifold*, is a $3n$-dimensional smooth manifold denoted here by $\mathcal{T}M$. Let us briefly recall some elementary notions about the big-tangent manifold $\mathcal{T}M$. For a detalied discussion about its geometry, we refer to [45].

Let $(U,(x^i))$ be a local chart on M. If $\{\frac{\partial}{\partial x^i}|_x\}$, $x\in U$ is a local frame of sections of the tangent bundle over U and $\{dx^i|_x\}$, $x\in U$ is a local frame of sections of the cotangent bundle over U, then, by definition of the Whitney sum, $\{\frac{\partial}{\partial x^i}|_x, dx^i|_x\}$, $x\in U$ is a local frame of sections of the big-tangent bundle $TM\oplus T^*M$ over U. Every section (y,p) of τ over U takes the form $(y,p)=y^i\frac{\partial}{\partial x^i}+p_idx^i$, and the local coordinates on $\tau^{-1}(U)$ will be defined as the triples (x^i,y^i,p_i), where $i=1,\ldots,n=\dim M$, (x^i) are local coordinates on M, (y^i) are vector coordinates, and (p_i) are covector coordinates. The local expressions of a vector field X and of a 1-form φ on $\mathcal{T}M$ are

$$X=\xi^i\frac{\partial}{\partial x^i}+\eta^i\frac{\partial}{\partial y^i}+\zeta_i\frac{\partial}{\partial p_i}\quad\text{and}\quad\varphi=\alpha_idx^i+\beta_idy^i+\gamma^idp_i. \tag{5.2}$$

For the big-tangent manifold $\mathcal{T}M$ we have the following projections:

$$\tau:\mathcal{T}M\to M,\quad \tau_1:\mathcal{T}M\to TM,\quad \tau_2:\mathcal{T}M\to T^*M$$

on M and on the total spaces of tangent and cotangent bundle, respectively. As usual, we denote by $V=V(\mathcal{T}M)$ the vertical bundle of the big-tangent manifold $\mathcal{T}M$ with respect to projection τ, and it has the decomposition

$$V=V_1\oplus V_2, \tag{5.3}$$

where $V_1 = \tau_1^{-1}(V(TM))$, $V_2 = \tau_2^{-1}(V(T^*M))$, with the local frames $\left\{\frac{\partial}{\partial y^i}\right\}$, $\left\{\frac{\partial}{\partial p_i}\right\}$, respectively. The subbundles V_1, V_2 are the vertical foliations of $\mathcal{T}M$ by fibers of τ_1, τ_2, respectively, and $\mathcal{T}M$ has a multi-foliate structure [42]. The *Liouville vector fields* are given by

$$\mathcal{E}_1 = y^i \frac{\partial}{\partial y^i} \in \Gamma(V_1), \quad \mathcal{E}_2 = p_i \frac{\partial}{\partial p_i} \in \Gamma(V_2), \quad \mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 \in \Gamma(V). \tag{5.4}$$

In the following, we consider a Riemannian metric $g = (g_{ij}(x))_{n\times n}$ on the paracompact manifold M, and we put

$$y_i = g_{ij} y^j, \quad p^i = g^{ij} p_j, \tag{5.5}$$

where $(g^{ij})_{n\times n}$ denotes the inverse matrix of $(g_{ij})_{n\times n}$. It is well known that g_{ij} determines in a natural way a Finsler metric on TM by putting $F^2(x,y) = g_{ij}(x)y^iy^j$, and similarly, g^{ij} determines a Cartan metric on T^*M by putting $K^2(x,p) = g^{ij}(x)p_ip_j$. Then the relations (5.5) imply

$$y_i y^i = F^2, \quad p_i p^i = K^2. \tag{5.6}$$

Also, the Riemannian metric g on M determines a metric structure G on V by setting

$$G(X,Y) = g_{ij}(x)X_1^i(x,y,p)Y_1^j(x,y,p) + g^{ij}(x)X_i^2(x,y,p)Y_j^2(x,y,p), \tag{5.7}$$

for every

$$X = X_1^i(x,y,p)\frac{\partial}{\partial y^i} + X_i^2(x,y,p)\frac{\partial}{\partial p_i},$$

and

$$Y = Y_1^j(x,y,p)\frac{\partial}{\partial y^j} + Y_j^2(x,y,p)\frac{\partial}{\partial p_j} \in \Gamma(V).$$

Let us define the linear operator $\phi : V \to V$ given in the local vertical frames $\left\{\frac{\partial}{\partial y^i}, \frac{\partial}{\partial p_i}\right\}$ by

$$\phi\left(\frac{\partial}{\partial y^i}\right) = -g_{ij}\frac{\partial}{\partial p_j}, \quad \phi\left(\frac{\partial}{\partial p_i}\right) = g^{ij}\frac{\partial}{\partial y^j}. \tag{5.8}$$

It is easy to see that ϕ defines an almost complex structure on V and

$$G(\phi(X), \phi(Y)) = G(X,Y), \quad \forall X, Y \in \Gamma(V). \tag{5.9}$$

As V is an integrable distribution on $\mathcal{T}M$, it follows that (V,ϕ,G) is a Hermitian Lie algebroid (foliation) over $\mathcal{T}M$ since $N_\phi=0$, where N_ϕ denotes the Nijenhuis vertical tensor field associated to ϕ.

Let us put

$$\xi_2=\frac{1}{\sqrt{F^2+K^2}}\left(y^i\frac{\partial}{\partial y^i}+p_i\frac{\partial}{\partial p_i}\right) \text{ and}$$
$$\xi_1=\phi(\xi_2)=\frac{1}{\sqrt{F^2+K^2}}\left(p^i\frac{\partial}{\partial y^i}-y_i\frac{\partial}{\partial p_i}\right), \tag{5.10}$$

where as before, $y_i=g_{ij}y^j$ and $p^i=g^{ij}p_j$.

Also, we consider the corresponding dual vertical 1-forms of ξ_1 and ξ_2, respectively, which are locally given by

$$\omega^1=\frac{1}{\sqrt{F^2+K^2}}(p_i\theta^i-y^ik_i),\quad \omega^2=\frac{1}{\sqrt{F^2+K^2}}(p^ik_i+y_i\theta^i), \tag{5.11}$$

where $\theta^i(\partial/\partial y^j)=\delta^i_j$, $\theta^i(\partial/\partial p_j)=0$, $k_i(\partial/\partial y^j)=0$ and $k_i(\partial/\partial p_j)=\delta^j_i$.

By direct calculations, we have

Lemma 5.1. *The following assertions hold:*

(i) $\phi(\xi_1)=-\xi_2$, $\phi(\xi_2)=\xi_1$;

(ii) $\omega^1\circ\phi=\omega^2$, $\omega^2\circ\phi=-\omega^1$;

(iii) $\omega^a(X)=G(X,\xi_a)$, $a=1,2$.

Now, we define a tensor field f of type $(1,1)$ on V by

$$f(X)=\phi(X)-\omega^2(X)\xi_1+\omega^1(X)\xi_2,\quad \forall X\in\Gamma(V). \tag{5.12}$$

Theorem 5.1. *The triplet* $(f,(\xi_a),(\omega^a)), a=1,2$ *provides a framed* $f(3,1)$*-structure on* V*, namely,*

(i) $\omega^a(\xi_b)=\delta^a_b$, $f(\xi_a)=0$, $\omega^a\circ f=0$;

(ii) $f^2(X)=-X+\omega^1(X)\xi_1+\omega^2(X)\xi_2$, *for any* $X\in\Gamma(V)$;

(iii) f *is of rank* $2n-2$ *and* $f^3+f=0$.

Proof. Using (5.12) and Lemma 5.1 (i) and (ii), by direct calculations we get (i) and (ii). Applying f to the equality (ii) and taking into account the equality (i), one obtains $f^3+f=0$. Now, from the second equations in (i), we see that $\operatorname{span}\{\xi_1,\xi_2\}\subset\ker f$. We prove now that $\ker f\subset\operatorname{span}\{\xi_1,\xi_2\}$. Indeed, let be $X\in\ker f$ written locally in the form $X=X^i\frac{\partial}{\partial y^i}+Y_i\frac{\partial}{\partial p_i}$. By a direct calculation, the condition $f(X)=0$ gives

$$X=\frac{p_iX^i-y^iY_i}{\sqrt{F^2+K^2}}\xi_1+\frac{y_iX^i+p^iY_i}{\sqrt{F^2+K^2}}\xi_2\in\operatorname{span}\{\xi_1,\xi_2\}$$

and $\operatorname{rank} f=2n-2$. □

Theorem 5.2. *The Riemannian metric G verifies*

$$G(f(X), f(Y)) = G(X, Y) - \omega^1(X)\omega^1(Y) - \omega^2(X)\omega^2(Y) \tag{5.13}$$

for any $X, Y \in \Gamma(V)$.

Proof. Since $G(\xi_1, \xi_2) = 0$ and $G(\xi_1, \xi_1) = G(\xi_2, \xi_2) = 1$, by using (5.12) and Lemma 5.1 (ii) and (iii), we get (5.13). □

Remark 5.1. The above theorem follows in a different way if we use the local expression of the vertical tensor field f in the local vertical frame $\left\{\frac{\partial}{\partial y^i}, \frac{\partial}{\partial p_i}\right\}$. Indeed, from (5.12) we have

$$f\left(\frac{\partial}{\partial y^i}\right) = \frac{p_i y^j - y_i p^j}{F^2 + K^2}\frac{\partial}{\partial y^j} - \left(g_{ij} - \frac{y_i y_j + p_i p_j}{F^2 + K^2}\right)\frac{\partial}{\partial p_j}, \tag{5.14}$$

$$f\left(\frac{\partial}{\partial p_i}\right) = \left(g^{ij} - \frac{p^i p^j + y^i y^j}{F^2 + K^2}\right)\frac{\partial}{\partial y^j} + \frac{p^i y_j - y^i p_j}{F^2 + K^2}\frac{\partial}{\partial p_j}, \tag{5.15}$$

and using (5.14) and (5.15) one finds

$$\begin{aligned} G\left(f\left(\frac{\partial}{\partial y^i}\right), f\left(\frac{\partial}{\partial y^j}\right)\right) &= g_{ij} - \frac{y_i y_j + p_i p_j}{F^2 + K^2}, \\ G\left(f\left(\frac{\partial}{\partial y^i}\right), f\left(\frac{\partial}{\partial p_j}\right)\right) &= \frac{p_i y^j - y_i p^j}{F^2 + K^2}, \\ G\left(f\left(\frac{\partial}{\partial p_i}\right), f\left(\frac{\partial}{\partial p_j}\right)\right) &= g^{ij} - \frac{y^i y^j + p^i p^j}{F^2 + K^2}. \end{aligned} \tag{5.16}$$

Now, from (5.16) easily follows (5.13).

Theorem 5.2 says that (f, G) is a Riemannian framed $f(3, 1)$-structure on V.

Let us put $\Phi(X, Y) = G(f(X), Y)$ for any $X, Y \in \Gamma(V)$. We have that Φ is bilinear since G is so, and using Lemma 5.1 (iii) and Theorems 5.1 and 5.2, by direct calculations we have $\Phi(Y, X) = -\Phi(X, Y)$, which says that Φ is a 2-form on V.

The Theorem shows that the annihilator of Φ is span$\{\xi_1, \xi_2\}$. Also, a direct calculation gives $[\xi_1, \xi_2] = \frac{1}{\sqrt{F^2+K^2}}\xi_1$, which says that the distribution $\{\xi_1, \xi_2\}$ is integrable even if Φ is not d_V-closed, where d_V is the (leafwise) vertical differential on $\mathcal{T}M$. We notice that the annihilator of a d_V-closed vertical 2-form is always integrable.

A direct calculus in local coordinates, using (5.14) and (5.15), leads to

$$\Phi\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \frac{p_i y_j - y_i p_j}{F^2 + K^2},$$

$$\Phi\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial p_j}\right)=-\delta_i^j+\frac{y_iy^j+p_ip^j}{F^2+K^2},$$
$$\Phi\left(\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_j}\right)=\frac{p^iy^j-y^ip^j}{F^2+K^2}. \tag{5.17}$$

On the other hand, we have

$$d_V\omega^1\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)=\frac{p_iy_j-y_ip_j}{2(F^2+K^2)\sqrt{F^2+K^2}},$$
$$d_V\omega^1\left(\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_j}\right)=\frac{p^iy^j-y^ip^j}{2(F^2+K^2)\sqrt{F^2+K^2}},$$
$$d_V\omega^1\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial p_j}\right)=\frac{1}{2\sqrt{F^2+K^2}}\left(-2\delta_i^j+\frac{y_iy^j+p_ip^j}{F^2+K^2}\right), \tag{5.18}$$

and comparing Φ with $d_V\omega^1$, it follows that

$$\Phi=2\sqrt{F^2+K^2}d_V\omega^1+\varphi, \tag{5.19}$$

where $\varphi=\delta_i^j\theta^i\wedge k_j$. We have that $\frac{\Phi}{\sqrt{F^2+K^2}}$ is d_V-closed if and only if $\frac{\varphi}{\sqrt{F^2+K^2}}$ is d_V-closed, and it defines an almost presymplectic structure on the vertical Lie algebroid V.

5.2. An almost contact structure on the vertical Liouville distribution. Let us begin by considering a vertical Liouville distribution on $\mathcal{T}M$ as the complementary orthogonal distribution in V to the line distribution spanned by the unitary Liouville vector field $\xi_2=\frac{1}{\sqrt{F^2+K^2}}\mathcal{E}$. In [23], this distribution is considered in a more general case, when the manifold M is endowed with a Finsler structure, and for this reason, certain proofs are omitted here.

Let us denote by $\{\xi_2\}$ the line vector bundle over $\mathcal{T}M$ spanned by ξ_2, and define the *vertical Liouville distribution* as the complementary orthogonal distribution V_{ξ_2} to $\{\xi_2\}$ in V with respect to G, that is, $V=V_{\xi_2}\oplus\{\xi_2\}$. Thus, V_{ξ_2} is defined by ω^2, that is

$$\Gamma\left(V_{\xi_2}\right)=\left\{X\in\Gamma(V):\omega^2(X)=0\right\}. \tag{5.20}$$

We get that every vertical vector field $X=X_1^i(x,y,p)\frac{\partial}{\partial y^i}+X_i^2(x,y,p)\frac{\partial}{\partial p_i}$ can be expressed as

$$X=PX+\omega^2(X)\xi_2, \tag{5.21}$$

where P is the projection morphism of V on V_{ξ_2}. Also, by direct calculus, we get

$$G(X,PY)=G(PX,PY)=G(X,Y)-\omega^2(X)\omega^2(Y),\quad\forall X,Y\in\Gamma(V). \tag{5.22}$$

With respect to the basis $\left\{\theta^j \otimes \frac{\partial}{\partial y^i}, \theta^j \otimes \frac{\partial}{\partial p_i}, k_j \otimes \frac{\partial}{\partial y^i}, k_j \otimes \frac{\partial}{\partial p_i}\right\}$ of $\Gamma(V \otimes V^*)$, the vertical tensor field P is locally given by

$$P = \overset{1}{P}{}^i_j\, \theta^j \otimes \frac{\partial}{\partial y^i} + \overset{2}{P}{}^j_i\, k_j \otimes \frac{\partial}{\partial p_i} + \overset{3}{P}_{ij}\, \theta^j \otimes \frac{\partial}{\partial p_i} + \overset{4}{P}{}^{ij}\, k_j \otimes \frac{\partial}{\partial y^i}, \tag{5.23}$$

where the local components are expressed by

$$\begin{aligned} &\overset{1}{P}{}^i_j = \delta^i_j - \frac{y_j y^i}{F^2 + K^2}, \quad &\overset{2}{P}{}^i_j = \delta^i_j - \frac{p^i p_j}{F^2 + K^2}, \\ &\overset{3}{P}_{ij} = -\frac{y_j p_i}{F^2 + K^2}, \quad &\overset{4}{P}{}^{ij} = -\frac{p^j y^i}{F^2 + K^2}. \end{aligned} \tag{5.24}$$

Theorem 5.3. *The vertical Liouville distribution $V_{\mathcal{E}}$ is integrable, and it defines a Lie algebroid structure on $\mathcal{T}M$, called a vertical Liouville Lie algebroid, over the big-tangent manifold $\mathcal{T}M$.*

Proof. Follows using an argument similar to that used in [4], [22]. It can be found in [23] for a more general case when the manifold M is endowed with a Finsler structure. □

Now, let us restrict to V_{ξ_2} all the geometrical structures introduced in Section 2 for V, and indicate this by overlines. Hence, we have

- $\overline{\xi_1} = \xi_1$ since ξ_1 lies in V_{ξ_2};
- $\overline{\omega^2} = 0$ since $\omega^2(X) = G(X, \xi_2) = 0$ for every vertical vector field $X \in V_{\xi_2}$;
- $\overline{G} = G|_{V_{\xi_2}}$;
- $\overline{f}(X) = \overline{\phi}(X) + \overline{\omega^1}(X) \otimes \xi_2$ is an endomorphism of V_{ξ_2} since

$$G\left(\overline{f}(X), \xi_2\right) = G(\overline{\phi}(X), \xi_2) + \overline{\omega^1}(X) G(\xi_2, \xi_2) = \omega^2(\overline{\phi}(X)) + \overline{\omega^1}(X) = 0.$$

We denote now $\overline{\xi} = \overline{\xi_1}$ and $\overline{\eta} = \overline{\omega^1}$. By Theorem 5.1, we obtain

Theorem 5.4. *The triple $(\overline{f}, \overline{\xi}, \overline{\eta})$ provides an almost contact structure on V_{ξ_2}, that is*

(i) $\overline{f}^3 + \overline{f} = 0$, $\operatorname{rank} \overline{f} = 2n - 2 = (2n-1) - 1$;

(ii) $\overline{\eta}(\overline{\xi}) = 1$, $\overline{f}(\overline{\xi}) = 0$, $\overline{\eta} \circ \overline{f} = 0$;

(iii) $\overline{f}^2(X) = -X + \overline{\eta}(X)\overline{\xi}$, *for* $X \in V_{\xi_2}$.

Also, by Theorem 5.2, we obtain

Theorem 5.5. *The Riemannian metric $\overline{G}$ verifies*

$$\overline{G}(\overline{f}(X),\overline{f}(Y)) = \overline{G}(X,Y) - \overline{\eta}(X)\overline{\eta}(Y), \tag{5.25}$$

for every vertical vector fields $X, Y \in V_{\xi_2}$.

Concluding, as V_{ξ_2} is an integrable distribution, the ensemble $(\overline{f},\overline{\xi},\overline{\eta},\overline{G})$ is an almost contact Riemannian structure on the Lie algebroid V_{ξ_2}.

Let us consider now $\overline{\Phi}(X,Y) = \overline{G}(\overline{f}(X),Y)$, for $X, Y \in \Gamma(V_{\xi_2})$, the vertical 2-form usually associated to the almost contact Riemannian structure from Theorem 5.5.

The vertical Liouville distribution V_{ξ_2} is spanned by $\left\{P(\frac{\partial}{\partial y^i}), P(\frac{\partial}{\partial p_i})\right\}$, where by using (5.23), we have

$$P\left(\frac{\partial}{\partial y^i}\right) = \overset{1}{P}{}_i^l \frac{\partial}{\partial y^l} + \overset{3}{P}{}_{li} \frac{\partial}{\partial p_l}, \quad P\left(\frac{\partial}{\partial p_j}\right) = \overset{2}{P}{}_k^j \frac{\partial}{\partial p_k} + \overset{4}{P}{}^{kj} \frac{\partial}{\partial y^k}. \tag{5.26}$$

Now, using the abbreviation $\overline{d}_V = d_V|_{V_{\xi_2}}$, by direct calculations in the basis $\left\{P\left(\frac{\partial}{\partial y^i}\right), P\left(\frac{\partial}{\partial p_i}\right)\right\}$, we get

$$\overline{d}_V\overline{\eta} = \frac{\overline{\Phi}}{\sqrt{F^2+K^2}}. \tag{5.27}$$

Remark 5.2. The relation (5.27) can be obtained directly from (5.19), since a straightforward computation shows that $\overline{\varphi} = \varphi|_{V_{\xi_2}} = 0$.

Finally, $\overline{\eta}\wedge\left(\overline{d}_V\overline{\eta}\right)^{n-1} = \overline{\eta}\wedge\left(\frac{\overline{\Phi}}{\sqrt{F^2+K^2}}\right)^{n-1} \neq 0$, which says that $\left(\overline{\eta}, \frac{\overline{\Phi}}{\sqrt{F^2+K^2}}\right)$ is a contact structure on the vertical Liouville Lie algebroid V_{ξ_2}.

Remark 5.3. If A is a Lie algebroid, then it is well known that $A \oplus A^*$ has a natural structure of a Courant algebroid, and contact structures on Courant Lie algebroids are recently considered in [18]. On the other hand, if we consider a Riemannian vector bundle (A, g_A), and $\mathcal{A}$ is the total space of the vector bundle $A \oplus A^* \to M$, then, similarly to our study, we can construct an almost contact Riemannian structure on an integrable vertical Liouville distribution over $\mathcal{A}$. However, the most techniques used in the study of the geometry of the total space of a vector bundle A (or its dual A^*) have some analogies (for the case of Lie algebroids) when investigating the geometry of the prolongations $\mathcal{T}^A A$ and $\mathcal{T}^A A^*$, respectively, and then, we can formulate the following problem: $\mathcal{T}^A A$ is a Lie algebroid over A, and $\mathcal{T}^A A^*$ is a Lie algebroid over A^*, and thus, in place of a direct sum, we can consider the direct product $\mathcal{T}^A A \times \mathcal{T}^A A^* \to A \times A^*$ (viewed as the direct sum of $pr_1^{-1}(\mathcal{T}^A A) \to A \times A^*$ and of $pr_2^{-1}(\mathcal{T}^A A^*) \to A \times A^*$, where $pr_1 : A \times A^* \to A$ and $pr_2 : A \times A^* \to A^*$). In this way, we can consider a vertical subbundle of $\mathcal{T}^A A \times \mathcal{T}^A A^*$ as $V\left(\mathcal{T}^A A \times \mathcal{T}^A A^*\right) = pr_1^{-1}(V^A A) \oplus pr_2^{-1}(V^A A^*)$, where $V^A A$ is the vertical subbundle of $\mathcal{T}^A A$, and $V^A A^*$ is the vertical subbundle of $\mathcal{T}^A A^*$. Moreover, we can consider the Liouville (Euler) section of $V\left(\mathcal{T}^A A \times \mathcal{T}^A A^*\right)$ as the direct sum of the canonical Liouville sections of $pr_1^{-1}(V^A A)$ and $pr_2^{-1}(V^A A^*)$, and a Liouville-type subbundle of $V\left(\mathcal{T}^A A \times \mathcal{T}^A A^*\right)$ defined as the orthogonal subbundle of $V\left(\mathcal{T}^A A \times \mathcal{T}^A A^*\right)$ to the line bundle generated by the Liouville section. Then, another problem to solve is the construction of an almost contact Riemannian structure on the vertical Liouville subbundle using the above procedure.

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CRISTIAN IDA
DEPARTMENT OF MATHEMATICS
AND COMPUTER SCIENCE
TRANSILVANIA UNIVERSITY OF BRAŞOV
IULIU MANIU, NO. 50
BRASOV 500091
ROMANIA

E-mail: cristian.ida@unitbv.ro

PAUL POPESCU
DEPARTMENT OF APPLIED MATHEMATICS
UNIVERSITY OF CRAIOVA
AL. CUZA, NO. 13
CRAIOVA 200585
ROMANIA

E-mail: paul_p_popescu@yahoo.com

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