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Spectrum of a nonautonomous dynamics for growth rates

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Abstract. For a nonautonomous dynamics defined by a sequence of matrices, we consider the notion of nonuniform spectrum defined in terms of nonuniform exponential dichotomies with an arbitrarily small nonuniform part. The exponential behavior may be given by an arbitrary growth rate, thus including dynamics for which the Lyapunov exponents are all zero or all infinite. We describe all possible nonuniform spectra and the asymptotic behavior of the dynamics on certain invariant subspaces. In addition, we obtain results for one-sided and two-sided dynamics. Finally, we describe all possible nonuniform spectra for a nonautonomous dynamics with continuous time.

1. Introduction

Our main aim is to describe all possible *nonuniform spectra* for a nonautonomous dynamics, both for discrete and continuous time, and both for a one-sided and a two-sided dynamics. More generally, we consider an exponential behavior given by an arbitrary growth rate $e^{\rho(m)}$ and not only by the usual exponential rate e^{cm}. The latter includes dynamics for which the Lyapunov exponents are all zero or are all infinite. Arbitrary growth rates were considered in [3], although for strong exponential dichotomies (and so none of the results in the two papers implies results in the other). In addition, we describe the asymptotic behavior of the dynamics on the associated invariant subspaces. This amounts to show that the lower and upper Lyapunov exponents of a vector in a given invariant subspace belong to the connected component of the nonuniform spectrum associated to that subspace.

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The notion of nonuniform spectrum is inspired on the original notion introduced by Sacker and Sell in [13] for cocycles over a compact base, replacing the uniform exponential dichotomies in their work by the nonuniform exponential dichotomies with an arbitrarily small nonuniform part, now for a single linear nonautonomous dynamics (and not for an entire cocycle). The Sacker–Sell spectrum can be seen as a generalization of the spectrum of a matrix (the set of its eigenvalues), or, equivalently, of the autonomous dynamics $x_m = A^m x_0$ defined by a matrix A, for an arbitrary nonautonomous dynamics

$$x_m = A_m A_{m-1} \cdots A_1 x_0$$

defined by a sequence of matrices $(A_m)_{m\in\mathbb{N}}$. Indeed, for a constant sequence, that is, for a sequence of matrices $A_m = A$ for $m \in \mathbb{N}$, a constant $\lambda \in \mathbb{R}$ is of the form $-\log|\mu|$ for some eigenvalue μ of A if and only if the dynamics defined by the matrix $e^{-\lambda}A$ admits a uniform exponential dichotomy. The construction of the associated invariant subspaces corresponding to each connected component of the spectrum follows a simple yet powerful idea apparently used first by Oseledets in [10], in his proof of the multiplicative ergodic theorem (see [4]).

There are many other works in the literature related with the study of various notions of spectra for a nonautonomous dynamics (note that for an autonomous dynamics all these spectra reduce essentially to the eigenvalues), both for discrete and continuous time, finite and infinite-dimensional systems, as well as invertible and noninvertible dynamics. In the case of the Sacker–Sell spectrum and its generalizations, for the study of uniform exponential dichotomies and its variations we refer the reader to [1], [2], [5], [6], [9], [14], [15] (in particular, [9] describes the relation to ergodic theory, [15] considers nonautonomous linear equations, [1], [2] consider systems of difference equations and [6], [14] study infinite-dimensional systems). For references related to other notions of spectra, which are out of the scope of our work, see [4] for the Lyapunov spectrum (which plays a role in smooth ergodic theory), and see [7], [8], [12] for the Morse spectrum (as well as for a discussion of the relation between the Sacker–Sell spectrum and the Morse spectrum).

Analogously to the construction for the Sacker–Sell spectrum, the *nonuniform spectrum* of a sequence of matrices $(A_m)_{m\in\mathbb{N}}$ is the set of all $\lambda \in \mathbb{R}$ such that the dynamics defined by the sequence $e^{-\lambda}A_m$ does not admit a nonuniform exponential dichotomy with an arbitrarily small nonuniform part (see Section 2 for the definition, and see Section 3 for examples). In particular, we describe completely the structure of the nonuniform spectrum and how the lower and upper Lyapunov exponents relate to the associated invariant subspaces. Namely, for a vector in a given invariant subspace, the Lyapunov exponents belong to the connected component of the nonuniform spectrum associated to that subspace.

The nonuniform exponential dichotomies with an arbitrarily small nonuniform part are ubiquitous in the context of ergodic theory and are present in much more dynamics than their uniform counterpart. For example, the dynamics of a diffeomorphism on a Smale horseshoe with a parabolic fixed point may have nonzero Lyapunov exponents but is not uniformly hyperbolic. In fact, almost all trajectories with nonzero Lyapunov exponents of a measure-preserving flow give rise to a linear variational equation admitting a nonuniform exponential dichotomy with an arbitrarily small nonuniform part. More precisely, let $f\colon \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism preserving a probability measure μ on $\mathbb{R}^d$. This means that

$$\mu(f^{-1}A) = \mu(A)$$

for every measurable set $A \subset \mathbb{R}^d$. For example, any time-1 map of a Hamiltonian flow preserves the Liouville measure on each energy level, and so there are many examples already in the somewhat classical context of mechanical systems. Consider the Lyapunov exponents

$$\lambda(x, v) = \limsup_{m\to\infty} \frac{1}{m} \log\|d_x f^m v\|$$

for $x, v \in \mathbb{R}^d$ with $v \neq 0$. If $\log^+\|df\| = \max\{0, \log\|df\|\}$ is μ-integrable (for example, if the measure μ has compact support, such as the Liouville measure on any compact energy level), then for μ-almost every x with $\lambda(x, v) \neq 0$ for all $v \neq 0$ the sequence of matrices

$$A_m = d_{f^m(x)} f, \quad m \in \mathbb{Z}$$

admits a nonuniform exponential dichotomy with an arbitrarily small nonuniform part. Our results can also be considered a contribution to the theory of nonuniform hyperbolicity, which is an important tool in the study of stochastic behavior. We refer the reader to the book [4] for a comprehensive exposition of the theory, which goes back to seminal works of Oseledets [10], and particularly, Pesin [11].

2. Dichotomies on the whole line

2.1. Preliminaries. Let $(A_m)_{m\in\mathbb{Z}}$ be a two-sided sequence of $d \times d$ matrices. We define

$$\mathcal{A}(m, n) = \begin{cases} A_{m-1} \cdots A_n & \text{if } m > n, \\ \mathrm{Id} & \text{if } m = n. \end{cases} \tag{1}$$

Moreover, let $\rho\colon \mathbb{Z} \to \mathbb{R}$ be an increasing function such that

$$\lim_{n\to-\infty} \rho(n) = -\infty \quad \text{and} \quad \lim_{n\to+\infty} \rho(n) = +\infty.$$

We say that $(A_m)_{m\in\mathbb{Z}}$ admits a *ρ-nonuniform exponential dichotomy with an arbitrarily small nonuniform part* or simply a *ρ-dichotomy* if:

(1) there exist projections $P_m\colon \mathbb{R}^d \to \mathbb{R}^d$ for $m \in \mathbb{Z}$ satisfying

$$A_m P_m = P_{m+1} A_m \tag{2}$$

for $m \in \mathbb{Z}$ such that each map

$$A_m|\operatorname{Ker} P_m\colon \operatorname{Ker} P_m \to \operatorname{Ker} P_{m+1} \tag{3}$$

is invertible;

(2) there exist a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|\mathcal{A}(m,n)P_n\| \le De^{-\lambda(\rho(m)-\rho(n))+\varepsilon|\rho(n)|} \quad \text{for } m \ge n, \tag{4}$$

and

$$\|\mathcal{A}(m,n)Q_n\| \le De^{-\lambda(\rho(n)-\rho(m))+\varepsilon|\rho(n)|} \quad \text{for } m \le n, \tag{5}$$

where $Q_n = \mathrm{Id} - P_n$, and where

$$\mathcal{A}(m,n) = (\mathcal{A}(n,m)|\operatorname{Ker} P_m)^{-1}\colon \operatorname{Ker} P_n \to \operatorname{Ker} P_m$$

for $m < n$.

We first show that the images of the projections P_n and Q_n are uniquely determined. We make the convention that $\log 0 = -\infty$.

Proposition 1. *For each $n \in \mathbb{Z}$, we have*

$$\operatorname{Im} P_n = \left\{ v \in \mathbb{R}^d : \limsup_{m\to+\infty} \frac{1}{\rho(m)} \log\|\mathcal{A}(m,n)v\| < 0 \right\}, \tag{6}$$

and $\operatorname{Im} Q_n$ consists of all vectors $v \in \mathbb{R}^d$ for which there is a sequence $(x_m)_{m\le n}$ in $\mathbb{R}^d$ such that $x_n = v$, $x_m = A_{m-1}x_{m-1}$ for $m \le n$, and

$$\limsup_{m\to-\infty} \frac{1}{|\rho(m)|} \log\|x_m\| < 0. \tag{7}$$

PROOF. It follows from (4) that

$$\limsup_{m\to+\infty} \frac{1}{\rho(m)} \log\|\mathcal{A}(m,n)v\| < 0 \tag{8}$$

for $v \in \operatorname{Im} P_n$. Conversely, if $v \in \mathbb{R}^d$ satisfies (8), then it follows from (4) that

$$\limsup_{m\to+\infty} \frac{1}{\rho(m)} \log\|\mathcal{A}(m,n)Q_n v\| < 0. \tag{9}$$

By (5), for $m \geq n$ we have

$$\|Q_n v\| \leq De^{-\lambda(\rho(m)-\rho(n))+\varepsilon|\rho(m)|}\|\mathcal{A}(m,n)Q_n v\|,$$

that is,

$$\frac{1}{D}e^{\lambda(\rho(m)-\rho(n))-\varepsilon|\rho(m)|}\|Q_n v\| \leq \|\mathcal{A}(m,n)Q_n v\|.$$

Whenever $Q_n v \neq 0$, we obtain

$$0 < \lambda - \varepsilon \leq \limsup_{m\to+\infty} \frac{1}{\rho(m)} \log\|\mathcal{A}(m,n)Q_n v\|$$

for any sufficiently small $\varepsilon > 0$, which contradicts to (9). Therefore, $Q_n v = 0$ and $v \in \operatorname{Im} P_n$. This establishes identity (6).

Now, take $v \in \operatorname{Im} Q_n$ and define a sequence $(x_m)_{m\leq n}$ in $\mathbb{R}^d$ by $x_m = \mathcal{A}(m,n)v$ for $m \leq n$. Clearly,

$$x_n = v \quad \text{and} \quad x_m = A_{m-1}x_{m-1} \text{ for } m \leq n.$$

Moreover, it follows from (5) that (7) holds. For the converse, it is sufficient to show that there exists no $v \in \operatorname{Im} P_n \setminus \{0\}$ for which there is a sequence $(x_m)_{m\leq n}$ in $\mathbb{R}^d$ as in the proposition. It follows from (2) and (4) that

$$\|v\| = \|\mathcal{A}(n,m)P_m x_m\| \leq De^{-\lambda(\rho(n)-\rho(m))+\varepsilon|\rho(m)|}\|x_m\|$$

for $m \leq n$. Therefore,

$$0 < \lambda - \varepsilon \leq \limsup_{m\to-\infty} \frac{1}{|\rho(m)|} \log\|x_m\|$$

for any sufficiently small $\varepsilon > 0$, which contradicts to (7). $\square$

The *nonuniform spectrum* of a sequence $(A_m)_{m\in\mathbb{Z}}$ of $d \times d$ matrices is the set Σ of all numbers $a \in \mathbb{R}$ such that the sequence $(B_m)_{m\in\mathbb{Z}}$, where

$$B_m = e^{-a(\rho(m+1)-\rho(m))} A_m,$$

does not admit a ρ-dichotomy. For each $a \in \mathbb{R}$ and $n \in \mathbb{Z}$, let

$$S_a(n) = \left\{ v \in \mathbb{R}^d : \limsup_{m\to+\infty} \frac{1}{\rho(m)} \log\|\mathcal{A}(m,n)v\| < a \right\}, \tag{10}$$

and let $U_a(n)$ be the set of all vectors $v \in \mathbb{R}^d$ for which there is a sequence $(x_m)_{m\le n}$ in $\mathbb{R}^d$ such that $x_n = v$, $x_m = A_{m-1}x_{m-1}$ for $m \le n$ and

$$\limsup_{m\to-\infty} \frac{1}{|\rho(m)|} \log\|x_m\| < -a.$$

It follows from Proposition 1 that if $a \in \mathbb{R} \setminus \Sigma$, then

$$\mathbb{R}^d = S_a(n) \oplus U_a(n) \quad \text{for } n \in \mathbb{Z}, \tag{11}$$

with the projections P_n and Q_n associated to the sequence $(B_m)_{m\in\mathbb{Z}}$ satisfying $\operatorname{Im} P_n = S_a(n)$ and $\operatorname{Ker} P_n = U_a(n)$ for $n \in \mathbb{Z}$. With the convention that

$$S_{-\infty}(n) = U_{+\infty}(n) = \{0\} \quad \text{and} \quad S_{+\infty}(n) = U_{-\infty}(n) = \mathbb{R}^d,$$

for each $a \in [-\infty, +\infty]$ and $n \in \mathbb{Z}$, we have

$$A_n S_a(n) \subset S_a(n+1) \quad \text{and} \quad A_n U_a(n) \subset U_a(n+1). \tag{12}$$

Moreover, if $a < b$, then

$$S_a(n) \subset S_b(n) \quad \text{and} \quad U_b(n) \subset U_a(n) \tag{13}$$

for $n \in \mathbb{Z}$. Finally, for each $a \notin \Sigma$, the numbers $\dim S_a(n)$ and $\dim U_a(n)$ are independent of n (and thus, we shall simply denote them by $\dim S_a$ and $\dim U_a$). Indeed, it follows from the invertibility assumption in the notion of a ρ-dichotomy that $\dim U_a(n) = \dim U_a(n+1)$, and so also $\dim S_a(n) = \dim S_a(n+1)$, for $n \in \mathbb{Z}$.

Proposition 2. *The set $\Sigma \subset \mathbb{R}$ is closed. For each $a \in \mathbb{R}\setminus\Sigma$, we have $S_a(n) = S_b(n)$ and $U_a(n) = U_b(n)$ for all $n \in \mathbb{Z}$ and all b in some open neighborhood of a.*

PROOF. For each $a \in \mathbb{R}\setminus\Sigma$, there exist projections P_n for $n \in \mathbb{Z}$ satisfying (2), a constant $\lambda > 0$, and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)P_n\| \le De^{-\lambda(\rho(m)-\rho(n))+\varepsilon|\rho(n)|} \quad \text{for } m \ge n,$$

and

$$\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)Q_n\| \le De^{-\lambda(\rho(n)-\rho(m))+\varepsilon|\rho(n)|} \quad \text{for } m \le n.$$

Therefore, given $b \in \mathbb{R}$,

$$\|e^{-b(\rho(m)-\rho(n))}\mathcal{A}(m,n)P_n\| \le De^{-(\lambda-a+b)(\rho(m)-\rho(n))+\varepsilon|\rho(n)|} \quad \text{for } m \ge n,$$

and

$$\|e^{-b(\rho(m)-\rho(n))}\mathcal{A}(m,n)Q_n\| \le De^{-(\lambda+a-b)(\rho(n)-\rho(m))+\varepsilon|\rho(n)|} \quad \text{for } m \le n.$$

This shows that if $|a-b| < \lambda$, then $b \in \mathbb{R} \setminus \Sigma$. Hence, by Proposition 1, $S_b(n) = S_a(n)$ and $U_b(n) = U_a(n)$ for $n \in \mathbb{Z}$. □

2.2. Structure of the spectrum. The following is our main result. It gives a complete description of the nonuniform spectrum of a sequence of matrices. We write $I_i = [a_i, b_i]$ for $i = 2, \dots, k-1$.

Theorem 3. *For a two-sided sequence $(A_m)_{m\in\mathbb{Z}}$ of $d \times d$ matrices:*

(1) *either $\Sigma = \emptyset$, $\Sigma = \mathbb{R}$ or*

$$\Sigma = I_1 \cup [a_2, b_2] \cup \cdots \cup [a_{k-1}, b_{k-1}] \cup I_k, \tag{14}$$

where $I_1 = [a_1, b_1]$ or $I_1 = (-\infty, b_1]$ and $I_k = [a_k, b_k]$ or $I_k = [a_k, +\infty)$, for some constants

$$a_1 \le b_1 < a_2 \le b_2 < \cdots < a_k \le b_k, \quad k \le d;$$

(2) *if Σ is given by (14), then, taking numbers*

$$c_i \in (b_i, a_{i+1}) \quad \text{for } i = 1, \dots, k-1$$

and

$$\delta > 0, \qquad c_0 = \inf \Sigma - \delta, \qquad c_k = \sup \Sigma + \delta,$$

for each $n \in \mathbb{Z}$ the subspaces $W_i(n) = U_{c_{i-1}}(n) \cap S_{c_i}(n)$ satisfy

$$A_n W_i(n) \subset W_i(n+1) \quad \text{for } i = 1, \dots, k \tag{15}$$

and

$$\mathbb{R}^d = \bigoplus_{i=0}^{k+1} W_i(n), \tag{16}$$

where $W_0(n) = S_{c_0}(n)$ and $W_{k+1}(n) = U_{c_k}(n)$;

(3) *the subspaces* $W_i(n)$ *are independent of the numbers* $\delta, c_1, \ldots, c_{k-1}$*;*

(4) *for each* $i \in \{1, \ldots, k\}$ *with* I_i *compact,* $n \in \mathbb{Z}$ *and* $v \in W_i(n) \setminus \{0\}$*, we have*

$$\left[\liminf_{m\to+\infty} \frac{1}{\rho(m)} \log\|x_m\|, \limsup_{m\to+\infty} \frac{1}{\rho(m)} \log\|x_m\|\right] \subset I_i, \tag{17}$$

where $x_m = \mathcal{A}(m,n)v$*, and there exists a sequence* $(x_m)_{m\le n} \subset \mathbb{R}^d$ *such that* $x_n = v$*,* $x_m = A_{m-1}x_{m-1}$ *for* $m \le n$ *and*

$$\left[\liminf_{m\to-\infty} \frac{1}{\rho(m)} \log\|x_m\|, \limsup_{m\to-\infty} \frac{1}{\rho(m)} \log\|x_m\|\right] \subset I_i. \tag{18}$$

Proof. We start with an auxiliary result.

Lemma 1. *Take* $a_1, a_2 \in \mathbb{R} \setminus \Sigma$ *such that* $a_1 < a_2$*. Then* $[a_1, a_2] \cap \Sigma \ne \emptyset$ *if and only if* $\dim S_{a_1} < \dim S_{a_2}$*.*

Proof of the lemma. Assume that $[a_1, a_2] \cap \Sigma \ne \emptyset$. If $\dim S_{a_1} = \dim S_{a_2}$, then

$$S_{a_1}(n) = S_{a_2}(n) \quad \text{and} \quad U_{a_1}(n) = U_{a_2}(n)$$

for $n \in \mathbb{Z}$. By Proposition 1, there exist projections P_n for $n \in \mathbb{Z}$ satisfying (2), constants $\lambda_1, \lambda_2 > 0$, and for each $\varepsilon > 0$ constants $D_1 = D_1(\varepsilon), D_2 = D_2(\varepsilon) > 0$ such that for $i = 1, 2$ we have

$$\|e^{-a_i(\rho(m)-\rho(n))}\mathcal{A}(m,n)P_n\| \le D_i e^{-\lambda_i(\rho(m)-\rho(n))+\varepsilon|\rho(n)|} \quad \text{for } m \ge n, \tag{19}$$

and

$$\|e^{-a_i(\rho(m)-\rho(n))}\mathcal{A}(m,n)Q_n\| \le D_i e^{-\lambda_i(\rho(n)-\rho(m))+\varepsilon|\rho(n)|} \quad \text{for } m \le n. \tag{20}$$

For each $a \in [a_1, a_2]$, by (19),

$$\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)P_n\| \le D_1 e^{-\lambda_1(\rho(m)-\rho(n))+\varepsilon|\rho(n)|} \quad \text{for } m \ge n,$$

and similarly, by (20),

$$\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)Q_n\| \le D_2 e^{-\lambda_2(\rho(n)-\rho(m))+\varepsilon|\rho(n)|} \quad \text{for } m \le n.$$

Thus, $[a_1, a_2] \subset \mathbb{R} \setminus \Sigma$, but this contradicts to the assumption that $[a_1, a_2] \cap \Sigma \ne \emptyset$.

For the converse, let

$$b = \inf\{a \in \mathbb{R} \setminus \Sigma : \dim S_a = \dim S_{a_2}\}.$$

Since $\dim S_{a_1} < \dim S_{a_2}$, it follows from Proposition 2 that $a_1 < b < a_2$. If $b \notin \Sigma$, then either $\dim S_b = \dim S_{a_2}$ or $\dim S_b \neq \dim S_{a_2}$. In the first case, by Proposition 2 we have $\dim S_{b'} = \dim S_{a_2}$ and $b' \in \mathbb{R} \setminus \Sigma$, for all $b' \in (b-\varepsilon, b]$ and some $\varepsilon > 0$. In the second case, by Proposition 2 we have $\dim S_{b'} \neq \dim S_{a_2}$ and $b' \in \mathbb{R} \setminus \Sigma$, for all $b' \in [b, b+\varepsilon)$ and some $\varepsilon > 0$. Both properties contradict to the definition of b. Hence, $b \in \Sigma$, and so $[a_1, a_2] \cap \Sigma \neq \emptyset$. $\square$

Now, assume that Σ contains $d+1$ disjoint closed intervals, and take numbers $c_1, \ldots, c_d \in \mathbb{R} \setminus \Sigma$ such that all the intervals

$$(-\infty, c_1), (c_1, c_2), \ldots, (c_d, +\infty)$$

intersect Σ. By Lemma 1, we have

$$0 \leq \dim S_{c_1} < \dim S_{c_2} < \cdots < \dim S_{c_d} \leq d. \tag{21}$$

If $\dim S_{c_1} = 0$, then $S_{c_1}(n) = \{0\}$ for $n \in \mathbb{Z}$. Since $c_1 \in \mathbb{R} \setminus \Sigma$, there exist a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|e^{-c_1(\rho(m)-\rho(n))}\mathcal{A}(m,n)\| \leq De^{-\lambda(\rho(n)-\rho(m))+\varepsilon|\rho(n)|} \quad \text{for } m \leq n.$$

Hence, for $b < c_1$ we have

$$\|e^{-b(\rho(m)-\rho(n))}\mathcal{A}(m,n)\| \leq De^{-\lambda(\rho(n)-\rho(m))+\varepsilon|\rho(n)|} \quad \text{for } m \leq n.$$

Thus, $(-\infty, c_1) \subset \mathbb{R} \setminus \Sigma$, which is impossible since $(-\infty, c_1)$ intersects Σ. This shows that $\dim S_{c_1} > 0$. Now we assume that $\dim S_{c_d} = d$. Then $S_{c_d}(n) = \mathbb{R}^d$ for $n \in \mathbb{Z}$. Since $c_d \in \mathbb{R} \setminus \Sigma$, there exist a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|e^{-c_d(\rho(m)-\rho(n))}\mathcal{A}(m,n)\| \leq De^{-\lambda(\rho(m)-\rho(n))+\varepsilon|\rho(n)|} \quad \text{for } m \geq n.$$

Hence, for $b > c_d$ we have

$$\|e^{-b(\rho(m)-\rho(n))}\mathcal{A}(m,n)\| \leq De^{-\lambda(\rho(m)-\rho(n))+\varepsilon|\rho(n)|} \quad \text{for } m \geq n.$$

Thus, $(c_d, +\infty) \subset \mathbb{R} \setminus \Sigma$, which is impossible since $(c_d, +\infty)$ intersects Σ. This shows that $\dim S_{c_d} < d$. Therefore, (21) cannot hold, and so Σ is composed of at most d disjoint closed intervals. This establishes the first property in the theorem.

Property (15) follows readily from (12). Moreover, by (13), we have

$$W_i(n) \cap W_j(n) = \{0\} \quad \text{for } i \neq j \text{ and } n \in \mathbb{Z}.$$

Indeed, for $i < j$ we have

$$W_i(n) \subset S_{c_i}(n) \subset S_{c_{j-1}}(n) \quad \text{and} \quad W_j(n) \subset U_{c_{j-1}}(n).$$

Finally, since

$$(A + B) \cap C = A + (B \cap C)$$

for any subspaces A, B and C with $A \subset C$, taking $A = S_{c_{k-1}}(n)$, $B = U_{c_{k-1}}(n)$ and $C = S_{c_k}(n)$, it follows from (11) that

$$\begin{aligned}\mathbb{R}^d &= S_{c_k}(n) \oplus W_{k+1}(n) = \big((S_{c_{k-1}}(n) \oplus U_{c_{k-1}}(n)) \cap S_{c_k}(n)\big) \oplus W_{k+1}(n) \\ &= S_{c_{k-1}}(n) \oplus (S_{c_k}(n) \cap U_{c_{k-1}}(n)) \oplus W_{k+1}(n) = S_{c_{k-1}}(n) \oplus W_k(n) \oplus W_{k+1}(n)\end{aligned}$$

for each $n \in \mathbb{Z}$. Identity (16) can now be obtained in finitely many steps. The independence of the spaces $W_i(n)$ on the choice of constants $\delta, c_1, \ldots, c_{k-1}$ follows readily from Lemma 1.

For the last statement in the theorem, we note that since $c_i \notin \Sigma$, the sequence $e^{-c_i(\rho(m+1)-\rho(m))}A_m$ admits a ρ-dichotomy, and so there exist projections P_n for $n \in \mathbb{Z}$ satisfying (2), a constant $\lambda > 0$, and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|\mathcal{A}(m, n)P_n\| \le De^{(c_i-\lambda)(\rho(m)-\rho(n))+\varepsilon|\rho(n)|} \quad \text{for } m \ge n, \tag{22}$$

and

$$\|\mathcal{A}(m, n)Q_n\| \le De^{-(\lambda+c_i)(\rho(n)-\rho(m))+\varepsilon|\rho(n)|} \quad \text{for } m \le n.$$

It follows from Proposition 1 that $\operatorname{Im} P_n = S_{c_i}(n)$ for $n \in \mathbb{Z}$. Hence, $W_i(n) \subset \operatorname{Im} P_n$, and it follows from (22) that

$$\limsup_{m\to+\infty} \frac{1}{\rho(m)} \log\|\mathcal{A}(m, n)v\| \le c_i - \lambda < c_i. \tag{23}$$

Letting $c_i \searrow b_i$, we obtain

$$\limsup_{m\to+\infty} \frac{1}{\rho(m)} \log\|\mathcal{A}(m, n)v\| \le b_i \tag{24}$$

for $i \in \{1, \ldots, k - 1\}$ and $i = k$ unless $c_k = +\infty$. Similarly, since $c_{i-1} \notin \Sigma$, the sequence $e^{-c_{i-1}(\rho(m+1)-\rho(m))}A_m$ admits a ρ-dichotomy, and so there exist projections P'_n for $n \in \mathbb{Z}$ satisfying (2), a constant $\mu > 0$, and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|\mathcal{A}(m, n)P'_n\| \le De^{(c_{i-1}-\mu)(\rho(m)-\rho(n))+\varepsilon|\rho(n)|} \quad \text{for } m \ge n,$$

and

$$\|\mathcal{A}(m,n)Q'_n\| \le De^{-(\mu+c_{i-1})(\rho(n)-\rho(m))+\varepsilon|\rho(n)|} \quad \text{for } m \le n, \tag{25}$$

where $Q'_n = \mathrm{Id} - P'_n$. It follows from Proposition 1 that $\operatorname{Im} Q'_n = U_{c_{i-1}}(n)$ for $n \in \mathbb{Z}$. Hence, $W_i(n) \subset \operatorname{Im} Q'_n$, and it follows from (22) that

$$\|v\| \le De^{-(\mu+c_{i-1})(\rho(m)-\rho(n))+\varepsilon|\rho(m)|}\|\mathcal{A}(m,n)v\| \quad \text{for } m \ge n,$$

and thus,

$$\liminf_{m\to+\infty} \frac{1}{\rho(m)} \log\|\mathcal{A}(m,n)v\| \ge \mu + c_{i-1} - \varepsilon > c_{i-1},$$

taking ε sufficiently small. Letting $c_{i-1} \nearrow a_i$, we obtain

$$\liminf_{m\to+\infty} \frac{1}{\rho(m)} \log\|\mathcal{A}(m,n)v\| \ge a_i \tag{26}$$

for $i \in \{2,\dots,k\}$ and $i = 1$ unless $c_0 = -\infty$. Property (17) follows from (24) and (26).

Now, take $v \in W_i(n) \setminus \{0\}$, and let $(x_m)_{m\le n}$ be a sequence such that $x_m \in \operatorname{Im} Q'_m$ and $v = \mathcal{A}(n,m)x_m$ for $m \le n$. By (25), we have

$$\limsup_{m\to-\infty} \frac{1}{|\rho(m)|} \log\|x_m\| \le -\mu - c_{i-1} < -c_{i-1},$$

and thus,

$$\liminf_{m\to-\infty} \frac{1}{\rho(m)} \log\|x_m\| > c_{i-1}.$$

Letting $c_{i-1} \nearrow a_i$, we obtain

$$\liminf_{m\to-\infty} \frac{1}{\rho(m)} \log\|x_m\| \ge a_i, \tag{27}$$

for $i \in \{2,\dots,k\}$ and $i = 1$ unless $c_0 = -\infty$. Moreover, since $v \in S_{c_i}(n) = \operatorname{Im} P_n$, we have $v = \mathcal{A}(n,m)P_m x_m$ for $m \le n$. By (22), we obtain

$$\liminf_{m\to-\infty} \frac{1}{|\rho(m)|} \log\|x_m\| \ge \lambda - c_i - \varepsilon > -c_i,$$

taking ε sufficiently small. Hence,

$$\limsup_{m\to-\infty} \frac{1}{\rho(m)} \log\|x_m\| < c_i,$$

and letting $c_i \searrow b_i$, we find that

$$\limsup_{m\to-\infty} \frac{1}{\rho(m)} \log\|x_m\| \le b_i \tag{28}$$

for $i \in \{1,\dots,k-1\}$ and $i = k$ unless $c_k = +\infty$. Property (18) follows from (27) and (28). $\square$

3. Examples

In this section, we give some examples of sequences of matrices and of their nonuniform spectra.

Example 1. Take $w > b > 0$. For each $n \in \mathbb{Z}$, let

$$A_n = \begin{cases} e^{(-w+b)(2n+1)+(n+1)\cos(n+1)-n\cos n} & \text{if } n \geq 0, \\ e^{-(w+b)(2n+1)+(|n|+1)\cos(n+1)-|n|\cos n} & \text{if } n < 0. \end{cases}$$

We have

$$\mathcal{A}(m,n) = \begin{cases} e^{-(w-b)(m^2-n^2)+m\cos m-n\cos n} & \text{if } m,n \geq 0, \\ e^{-w(m^2-n^2)+b(m^2+n^2)+m\cos m-|n|\cos n} & \text{if } m \geq 0,\ n < 0, \\ e^{-(w+b)(m^2-n^2)+|m|\cos m-|n|\cos n} & \text{if } m,n < 0, \end{cases}$$

for $m \geq n$. First, we show that $\Sigma \subset [e^{-w-b}, e^{-w+b}]$. Take $a > -w+b$. Then

$$e^{-a(m^2-n^2)}\mathcal{A}(m,n) \leq e^{-(a+w-b)(m^2-n^2)+|m|+|n|} \quad \text{for } m \geq n. \tag{29}$$

Given $\delta > 0$, take $D = D(\delta) > 0$ such that

$$e^{|n|} \leq De^{\delta n^2} \quad \text{for } n \in \mathbb{Z}. \tag{30}$$

It follows from (29) that

$$\begin{aligned} e^{-a(m^2-n^2)}\mathcal{A}(m,n) &\leq D^2 e^{-(a+w-b)(m^2-n^2)+\delta m^2+\delta n^2} \\ &\leq D^2 e^{-(a+w-b-\delta)(m^2-n^2)+2\delta n^2} \end{aligned}$$

for $m \geq n$. Since $a+w-b > 0$ and δ is arbitrary, this shows that $(e^{-a}A_m)_{m\in\mathbb{Z}}$ admits a ρ-dichotomy with $\rho(n) = n^2$ and $P_m = \mathrm{Id}$.

Similarly, for $a < -w-b$ the sequence $(e^{-a}A_m)_{m\in\mathbb{Z}}$ admits a ρ-dichotomy with $\rho(n) = n^2$ and $P_m = 0$. Indeed, by (30), for $m \leq n$ we obtain

$$\begin{aligned} e^{-a(n^2-m^2)}\mathcal{A}(n,m) &\geq e^{-(a+w+b)(n^2-m^2)-|m|-|n|} \\ &\geq D^{-2} e^{-(a+w+\delta)(n^2-m^2)-2\delta n^2}. \end{aligned}$$

Since $a+w+b < 0$ and δ is arbitrary, we obtain the desired property. Therefore, $\Sigma \subset [e^{-w-b}, e^{-w+b}]$.

For the reverse inclusion, assume that the sequence $(e^{w-b}A_m)_{m\in\mathbb{Z}}$ admits a ρ-dichotomy with $\rho(n)=n^2$ and $P_m=\mathrm{Id}$. Then there exists $\lambda>0$, and for each $\varepsilon>0$ there exists $D=D(\varepsilon)>0$ such that

$$e^{(w-b)(m^2-n^2)}\mathcal{A}(m,n)\le De^{-\lambda(m^2-n^2)+\varepsilon n^2}\quad \text{for } m\ge n.$$

For $n=(2l-1)\pi$ and $m=2l\pi$ with $l\in\mathbb{N}$, we obtain

$$e^{(w-b)(m^2-n^2)}\mathcal{A}(m,n)=e^{m^2+n^2}\le De^{-\lambda(m^2-n^2)+\varepsilon n^2}.$$

But this is impossible for ε sufficiently small. One can also show that the sequence $(e^{w-b}A_m)_{m\in\mathbb{Z}}$ does not admit a ρ-dichotomy with $\rho(n)=n^2$ and $P_m=0$. Therefore, $-w+b\in\Sigma$. One can show in a similar manner that $-w-b\in\Sigma$.

Since $\Sigma\ne\emptyset$ and $\Sigma\ne\mathbb{R}$, it follows from Theorem 3 that Σ is a closed interval. Therefore, $\Sigma=[e^{-w-b},e^{-w+b}]$.

Example 2. Take $w>b>0$. For each $n\in\mathbb{Z}$, let

$$A_n=\begin{cases} e^{(-w+b)(2n+1)+(n+1)\cos(n+1)-n\cos n} & \text{if } n\ge 0,\\ 0 & \text{if } n<0.\end{cases}$$

We have

$$\mathcal{A}(m,n)=\begin{cases} e^{-(w-b)(m^2-n^2)+m\cos m-n\cos n} & \text{if } m,n\ge 0,\\ 0 & \text{otherwise},\end{cases}$$

for $m\ge n$. First, we show that $\Sigma\subset(-\infty,e^{-w+b}]$. Take $a>-w+b$. Then

$$e^{-a(m^2-n^2)}\mathcal{A}(m,n)\le e^{-(a+w-b)(m^2-n^2)+|m|+|n|}\tag{31}$$

for $m\ge n\ge 0$. By (30), it follows from (31) that

$$\begin{aligned} e^{-a(m^2-n^2)}\mathcal{A}(m,n)&\le D^2e^{-(a+w-b)(m^2-n^2)+\delta m^2+\delta n^2}\\ &\le D^2e^{-(a+w-b-\delta)(m^2-n^2)+2\delta n^2}\end{aligned}$$

for $m\ge n$. Since $a+w-b>0$ and δ is arbitrary, this shows that $(e^{-a}A_m)_{m\in\mathbb{Z}}$ admits a ρ-dichotomy with $\rho(n)=n^2$ and $P_m=\mathrm{Id}$. Hence, $\Sigma\subset(-\infty,-w+b]$.

For the reverse inclusion, take $a\le -w+b$, and assume that $(e^aA_m)_{m\in\mathbb{Z}}$ admits a ρ-dichotomy with $\rho(n)=n^2$ and $P_m=\mathrm{Id}$. Then there exists $\lambda>0$, and for each $\varepsilon>0$ there exists $D=D(\varepsilon)>0$ such that

$$e^{a(m^2-n^2)}\mathcal{A}(m,n)\le De^{-\lambda(m^2-n^2)+\varepsilon n^2}\quad \text{for } m\ge n.$$

For $n = (2l-1)\pi$ and $m = 2l\pi$ with $l \in \mathbb{N}$, we obtain

$$e^{-a(m^2-n^2)}\mathcal{A}(m,n) = e^{-(a+w-b)(m^2-n^2)}e^{|m|+|n|} \le De^{-\lambda(m^2-n^2)+\varepsilon n^2}.$$

But this is impossible for ε sufficiently small since $a + w - b \le 0$. On the other hand, $(e^{-a}A_m)_{m\in\mathbb{Z}}$ does not admit a ρ-dichotomy with $\rho(n) = n^2$ and $P_m = 0$ since A_m vanishes for $m < 0$. Therefore, $a \in \Sigma$ for any $a \le -w + b$, and so $\Sigma = (-\infty, -w + b]$.

4. Dichotomies on the half-line

In this section, we obtain corresponding results for a nonautonomous dynamics on the half-line. Let $(A_m)_{m\in\mathbb{N}}$ be a one-sided sequence of $d \times d$ matrices. We continue to define $\mathcal{A}(m,n)$ by (1). Let $\rho\colon \mathbb{N} \to \mathbb{R}$ be an increasing function such that

$$\lim_{n\to+\infty} \rho(n) = +\infty.$$

We say that the sequence $(A_m)_{m\in\mathbb{N}}$ admits a ρ-dichotomy if there exist projections P_m for $m \in \mathbb{N}$, satisfying (2) for $m \in \mathbb{N}$ such that each map in (3) is invertible, a constant $\lambda > 0$, and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that (4) and (5) hold.

The following result can be obtained repeating arguments in the proof of Proposition 1.

Proposition 4. *For each $n \in \mathbb{N}$, we have*

$$\operatorname{Im} P_n = \left\{ v \in \mathbb{R}^d : \limsup_{m\to+\infty} \frac{1}{\rho(m)} \log\|\mathcal{A}(m,n)v\| < 0 \right\}.$$

In the one-sided case, the images of the projections $Q_n = \mathrm{Id} - P_n$ need not be uniquely determined.

Proposition 5. *Assume that the sequence $(A_m)_{m\in\mathbb{N}}$ admits a ρ-dichotomy with respect to projections P_m. Moreover, let P'_m, for $m \in \mathbb{N}$, be projections such that:*

(1) $\operatorname{Im} P_m = \operatorname{Im} P'_m$ *and* $P'_{m+1}A_m = A_m P'_m$ *for* $m \in \mathbb{N}$*;*
(2) *the map* $A_m | \ker P'_m \colon \ker P'_m \to \ker P'_{m+1}$ *is invertible for* $m \in \mathbb{N}$*;*
(3) *for each* $\varepsilon > 0$ *there exists* $C = C(\varepsilon)$ *such that*

$$\|P'_m\| \le Ce^{\varepsilon|\rho(m)|}, \quad m \in \mathbb{N}. \tag{32}$$

Then $(A_m)_{m\in\mathbb{N}}$ admits a ρ-dichotomy with respect to the projections P'_m.

Proof. It follows from the assumptions that

$$P_m(P_m - P'_m) = P_m - P'_m \quad \text{for } m \in \mathbb{N}.$$

Hence, by (4) and (32), we have

$$\begin{aligned}
\|\mathcal{A}(m,n)P'_n\| &\le \|\mathcal{A}(m,n)P_n x\| + \|\mathcal{A}(m,n)(P_n - P'_n)\| \\
&\le De^{-\lambda(\rho(m)-\rho(n))+\varepsilon|\rho(n)|} + \|\mathcal{A}(m,n)P_n(P_n - P'_n)\| \\
&= De^{-\lambda(\rho(m)-\rho(n))+\varepsilon|\rho(n)|} + De^{-\lambda(\rho(m)-\rho(n))+\varepsilon|\rho(n)|}\|P_n - P'_n\| \\
&\le De^{-\lambda(\rho(m)-\rho(n))+\varepsilon|\rho(n)|} + De^{-\lambda(\rho(m)-\rho(n))+\varepsilon|\rho(n)|}(D+C)e^{\varepsilon|\rho(n)|} \\
&\le (D + D^2 + CD)e^{-\lambda(\rho(m)-\rho(n))+2\varepsilon|\rho(n)|}
\end{aligned}$$

for $m \ge n$. Similarly, since

$$P_m - P'_m = (P_m - P'_m)(\mathrm{Id} - P_m),$$

it follows from (32) that

$$\begin{aligned}
\|(\mathrm{Id} - P'_n)v\| &\le \|(\mathrm{Id} - P_n)v\| + \|(P_n - P'_n)v\| \\
&\le \|(\mathrm{Id} - P_n)v\| + \|P_n - P'_n\| \cdot \|(\mathrm{Id} - P_n)v\| \\
&\le (1 + \|P_n - P'_n\|) \cdot \|(\mathrm{Id} - P_n)v\| \\
&\le (1 + \|P_n - P'_n\|)e^{-\lambda(\rho(m)-\rho(n))+\varepsilon|\rho(m)|}\|\mathcal{A}(m,n)(\mathrm{Id} - P_n)v\|
\end{aligned}$$

for $m \ge n$. Using again (32), we conclude that

$$\|(\mathrm{Id} - P'_n)v\| \le K'e^{-\lambda(\rho(m)-\rho(n))+2\varepsilon|\rho(m)|}\|\mathcal{A}(m,n)(\mathrm{Id} - P_n)v\| \tag{33}$$

for $m \ge n$ and some constant $K' = K'(\varepsilon) > 0$. On the other hand,

$$\begin{aligned}
\|\mathcal{A}(m,n)(\mathrm{Id} - P_n)v\| &\le \|\mathcal{A}(m,n)(\mathrm{Id} - P'_n)v\| + \|\mathcal{A}(m,n)(P_n - P'_n)v\| \\
&= \|\mathcal{A}(m,n)(\mathrm{Id} - P'_n)v\| + \|(P_m - P'_m)\mathcal{A}(m,n)(\mathrm{Id} - P'_n)v\| \\
&\le (1 + \|P_m - P'_m\|)\|\mathcal{A}(m,n)(\mathrm{Id} - P'_n)v\|.
\end{aligned}$$

Using (32), we conclude that

$$\|\mathcal{A}(m,n)(\mathrm{Id} - P_n)v\| \le K''e^{\varepsilon|\rho(m)|}\|\mathcal{A}(m,n)(\mathrm{Id} - P'_n)v\| \tag{34}$$

for $m \geq n$ and some constant $K'' = K''(\varepsilon) > 0$. Inequalities (33) and (34) imply that

$$\|(\mathrm{Id} - P'_n)v\| \leq K'K''e^{-\lambda(\rho(m)-\rho(n))+3\varepsilon|\rho(m)|}\|\mathcal{A}(m,n)(\mathrm{Id} - P'_n)v\|,$$

and thus,

$$\begin{aligned}\|\mathcal{A}(n,m)(\mathrm{Id} - P'_m)v\| &\leq K'K''e^{-\lambda(\rho(m)-\rho(n))+3\varepsilon|\rho(m)|}\|(\mathrm{Id} - P'_m)v\| \\ &\leq (1+C)K'K''e^{-\lambda(\rho(m)-\rho(n))+3\varepsilon|\rho(m)|}\|v\|\end{aligned}$$

for $m \geq n$. This completes the proof of the lemma. □

The *nonuniform spectrum* of a sequence $(A_m)_{m\in\mathbb{N}}$ of $d \times d$ matrices is the set Σ of all $a \in \mathbb{R}$ such that the sequence $(B_m)_{m\in\mathbb{N}}$, where

$$B_m = e^{-a(\rho(m+1)-\rho(m))}A_m,$$

does not admit a ρ-dichotomy. For each $a \in \mathbb{R}$ and $n \in \mathbb{N}$, we continue to define $S_a(n)$ as in (10). For $a < b$, the first inclusion in (13) holds. Moreover, repeating the proof of Proposition 2, one can show that the set $\Sigma \subset \mathbb{R}$ is closed, and that for each $a \in \mathbb{R} \setminus \Sigma$ we have $S_a(n) = S_b(n)$ for all $n \in \mathbb{N}$ and all b in some open neighborhood of a.

The following result is a version of Theorem 3 for exponential dichotomies on the half-line. The main difference is that the direct sum in (16) is replaced by a filtration of subspaces. We write $I_i = [a_i, b_i]$ for $i = 2, \dots, k-1$.

Theorem 6. *For a one-sided sequence $(A_m)_{m\in\mathbb{N}}$ of $d \times d$ matrices:*

(1) *statement 1 in Theorem 3 holds;*

(2) *when* (14) *holds, taking numbers as in Theorem 3, we have*

$$S_{c_0}(n) \subset S_{c_1}(n) \subset \cdots \subset S_{c_k}(n);$$

(3) *for each $i \in \{1, \dots, k\}$ with I_i compact, $n \in \mathbb{N}$ and $v \in S_{c_i}(n) \setminus S_{c_{i-1}}(n)$, we have*

$$\limsup_{m\to+\infty} \frac{1}{\rho(m)} \log\|\mathcal{A}(m,1)v\| \leq b_i.$$

Proof. We first show that Lemma 1 holds in the present setting. Indeed, assume that $[a_1, a_2] \cap \Sigma \neq \emptyset$ and $\dim S_{a_1} = \dim S_{a_2}$. It follows from Propositions 4 and 5 that the sequences

$$B_m = e^{-a_1(\rho(m+1)-\rho(m))}A_m \quad \text{and} \quad C_m = e^{-a_2(\rho(m+1)-\rho(m))}A_m$$

admit ρ-dichotomies with respect to the same sequence of projections P_m. Proceeding as in the proof of Lemma 1, we obtain a contradiction. The converse can also be obtained as in the proof of Lemma 1.

Proceeding as in the proof of Theorem 3, one can show that Σ consists of at most d disjoint closed intervals. Since $c_i \notin \Sigma$, the sequence $e^{-c_i(\rho(m+1)-\rho(m))}A_m$ admits a ρ-dichotomy, and so there exist projections P_n for $n \in \mathbb{N}$ satisfying (2), a constant $\lambda > 0$, and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|\mathcal{A}(m,n)P_n\| \le De^{(c_i-\lambda)(\rho(m)-\rho(n))+\varepsilon\rho(n)} \quad \text{for } m \ge n, \tag{35}$$

and

$$\|\mathcal{A}(m,n)Q_n\| \le De^{-(\lambda+c_i)(\rho(n)-\rho(m))+\varepsilon\rho(n)} \quad \text{for } m \le n.$$

It follows from Proposition 4 that $\operatorname{Im} P_n = S_{c_i}(n)$ for $n \in \mathbb{N}$. Hence, $S_{c_i}(1) \subset \operatorname{Im} P_1$, and it follows from (35) that (23) and (24) hold. $\square$

5. The case of continuous time

In this section, we describe briefly versions of our results for continuous time. We first consider exponential dichotomies on the whole line. Let $T(t,s)$ be an evolution family for $t, s \in \mathbb{R}$ with $t \ge s$, and let $\rho\colon \mathbb{R} \to \mathbb{R}$ be an increasing function such that

$$\lim_{t\to-\infty} \rho(t) = -\infty \quad \text{and} \quad \lim_{t\to+\infty} \rho(t) = +\infty.$$

We say that $T(t,s)$ admits a *ρ-nonuniform exponential dichotomy with an arbitrarily small nonuniform part* or simply a *ρ-dichotomy* if:

(1) there exist projections $P(t)$ for $t \in \mathbb{R}$ satisfying

$$P(t)T(t,s) = T(t,s)P(s) \tag{36}$$

for $t \ge s$ such that each map

$$T(t,s)|\operatorname{Ker} P(s)\colon \operatorname{Ker} P(s) \to \operatorname{Ker} P(t) \tag{37}$$

is invertible;

(2) there exist a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|T(t,s)P(s)\| \le De^{-\lambda(\rho(t)-\rho(s))+\varepsilon|\rho(s)|} \quad \text{for } t \ge s, \tag{38}$$

and

$$\|T(t,s)Q(s)\| \le De^{\mu(\rho(t)-\rho(s))+\varepsilon|\rho(s)|} \quad \text{for } t \le s, \tag{39}$$

where $Q(t) = \mathrm{Id} - P(t)$, and where

$$T(t,s) = \big(T(s,t)|\operatorname{Ker} P(t)\big)^{-1} \colon \operatorname{Ker} P(s) \to \operatorname{Ker} P(t) \quad \text{for } t \le s$$

The *nonuniform spectrum* of $T(t,s)$ is the set Σ of all numbers $a \in \mathbb{R}$ for which the evolution family

$$T_a(t,s) = e^{-a(\rho(t)-\rho(s))}T(t,s)$$

does not admit a ρ-dichotomy.

The following result is a version of Theorem 3 for continuous time. The proof is analogous, and so we omit it. We write $I_i = [a_i, b_i]$ for $i = 2, \dots, k-1$.

Theorem 7. *For an evolution family $T(t,s)$ for $t, s \in \mathbb{R}$ with $t \ge s$, the following properties hold:*

(1) *statement 1 in Theorem 3 holds;*

(2) *when* (14) *holds, taking numbers as in Theorem 3, for each $t \in \mathbb{R}$ the subspaces $W_i(t) = U_{c_{i-1}}(t) \cap S_{c_i}(t)$ satisfy*

$$T(t,s)W_i(s) \subset W_i(t) \quad \text{for } i = 1, \dots, k,\ t \ge s,$$

and $\mathbb{R}^d = \bigoplus_{i=0}^{k+1} W_i(t)$, where $W_0(t) = S_{c_0}(t)$ and $W_{k+1}(t) = U_{c_k}(t)$;

(3) *the subspaces $W_i(t)$ are independent of the numbers $\delta, c_1, \dots, c_{k-1}$;*

(4) *for each $i \in \{1, \dots, k\}$ with I_i compact, $s \in \mathbb{R}$ and $v \in W_i(s) \setminus \{0\}$, we have*

$$\left[\liminf_{t\to+\infty} \frac{1}{\rho(t)} \log\|x(t)\|, \limsup_{t\to+\infty} \frac{1}{\rho(t)} \log\|x(t)\|\right] \subset I_i,$$

where $x(t) = T(t,s)v$, and there exists a function $x(t)$ such that $x(s) = v$, $x(t) = T(t,r)x(r)$ for $r \le t \le s$ and

$$\left[\liminf_{t\to-\infty} \frac{1}{\rho(t)} \log\|x(t)\|, \limsup_{t\to-\infty} \frac{1}{\rho(t)} \log\|x(t)\|\right] \subset I_i.$$

Now, we consider exponential dichotomies on the half-line. Let $T(t,s)$ be an evolution family for $t \geq s \geq 0$, and let $\rho\colon \mathbb{R}_0^+ \to \mathbb{R}$ be an increasing function such that

$$\lim_{t\to+\infty} \rho(t) = +\infty.$$

We say that $T(t,s)$ admits a *ρ-dichotomy* if there exist projections $P(t)$ for $t \in \mathbb{R}$ satisfying (36) such that each map in (37) is invertible, a constant $\lambda > 0$, and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that (38) and (39) hold. The *nonuniform spectrum* of $T(t,s)$ is the set Σ of all numbers $a \in \mathbb{R}$ for which the evolution family $T_a(t,s)$ does not admit a ρ-dichotomy.

The following result is a version of Theorem 6 for continuous time.

Theorem 8. *For an evolution family $T(t,s)$ for $t \geq s \geq 0$ the following properties hold:*

(1) *statement 1 of Theorem 3 holds;*

(2) *when* (14) *holds, taking numbers as in Theorem 3, we have*

$$S_{c_0}(t) \subset S_{c_1}(t) \subset \cdots \subset S_{c_k}(t);$$

(3) *for each $i \in \{1,\dots,k\}$ with I_i compact, $s \in \mathbb{R}$ and $v \in S_{c_i}(s) \setminus S_{c_{i-1}}(s)$, we have*

$$\limsup_{t\to+\infty} \frac{1}{\rho(t)} \log\|T(t,s)v\| \leq b_i.$$

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