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Ordinal sums of binary conjunctive operations based on the product

By SUSANNE SAMINGER-PLATZ (Linz), MICHAL DIBALA (Bratislava), ERICH PETER KLEMENT (Linz) and RADKO MESIAR (Bratislava)

Abstract. We discuss several types of ordinal sums for conjunctive operations for an infinite set of truth values (modeled by the real unit interval). In some cases, they can be seen as both a construction method and a representation (for example, when considering copulas), this is no more true for the product-based ordinal sums when considering quasi-copulas or semicopulas. For each of the three product-based ordinal sums discussed here, we characterize the smallest set of conjunctive operations containing all quasi-copulas and for which the considered ordinal sum is both a construction method and a representation. In particular, the set of all Lipschitz conjunctive operations is the smallest superclass of the set of quasi-copulas for which all three product-based ordinal sums under consideration are a construction method and a representation.

1. Introduction

The classical *Boolean conjunction* $\wedge_{\{0,1\}}\colon \{0,1\}^2 \to \{0,1\}$ acting on the two *truth values* 1 (= *true*) and 0 (= *false*) is given by $\wedge_{\{0,1\}}(1,1) = 1$, and $\wedge_{\{0,1\}}(x,y) = 0$ otherwise. When we consider a more general set of truth values such as the graded scale $[0,1]$ as in many-valued and fuzzy logics [16], [35], we have to find appropriate axioms for an operation $B\colon [0,1]^2 \to [0,1]$ modeling the

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conjunction on $[0,1]$. Obviously, in any case, B should be an *extension* of $\wedge_{\{0,1\}}$, i.e., $B\restriction_{\{0,1\}^2} = \wedge_{\{0,1\}}$.

We shall use the symbol $\wedge$ for the infix form of the function $M\colon [0,1]^2 \to [0,1]$ given by $M(x,y) = \min(x,y) = x \wedge y$ (which, obviously, is an extension of $\wedge_{\{0,1\}}$) and, similarly, we also shall write $x \vee y = \max(x,y)$.

Definition 1.1. Let us fix several sets of specific extensions of the Boolean conjunction $\wedge_{\{0,1\}}$ to the unit square $[0,1]^2$ for the rest of the paper:

(i) Denote by $\mathfrak{B}$ the set of all functions $B\colon [0,1]^2 \to \mathbb{R}$ satisfying the following *boundary conditions* (implying that each $B \in \mathfrak{B}$ is an extension of $\wedge_{\{0,1\}}$): 1 is a *neutral element* of B, i.e., $B(x,y) = x \wedge y$ whenever $x \vee y = 1$, and 0 is an *annihilator* of B, i.e., $B(x,y) = 0$ whenever $x \wedge y = 0$.

(ii) Each operation $B \in \mathfrak{B}$ with $\mathrm{Ran}(B) \subseteq [0,1]$ is called a *conjunctive operation* (on $[0,1]$), and the set of all conjunctive operations will be denoted by $\mathfrak{B}_c$.

(iii) Adding additional properties, particular subsets of $\mathfrak{B}_c$ are obtained. Some of them are:

[S] Monotone conjunctive operations are called *semicopulas* [3], [12], and the set of semicopulas will be denoted by $\mathfrak{S}$.

[C] Supermodular conjunctive operations are called *copulas* [13], [27], [34], and the set of copulas will be denoted by $\mathfrak{C}$. Recall that a function $f\colon [0,1]^2 \to [0,1]$ is *supermodular* if, for all $(x_1,x_2),(y_1,y_2) \in [0,1]^2$,

$$f(x_1 \wedge y_1, x_2 \wedge y_2) + f(x_1 \vee y_1, x_2 \vee y_2) \geq f(x_1,x_2) + f(y_1,y_2).$$

[L] *Lipschitz conjunctive operations*; the set of Lipschitz conjunctive operations will be denoted by $\mathfrak{L}$. A function $f\colon [0,1]^2 \to [0,1]$ is *Lipschitz* (with respect to the L_1-norm) if there is a constant $K \in \,]0,\infty[$ such that for all $(x_1,x_2),(y_1,y_2) \in [0,1]^2$

$$|f(x_1,x_2) - f(y_1,y_2)| \leq K \cdot (|x_1 - y_1| + |x_2 - y_2|).$$

[L_1] 1-*Lipschitz conjunctive operations* [19], [21], [22], [23], i.e., Lipschitz conjunctive operations with $K = 1$; the set of 1-Lipschitz conjunctive operations will be denoted by $\mathfrak{L}_1$.

[Q] 1-Lipschitz semicopulas are called *quasi-copulas* [2], [15], and the set of quasi-copulas will be denoted by $\mathfrak{Q}$ (observe that $\mathfrak{Q} = \mathfrak{S} \cap \mathfrak{L}_1$).

[T] Symmetric associative semicopulas are called *triangular norms* (*t-norms* for short) [1], [20], [29], [30], [31], [32], and the set of triangular norms will be denoted by $\mathfrak{T}$. Note that we have $\mathfrak{T} \cap \mathfrak{Q} = \mathfrak{T} \cap \mathfrak{C}$.

Obviously, we have the following strict inequalities: $\mathfrak{C} \subset \mathfrak{Q} \subset \mathfrak{L}_1 \subset \mathfrak{L} \subset \mathfrak{B}_c$, $\mathfrak{C} \subset \mathfrak{Q} \subset \mathfrak{S} \subset \mathfrak{B}_c$, as well as $\mathfrak{T} \subset \mathfrak{S} \subset \mathfrak{B}_c$. Neither $\mathfrak{L}$ nor $\mathfrak{L}_1$ is comparable with $\mathfrak{S}$ with respect to set inclusion.

For each function $F\colon [0,1]^2 \to \mathbb{R}$ it is possible to define the *F-volume* of a rectangle $[a,b] \times [c,d] \subseteq [0,1]^2$ by

$$V_F([a,b] \times [c,d]) = F(b,d) - F(b,c) - F(a,d) + F(a,c). \tag{1.1}$$

Then we get immediately the following relationships of the sets $\mathfrak{L}_1$, $\mathfrak{S}$, $\mathfrak{Q}$ and $\mathfrak{C}$ with the nonnegativity of the volume of (some) rectangles:

(i) If a function $L \in \mathfrak{B}$ is 1-Lipschitz, then $V_L([a,b]\times[c,d]) \geq 0$ for each rectangle $[a,b] \times [c,d] \subseteq [0,1]^2$ with $1 \in \{b,d\}$.

(ii) A function $S\colon [0,1]^2 \to [0,1]$ is a semicopula if and only if $S \in \mathfrak{B}_c$, and if $V_S([a,b] \times [c,d]) \geq 0$ for each rectangle $[a,b] \times [c,d] \subseteq [0,1]^2$ satisfying $0 \in \{a,c\}$.

(iii) A function $Q\colon [0,1]^2 \to [0,1]$ is a quasi-copula if and only if $Q \in \mathfrak{B}_c$, and if $V_Q([a,b] \times [c,d]) \geq 0$ for each rectangle $[a,b] \times [c,d] \subseteq [0,1]^2$ satisfying $\{0,1\} \cap \{a,b,c,d\} \neq \emptyset$.

(iv) A function $C\colon [0,1]^2 \to [0,1]$ is a copula if and only if $C \in \mathfrak{B}_c$, and if $V_C([a,b] \times [c,d]) \geq 0$ for each rectangle $[a,b] \times [c,d] \subseteq [0,1]^2$.

Observe that for a function $L \in \mathfrak{B}$ to be 1-Lipschitz, the nonnegativity of the volume of the rectangles mentioned in (i) is only necessary but not sufficient.

An important tool for the construction of conjunctive operations are the so-called *ordinal sums*. Based on earlier results in the context of partially ordered sets [4] and of abstract semigroups [5], [6], [7], [8], the ordinal sum of t-norms [14], [24], [31] was introduced as follows (compare also [1], [20], [32]):

Definition 1.2. Let $(]a_i,b_i[)_{i\in I}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$, and let $(T_i)_{i\in I}$ be a family of t-norms. Then, the function $T\colon [0,1]^2 \to [0,1]$ given by

$$T(x,y) = \begin{cases} a_i + (b_i - a_i) \cdot T_i\left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}\right) & \text{if } (x,y) \in [a_i,b_i]^2, \\ M(x,y) & \text{otherwise,} \end{cases}$$

is well-defined, and we have $T \in \mathfrak{T}$. The t-norm T is called the *M-ordinal sum* of the summands $(]a_i,b_i[\,,T_i)_{i\in I}$, and we often shall write $T = M\text{-}(\langle a_i,b_i,T_i\rangle)_{i\in I}$.

An M-ordinal sum $T = M\text{-}(\langle a_i,b_i,T_i\rangle)_{i\in I}$ will be called *non-trivial* if the family $(]a_i,b_i[)_{i\in I}$ does not consist of $]0,1[$ only, i.e., if $\{]a_i,b_i[\mid i \in I\} \neq \{]0,1[\}$.

The concept of M-ordinal sums as a construction method can be carried over in a straightforward way to the set $\mathfrak{B}$, and to the other sets of conjunctive operations $\mathfrak{B}_c$, $\mathfrak{S}$, $\mathfrak{L}$, $\mathfrak{L}_1$, $\mathfrak{C}$ and $\mathfrak{Q}$ considered in Definition 1.1.

Observe that the M-ordinal sum given in Definition 1.2 is not only a construction method, but also a representation of t-norms in the following sense:

Proposition 1.3. *$T \in \mathfrak{T}$ if and only if there is an index set I such that $T = M\text{-}(\langle a_i, b_i, T_i\rangle)_{i\in I}$ for some family $(]a_i, b_i[)_{i\in I}$ of non-empty, pairwise disjoint open subintervals of $[0,1]$ and for some subfamily $(T_i)_{i\in I}$ of $\mathfrak{T}$.*

Proof. Fix an arbitrary $T \in \mathfrak{T}$. If there is an $a \in]0,1[$ such that we have $T(x,y) = M(x,y)$ whenever $a \in \{x,y\}$, then $T = M\text{-}(\langle 0, a, T_1\rangle, \langle a, 1, T_2\rangle)$, where $T_1, T_2 \in \mathfrak{T}$ are given by

$$T_1(x,y) = \frac{T(a\cdot x, a\cdot y)}{a} \text{ and } T_2(x,y) = \frac{T(a+(1-a)\cdot x, a+(1-a)\cdot y) - a}{1-a}.$$

If there is no $a \in]0,1[$ such that $T(x,y) = M(x,y)$ whenever $a \in \{x,y\}$, then the only M-ordinal sum representation of T is the trivial one, i.e., $T = M\text{-}(\langle 0,1,T\rangle)$.

This shows that each $T \in \mathfrak{T}$ is an M-ordinal sum of t-norms, the converse being an immediate consequence of Definition 1.2. □

Note that, for an M-ordinal sum of t-norms $T = M\text{-}(\langle a_i, b_i, T_i\rangle)_{i\in I}$, repeating the procedure in the proof of Proposition 1.3 at most two times allows us to reconstruct each summand $T_i \in \mathfrak{T}$ from T via

$$T_i(x,y) = \frac{T(a_i + (b_i - a_i)\cdot x, a_i + (b_i - a_i)\cdot y) - a_i}{b_i - a_i}.$$

Remark 1.4. It is not difficult to show that Proposition 1.3 remains valid if we replace the set $\mathfrak{T}$ by one of the sets $\mathfrak{B}$, $\mathfrak{S}$, $\mathfrak{L}$, $\mathfrak{L}_1$, $\mathfrak{C}$ or $\mathfrak{Q}$.

On the set $\mathfrak{B}_c$, however, the M-ordinal sum is not a representation, in general (it is a representation only for conjunctive operations $B \in \mathfrak{B}_c$ satisfying, for some $a \in]0,1[$, $B(x,y) = M(x,y)$ whenever $a \in \{x,y\}$, if we have $B(x,y) \le x \wedge y$ whenever $x \wedge y \le a$, and if $B(x,y) \ge x \wedge y \wedge a$ whenever $x \vee y \ge a$).

There are also generalizations of Definition 1.2, dealing with more general binary operations on $[0,1]$ (the so-called *t-subnorms* [17], [18]) and still leading to a t-norm. Since these t-subnorms are not conjunctive operations, we shall not consider these constructions here.

Another type of ordinal sums for copulas, based on the smallest copula W defined by $W(x,y) = \max(x+y-1, 0)$, was introduced in [26]:

Definition 1.5. Let $(]a_i, b_i[)_{i\in I}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$, and let $(C_i)_{i\in I}$ be a family of copulas. Then, the function $C\colon [0,1]^2 \to [0,1]$ given by

$$C(x,y) = \begin{cases} (b_i - a_i)\cdot C_i\left(\frac{x-a_i}{b_i-a_i}, \frac{y+b_i-1}{b_i-a_i}\right) & \text{if } (x,y) \in [a_i, b_i] \times [1-b_i, 1-a_i]\,, \\ W(x,y) & \text{otherwise,} \end{cases}$$

is well-defined, and we have $C \in \mathfrak{C}$. The copula C is called the *W-ordinal sum* of the summands $(]a_i, b_i[\,, C_i)_{i\in I}$, and we often shall write $C = W\text{-}(\langle a_i, b_i, C_i\rangle)_{i\in I}$.

The concept of W-ordinal sums as a construction method can be carried over in a straightforward way to the set $\mathfrak{B}$, and to the sets of conjunctive operations $\mathfrak{B}_c$, $\mathfrak{S}$, $\mathfrak{L}$, $\mathfrak{L}_1$ and $\mathfrak{Q}$ considered in Definition 1.1 (but not to $\mathfrak{T}$: the W-ordinal sum preserves neither the associativity nor the symmetry of the summands).

Observe that the W-ordinal sum given in Definition 1.5 is not only a construction method, but also a representation of copulas in the following sense:

Proposition 1.6. *$C \in \mathfrak{C}$ if and only if there is an index set I such that $C = W\text{-}(\langle a_i, b_i, C_i\rangle)_{i\in I}$ for some family $(]a_i, b_i[)_{i\in I}$ of non-empty, pairwise disjoint open subintervals of $[0,1]$ and for some subfamily $(C_i)_{i\in I}$ of $\mathfrak{C}$.*

Proof. Consider, for $C \in \mathfrak{C}$, the flipping $C^- \in \mathfrak{C}$ (see, e.g., [27]) defined by $C^-(x,y) = x - C(x, 1-y)$, and observe that the flipping of a W-ordinal sum of copulas C_i equals the M-ordinal sum of the flipped copulas C_i^-. Then the assertion follows immediately from Remark 1.4. □

Remark 1.7. It is not difficult to show that Proposition 1.6 remains valid if we replace the set $\mathfrak{C}$ by one of the sets $\mathfrak{B}$, $\mathfrak{S}$, $\mathfrak{L}$, $\mathfrak{L}_1$ or $\mathfrak{Q}$. On the set $\mathfrak{B}_c$, however, the W-ordinal sum is not a representation, in general.

The copulas M and W are the two extremal copulas (sometimes called the *Fréchet–Hoeffding bounds*) since we have $W \le C \le M$ for each $C \in \mathfrak{C}$. A third distinguished copula is the *product copula* $\Pi \in \mathfrak{C}$ given by $\Pi(x,y) = x \cdot y$.

These three copulas have nice interpretations in the context of statistical dependence: Π models the independence of random variables, M their comonotone dependence, and W their countermonotone dependence (for more information, see, e.g., [27]).

Recently, Π-*vertical* and Π-*horizontal ordinal sums* of copulas were introduced in [25] (the details will be given in Section 2), based on some patchwork techniques [9], [10], [11] and methods for gluing copulas [33].

A third type of ordinal sum of copulas related to the product Π, here called Π-*diagonal ordinal sums*, was introduced in [22] (the details will be given in Section 4), which, in a particular case (when there is only one summand) can be obtained via consecutive application of Π-vertical and Π-horizontal ordinal sums.

The construction of all three types of Π-ordinal sums can be carried over in a straightforward way to the other sets mentioned in Definition 1.1, with the exception of $\mathfrak{T}$. However, when thinking about them as a representation, this works only for the sets $\mathfrak{B}$ and $\mathfrak{L}$, but neither for the set of all conjunctive operations nor for quasi-copulas or semicopulas.

This fact was the main motivation for this paper, which was written with the intention to provide a better understanding of conjunctive operations on $[0, 1]$ at large, and in particular, of their relationship with the different types of ordinal sums related to the product Π.

The paper is organized as follows: in Section 2, we recall the Π-vertical and Π-horizontal ordinal sums of copulas and present some first properties and results. When extending these concepts to other sets given in Definition 1.1, we see that they are no representation for (1-Lipschitz) conjunctive operations nor for semicopulas or quasi-copulas.

In Section 3, we show that there is a smallest set $\mathfrak{Q}_v$ satisfying $\mathfrak{Q} \subseteq \mathfrak{Q}_v \subseteq \mathfrak{B}_c$ (i.e., a set of conjunctive operations containing all quasi-copulas) such that the Π-vertical ordinal sum is both a construction method and a representation on $\mathfrak{Q}_v$. The corresponding result for the Π-horizontal ordinal sum is also given.

Finally, we study the Π-diagonal ordinal sum of copulas, which is again no representation for (1-Lipschitz) conjunctive operations, nor for semicopulas or quasi-copulas. However, the set $\mathfrak{L}$ of all Lipschitz conjunction operators turns out to be the smallest set of conjunction operators which contains all quasi-copulas and for which the Π-diagonal ordinal sum is both a construction method and a representation (Section 4).

2. Π-vertical and Π-horizontal ordinal sums

When looking at ordinal sums of copulas related to the product Π, we first mention the concept introduced in [25], compare also [9], [11], [33]. Such an ordinal sum of copulas is different from Π only on vertical or horizontal stripes in the unit interval. That is why we speak about a Π-vertical or a Π-horizontal ordinal sum, respectively. In a natural way, the relationship of these two concepts is based on the switching of the order of the coordinates of the points in the domain. Consider, therefore, for an arbitrary function $F\colon \mathbb{R}^2 \to \mathbb{R}$, the function $F^{\circlearrowleft}\colon \mathbb{R}^2 \to \mathbb{R}$ given by $F^{\circlearrowleft}(x,y) = F(y,x)$.

Definition 2.1. Let $(]a_i, b_i[)_{i\in I}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$, and let $(C_i)_{i\in I}$ be a family of copulas.

(i) The function $C\colon [0,1]^2 \to [0,1]$ given by

$$C(x,y) = \begin{cases} a_i \cdot y + (b_i - a_i) \cdot C_i\left(\frac{x-a_i}{b_i-a_i}, y\right) & \text{if } x \in [a_i, b_i]\,, \\ \Pi(x,y) & \text{otherwise,} \end{cases}$$

is well-defined, and we have $C \in \mathfrak{C}$. The copula C is called the Π-*vertical ordinal sum* of the summands $(]a_i, b_i[\,, C_i)_{i\in I}$, and as an abbreviation, we often shall use $C = \Pi_v\text{-}(\langle a_i, b_i, C_i\rangle)_{i\in I}$.

(ii) The function $C\colon [0,1]^2 \to [0,1]$ given by

$$C = \left(\Pi_v\text{-}(\langle a_i, b_i, (C_i)^{\circlearrowleft}\rangle)_{i\in I}\right)^{\circlearrowleft}$$

is also a copula, and it is called the Π-*horizontal ordinal sum* of the summands $(]a_i, b_i[\,, C_i)_{i\in I}$, briefly $C = \Pi_h\text{-}(\langle a_i, b_i, C_i\rangle)_{i\in I}$.

The concepts of Π-vertical and Π-horizontal ordinal sums as a construction method can be carried over in a straightforward way to the set $\mathfrak{B}$, and to the sets of conjunctive operations $\mathfrak{B}_c$, $\mathfrak{S}$, $\mathfrak{L}$, $\mathfrak{L}_1$ and $\mathfrak{Q}$ considered in Definition 1.1 (but not to $\mathfrak{T}$).

Observe that both the Π-vertical and the Π-horizontal ordinal sum given in Definition 2.1 are not only construction methods, but also representations of copulas in the following sense:

Proposition 2.2. *The following are equivalent:*

(i) $C \in \mathfrak{C}$*;*

(ii) *there is an index set I such that $C = \Pi_v\text{-}(\langle a_i, b_i, C_i\rangle)_{i\in I}$ for some family $(]a_i, b_i[)_{i\in I}$ of non-empty, pairwise disjoint open subintervals of* $[0,1]$ *and for some subfamily $(C_i)_{i\in I}$ of* $\mathfrak{C}$*;*

(iii) *there is an index set J such that $C = \Pi_h\text{-}(\langle a_j, b_j, C_j\rangle)_{j\in J}$ for some family $(]a_j, b_j[)_{j\in J}$ of non-empty, pairwise disjoint open subintervals of* $[0,1]$ *and for some subfamily $(C_j)_{j\in J}$ of* $\mathfrak{C}$.

Proof. Fix $C \in \mathfrak{C}$. If there is an $a \in\,]0,1[$ such that $C(a,y) = \Pi(a,y)$ for each $y \in [0,1]$, then we have $C = \Pi_v\text{-}(\langle 0, a, C_1\rangle, \langle a, 1, C_2\rangle)$, where $C_1, C_2 \in \mathfrak{C}$ are given by

$$C_1(x,y) = \frac{C(a\cdot x, y)}{a} \quad\text{and}\quad C_2(x,y) = \frac{C(a + (1-a)\cdot x, y) - a\cdot y}{1-a}.$$

If there is no $a \in]0,1[$ such that $C(a,y) = \Pi(a,y)$ for each $y \in [0,1]$, then $C = \Pi_v$-$(\langle 0,1,C\rangle)$, showing that (i) implies (ii). The converse is an immediate consequence of Definition 2.1, and from the one-to-one correspondence between Π-vertical and Π-horizontal ordinal sums, we obtain the equivalence of (ii) and (iii). □

Remark 2.3. It is not difficult to show that Proposition 2.2 remains valid if we replace the set $\mathfrak{C}$ by the sets $\mathfrak{B}$ or $\mathfrak{L}$. On the sets $\mathfrak{B}_c$, $\mathfrak{S}$, $\mathfrak{L}_1$ and $\mathfrak{Q}$ neither the Π-vertical nor the Π-horizontal ordinal sum is a representation, in general. Indeed, neither the monotonicity nor the 1-Lipschitz property of the summands is preserved, in general (see Example 2.4 for the sets $\mathfrak{Q}$ and $\mathfrak{S}$).

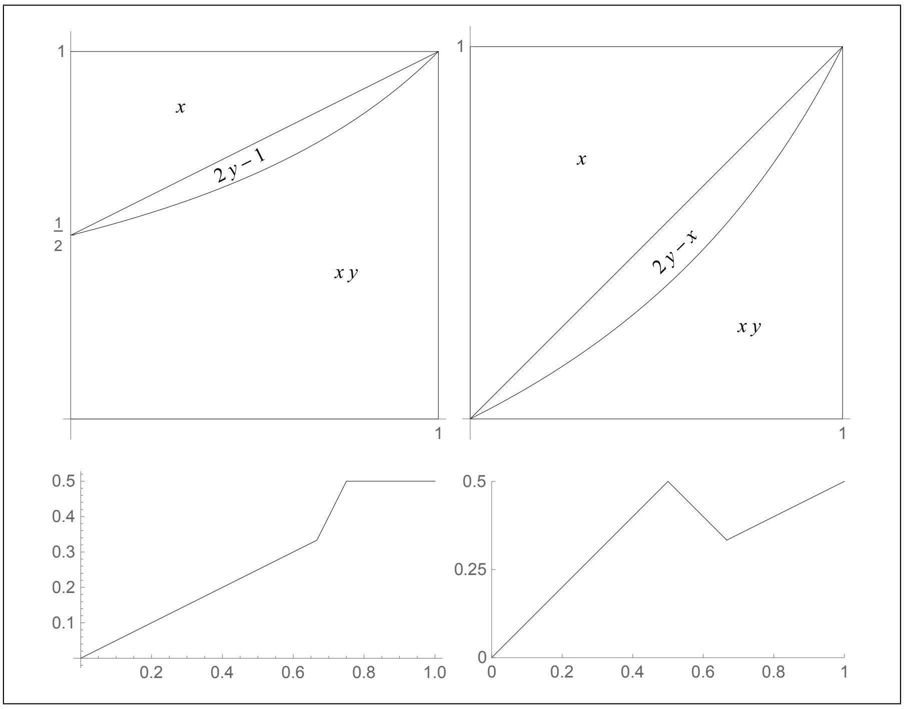

Figure 1. The semicopula B_1 (left) with its vertical section $B_1\left(\frac{1}{2},\cdot\right)$ which is not 1-Lipschitz, and the conjunctive operation B_4 (right) with its non-monotone horizontal section $B_4\left(\cdot,\frac{1}{2}\right)$ in Example 2.4.

Example 2.4. Consider the quasi-copula Q given by (using "med" as a shortcut for the median)

$$Q(x,y) = \operatorname{med}\Big(x \cdot y, y - \tfrac{1}{2}, x\Big).$$

Then, for each $y \in [0,1]$, we have $Q\left(\frac{1}{2}, y\right) = \Pi\left(\frac{1}{2}, y\right)$, and thus we can represent Q as a Π-vertical ordinal sum as follows: $Q = \Pi_v\text{-}\big(\langle 0, \frac{1}{2}, B_1\rangle, \langle \frac{1}{2}, 1, B_2\rangle\big)$ with $B_1, B_2 \in \mathfrak{B}_c$ being given by $B_1(x,y) = \operatorname{med}(x \cdot y, 2y-1, x)$ and $B_2 = \Pi$. Then B_1 is a semicopula, but obviously not 1-Lipschitz and, therefore, not a quasi-copula (for instance, the vertical section $B_1\left(\frac{1}{2}, \cdot\right)$ is not 1-Lipschitz, see Figure 1 bottom left).

Because of $B_1\left(x, \frac{1}{2}\right) = \Pi\left(x, \frac{1}{2}\right)$ for all $x \in [0,1]$, we can write B_1 as a Π-horizontal sum of conjunctive operations: $B_1 = \Pi_h\text{-}\big(\langle 0, \frac{1}{2}, B_3\rangle, \langle \frac{1}{2}, 1, B_4\rangle\big)$, where $B_3, B_4 \in \mathfrak{B}_c$ are given by $B_3 = \Pi$ and $B_4(x,y) = \operatorname{med}(x \cdot y, 2y - x, x)$. Clearly, B_4 is not monotone and, therefore, not a semicopula (for instance, the horizontal section $B_4\left(\cdot, \frac{1}{2}\right)$ is not monotone, see Figure 1 bottom right).

3. Π-vertical and Π-horizontal ordinal sums as representation tools

Motivated by Example 2.4, we are looking now for the smallest set $\mathfrak{Q}_v$ satisfying $\mathfrak{Q} \subseteq \mathfrak{Q}_v \subseteq \mathfrak{B}_c$ (i.e., a set of conjunctive operations containing all quasi-copulas) such that the Π-vertical ordinal sum given in Definition 2.1 is both a construction method and a representation on $\mathfrak{Q}_v$. From Example 2.4, we already know that $\mathfrak{Q}_v$ is a proper superset of $\mathfrak{Q}$, i.e., $\mathfrak{Q} \subset \mathfrak{Q}_v$.

Example 3.1. Here are two examples showing that $\mathfrak{Q}_v$ is a proper subset of $\mathfrak{B}_c$.

(i) Consider the conjunctive operator $B \in \mathfrak{B}_c$ given by

$$B(x,y) = \begin{cases} \Pi(x,y) & \text{if } (x,y) \in [0,1]^2 \setminus]0,1[^2 \text{ or } x = \frac{1}{2}, \\ 1 & \text{if } (x,y) \in \left]0, \frac{1}{2}\right[\times]0,1[\,, \\ 0 & \text{otherwise.} \end{cases}$$

Then we may write $B = \Pi_v\text{-}\big(\langle 0, \frac{1}{2}, B_1\rangle, \langle \frac{1}{2}, 1, B_2\rangle\big)$, where the two functions $B_1, B_2 \colon [0,1]^2 \to \mathbb{R}$ are given by

$$B_1(x,y) = \begin{cases} \Pi(x,y) & \text{if } (x,y) \in [0,1]^2 \setminus]0,1[^2\,, \\ 2 & \text{otherwise,} \end{cases}$$

$$B_2(x,y) = \begin{cases} \Pi(x,y) & \text{if } (x,y) \in [0,1]^2 \setminus]0,1[^2\,, \\ -y & \text{otherwise,} \end{cases}$$

i.e., $B_1, B_2 \in \mathfrak{B}$, but neither B_1 nor B_2 is a conjunctive operation, and, as a consequence, we have $B \in \mathfrak{B}_c \setminus \mathfrak{Q}_v$.

(ii) However, if $(\mathfrak{B}_c)_v$ is the set of all $B \in \mathfrak{B}_c$ satisfying the statement

$$\text{if } B(a, y) = a \cdot y \text{ for some } a \in \,]0, 1[\, \text{ and all } y \in [0, 1],$$
$$\text{then } B(x, y) \in \begin{cases} [0, a \cdot y] & \text{if } x \leq a, \\ [a \cdot y, 1 - a + a \cdot y] & \text{if } x \geq a, \end{cases}$$

then the Π-vertical ordinal sum is both a construction method and a representation on $(\mathfrak{B}_c)_v$. Note that $\mathfrak{L}_1$ and $\mathfrak{S}$ (and, as a consequence, also $\mathfrak{C}$ and $\mathfrak{Q}$) are subsets of $(\mathfrak{B}_c)_v$, and, therefore, also $\mathfrak{Q}_v \subseteq (\mathfrak{B}_c)_v$. Moreover, for B considered in (i), we even have $B \in \mathfrak{B}_c \setminus (\mathfrak{B}_c)_v$.

Lemma 3.2. *Let $Q\colon [0, 1]^2 \to [0, 1]$ be given by $Q = \Pi_v$-$(\langle a, b, B\rangle)$ for some $[a, b] \subseteq [0, 1]$ and for some $B \in \mathfrak{B}_c$. Then the following are equivalent:*

(i) *Q is a quasi-copula;*

(ii) *B is monotone non-decreasing and 1-Lipschitz in the first coordinate, i.e., for all $(x, y) \in [0, 1[\times [0, 1]$ and for all $\varepsilon \in \,]0, 1 - x]$ we have*

$$0 \leq B(x + \varepsilon, y) - B(x, y) \leq \varepsilon,$$

and, concerning the second coordinate of B, for all $(x, y) \in [0, 1] \times [0, 1[$ and for all $\varepsilon \in \,]0, 1 - y]$ we have

$$-\frac{a}{b-a} \cdot \varepsilon \leq B(x, y + \varepsilon) - B(x, y) \leq \frac{1-a}{b-a} \cdot \varepsilon.$$

Proof. Recall first that (see Definition 2.1) $Q = \Pi_v$-$(\langle a, b, B\rangle)$ implies that for all $(x, y) \in [0, 1]^2$ we have $B(x, y) = \frac{Q(a+(b-a)\cdot x, y) - a\cdot y}{b-a}$.

If Q is a quasi-copula, then the monotonicity and the 1-Lipschitz property of Q imply, for the first coordinate of B,

$$\begin{aligned} 0 &\leq B(x + \varepsilon, y) - B(x, y) \\ &= \frac{Q(a + (b - a) \cdot x + (b - a) \cdot \varepsilon, y) - Q(a + (b - a) \cdot x, y)}{b - a} \leq \varepsilon \end{aligned}$$

for all $(x, y) \in [0, 1[\times [0, 1]$ and for all $\varepsilon \in \,]0, 1 - x]$, and, for the second coordinate of B,

$$B(x, y + \varepsilon) - B(x, y) = \frac{Q(a + (b - a) \cdot x, y + \varepsilon) - Q(a + (b - a) \cdot x, y) - a \cdot \varepsilon}{b - a},$$

the latter expression being an element of $\left[-\frac{a\cdot\varepsilon}{b-a}, \frac{\varepsilon - a\cdot\varepsilon}{b-a}\right] = \left[-\frac{a}{b-a} \cdot \varepsilon, \frac{1-a}{b-a} \cdot \varepsilon\right]$, for all $(x, y) \in [0, 1] \times [0, 1[$ and for all $\varepsilon \in \,]0, 1 - y]$, thus proving that (i) ⇒ (ii).

To show (ii) ⇒ (i), suppose that B satisfies the conditions in (ii). Clearly, B is a Lipschitz function implying that Q is also Lipschitz and, therefore, continuous. The monotonicity and the 1-Lipschitz property of Q on the rectangles $[0,a]\times[0,1]$ and $[b,1]\times[0,1]$ follow from the validity of these properties for the product Π. On the set $[a,b]\times[0,1]$ the monotonicity and the 1-Lipschitz property of Q in the first coordinate are an immediate consequence of the validity of the respective properties for B. In the second coordinate, for each $(x,y)\in[a,b]\times[0,1[$ and for each $\varepsilon\in\,]0,1-y]$ we have

$$\begin{aligned}&Q(x,y+\varepsilon)-Q(x,y)\\&\quad=a\cdot(y+\varepsilon)+(b-a)\cdot B\left(\tfrac{x-a}{b-a},y+\varepsilon\right)-\left(a\cdot y+(b-a)\cdot B\left(\tfrac{x-a}{b-a},y\right)\right)\\&\quad=a\cdot\varepsilon+(b-a)\cdot\left(B\left(\tfrac{x-a}{b-a},y+\varepsilon\right)-B\left(\tfrac{x-a}{b-a},y\right)\right),\end{aligned}$$

the latter being an element of

$$\left[a\cdot\varepsilon-(b-a)\cdot\frac{a}{b-a}\cdot\varepsilon,a\cdot\varepsilon+(b-a)\cdot\frac{1-a}{b-a}\cdot\varepsilon\right]=[0,\varepsilon]\,.$$

This shows that Q is monotone non-decreasing and 1-Lipschitz on $[a,b]\times[0,1]$ in the second coordinate. □

Based on Lemma 3.2, we obtain the following characterization of the set $\mathfrak{Q}_v$.

Theorem 3.3. *Let $\mathfrak{Q}_v\subseteq(\mathfrak{B}_c)_v$ be a set of conjunctive operations. Then the following are equivalent:*

(i) *$\mathfrak{Q}_v$ is the smallest set containing all quasi-copulas such that we have $B\in\mathfrak{Q}_v$ if and only if there is an index set I such that $B=\Pi_v$-$(\langle a_i,b_i,B_i\rangle)_{i\in I}$ for some family $(]a_i,b_i[)_{i\in I}$ of non-empty, pairwise disjoint open subintervals of $[0,1]$ and for some subfamily $(B_i)_{i\in I}$ of $\mathfrak{Q}_v$;*

(ii) *each element $B\in\mathfrak{Q}_v$ is a Lipschitz function, and, in its first coordinate, B is monotone non-decreasing and 1-Lipschitz.*

Proof. Evidently, we have $\mathfrak{Q}\subseteq\mathfrak{Q}_v\subseteq(\mathfrak{B}_c)_v$, where the elements of $\mathfrak{Q}_v$ are characterized by (ii). Fix an arbitrary element $B\in\mathfrak{Q}_v$. Since B is, in its first coordinate, monotone non-decreasing and 1-Lipschitz, and, in its second coordinate, a Lipschitz function, there are constants $\alpha,\beta\in[0,\infty[$ such that for all $(x,y)\in[0,1]\times[0,1[$ and for all $\varepsilon\in\,]0,1-y]$ we have

$$-\alpha\cdot\varepsilon\le B(x,y+\varepsilon)-B(x,y)\le\beta\cdot\varepsilon. \tag{3.1}$$

Evaluating (3.1) for $x = 1$, we obtain $-\alpha \cdot \varepsilon \leq \varepsilon \leq \beta \cdot \varepsilon$, i.e., $\beta \geq 1$. Putting $a = \frac{\alpha}{\alpha+\beta}$ and $b = \frac{1+\alpha}{\alpha+\beta}$, we get $0 \leq a < b \leq 1$, and it is easy to see that the function $Q = \Pi_v\text{-}(\langle a, b, B\rangle)$ is a quasi-copula. Therefore, each set $\mathfrak{H}$ satisfying $\mathfrak{Q} \subseteq \mathfrak{H} \subseteq \mathfrak{B}_c$ for which the Π-vertical ordinal sum is a representation necessarily satisfies $\mathfrak{Q}_v \subseteq \mathfrak{H}$.

The Π-vertical ordinal sum preserves both the monotonicity and the 1-Lipschitz property of the summands in the first coordinate, as well as the Lipschitz property of the summands in the second coordinate, i.e., it is a construction method on $\mathfrak{Q}_v$. Moreover, using similar arguments as in the proof of Lemma 3.2, one can show that the Π-vertical ordinal sum is also a representation on $\mathfrak{Q}_v$.

Summarizing, $\mathfrak{Q}_v$ is the smallest set containing all quasi-copulas for which the Π_v-vertical ordinal sum is both a construction method and a representation. □

Remark 3.4. Because of the one-to-one correspondence between Π-vertical and Π-horizontal ordinal sums, we have an analogous characterization of the smallest subset $\mathfrak{Q}_h$ of $\mathfrak{B}_c$ containing all quasi-copulas for which the Π-horizontal ordinal sum is both a construction method and a representation. To be precise, $\mathfrak{Q}_h$ consists of all Lipschitz conjunction operations which are monotone non-decreasing and 1-Lipschitz in the second coordinate.

Using similar arguments as in the proof of Theorem 3.3, we obtain the following result.

Corollary 3.5. *The set $\mathfrak{L}$ of all Lipschitz conjunctive operations is the smallest set which contains all quasi-copulas and for which both the Π-vertical and the Π-horizontal ordinal sum are construction methods and representations.*

Remark 3.6. Observe that the sets $\mathfrak{Q}_v$ and $\mathfrak{Q}_h$ can be characterized as follows:

(i) $\mathfrak{Q}_v$ consists of all elements $B \in \mathfrak{L}_1$ satisfying $V_B([a, b] \times [c, d]) \geq 0$ for each rectangle $[a, b] \times [c, d] \subseteq [0, 1]$ with $c = 0$ or $d = 1$;

(ii) $\mathfrak{Q}_h$ consists of all elements $B \in \mathfrak{L}_1$ satisfying $V_B([a, b] \times [c, d]) \geq 0$ for each rectangle $[a, b] \times [c, d] \subseteq [0, 1]$ with $a = 0$ or $b = 1$.

As an immediate consequence, we obtain $\mathfrak{Q}_v \cap \mathfrak{Q}_h = \mathfrak{Q}$.

Coming back to Example 2.4, note that for the semicopula B_1 and for the conjunctive operation B_4 considered there, we have

$$B_1 \in \mathfrak{Q}_v \setminus \mathfrak{Q} \text{ and } B_4 \in \mathfrak{L}_1 \setminus (\mathfrak{Q}_v \cup \mathfrak{Q}_h).$$

4. Π-diagonal ordinal sums

A third type of ordinal sum of copulas related to the product Π was proposed in [22] and relates to patchwork approaches also discussed in [9], [11]. Such an ordinal sum differs from Π only on squares along the main diagonal of the unit square $[0,1]^2$, and we therefore speak about Π-diagonal ordinal sums.

Definition 4.1. Let $(]a_i,b_i[)_{i\in I}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$, and let $(C_i)_{i\in I}$ be a family of copulas. Then the function $C\colon [0,1]^2 \to [0,1]$ given by

$$C(x,y)=\begin{cases} x\cdot y-(x-a_i)\cdot(y-a_i)+(b_i-a_i)^2\cdot C_i\left(\frac{x-a_i}{b_i-a_i},\frac{y-a_i}{b_i-a_i}\right) & \text{if } (x,y)\in[a_i,b_i]^2,\\ \Pi(x,y) & \text{otherwise,}\end{cases}$$

is well-defined, and we have $C \in \mathfrak{C}$. The copula C is called the Π-*diagonal ordinal sum* of the summands $(]a_i,b_i[\,,C_i)_{i\in I}$, and we shall write $C = \Pi_d\text{-}(\langle a_i,b_i,C_i\rangle)_{i\in I}$.

The concept of Π-diagonal ordinal sums as a construction method can be carried over in a straightforward way to the set $\mathfrak{B}$, and to the sets of conjunctive operations $\mathfrak{B}_c$, $\mathfrak{S}$, $\mathfrak{L}$, $\mathfrak{L}_1$ and $\mathfrak{Q}$ considered in Definition 1.1 (but not to $\mathfrak{T}$).

Observe that the Π-diagonal ordinal sum given in Definition 4.1 is not only a construction method, but also a representation of copulas in the following sense:

Proposition 4.2. *$C \in \mathfrak{C}$ if and only if there is an index set I such that $C = \Pi_d\text{-}(\langle a_i,b_i,C_i\rangle)_{i\in I}$ for some family $(]a_i,b_i[)_{i\in I}$ of non-empty, pairwise disjoint open subintervals of $[0,1]$ and for some subfamily $(C_i)_{i\in I}$ of $\mathfrak{C}$.*

Proof. Fix $C \in \mathfrak{C}$. If there is an $a \in\,]0,1[$ such that $C(x,y) = \Pi(x,y)$ whenever $a \in \{x,y\}$, then we have $C = \Pi_d\text{-}(\langle 0,a,C_1\rangle,\langle a,1,C_2\rangle)$, where $C_1,C_2 \in \mathfrak{C}$ are given by

$$C_1(x,y) = \frac{C(a\cdot x, a\cdot y)}{a^2},$$
$$C_2(x,y) = \frac{C(a+(1-a)\cdot x, a+(1-a)\cdot y)-a\cdot(1-a)\cdot(x+y)-a^2}{(1-a)^2}.$$

If there is no $a \in\,]0,1[$ such that $C(x,y) = \Pi(x,y)$ whenever $a \in \{x,y\}$, then $C = \Pi_d\text{-}(\langle 0,1,C\rangle)$. □

Remark 4.3. It is not difficult to show that Proposition 4.2 remains valid if we replace the set $\mathfrak{C}$ by the sets $\mathfrak{B}$ or $\mathfrak{L}$. On the sets $\mathfrak{B}_c$, $\mathfrak{S}$, $\mathfrak{L}_1$ and $\mathfrak{Q}$, the Π-diagonal ordinal sum is not a representation, in general. Again, neither the monotonicity nor the 1-Lipschitz property of the summands is preserved, in general.

The Π-diagonal ordinal sum of copulas is closely related to the Π-vertical and the Π-horizontal ordinal sum of copulas:

Lemma 4.4. *For each copula* $C \in \mathfrak{C}$ *and for each interval* $[a, b] \subseteq [0, 1]$*, we have*

$$\Pi_d\text{-}(\langle a, b, C\rangle) = \Pi_h\text{-}(\langle a, b, \Pi_v\text{-}(\langle a, b, C\rangle)\rangle) = \Pi_v\text{-}(\langle a, b, \Pi_h\text{-}(\langle a, b, C\rangle)\rangle).$$

Proof. Fix an arbitrary $C \in \mathfrak{C}$ and an interval $[a, b] \subseteq [0, 1]$. For the sake of brevity, let us use the following shortcuts in this proof: $C_d = \Pi_d\text{-}(\langle a, b, C\rangle)$, $C_v = \Pi_v\text{-}(\langle a, b, C\rangle)$, and $C_{v,h} = \Pi_h\text{-}(\langle a, b, \Pi_v\text{-}(\langle a, b, C\rangle)\rangle)$.

For all $x \in [a, b]$ and $y \in [0, 1]$ we have, according to Definition 2.1(i),

$$C_v(x, y) = a \cdot y + (b - a) \cdot C\left(\tfrac{x-a}{b-a}, y\right),$$

and, because of Definition 2.1(ii), for all $(x, y) \in [a, b]^2$

$$\begin{aligned} C_{v,h}(x, y) &= a \cdot x + (b - a) \cdot C_v\left(x, \tfrac{y-a}{b-a}\right) \\ &= a \cdot x + (b - a) \cdot \left(a \cdot \frac{y - a}{b - a} + (b - a) \cdot C\left(\tfrac{x-a}{b-a}, \tfrac{y-a}{b-a}\right)\right) \\ &= x \cdot y - (x - a) \cdot (y - a) + (b - a)^2 \cdot C\left(\tfrac{x-a}{b-a}, \tfrac{y-a}{b-a}\right) = C_d(x, y), \end{aligned}$$

where the latter equality holds because of Definition 4.1.

It is not difficult to check that for each $(x, y) \in [0, 1]^2 \setminus [a, b]^2$ we obtain $C_{v,h}(x, y) = \Pi(x, y)$, thus showing that $\Pi_d\text{-}(\langle a, b, C\rangle) = \Pi_h\text{-}(\langle a, b, \Pi_v\text{-}(\langle a, b, C\rangle)\rangle)$.

The remaining equality in the claim of the lemma is shown in complete analogy. □

As an immediate consequence of Corollary 3.5 and Lemma 4.4, we obtain the following characterization:

Theorem 4.5. *The set* $\mathfrak{L}$ *of all Lipschitz conjunction operators is the smallest set of conjunction operators which contains all quasi-copulas such that we have:* $L \in \mathfrak{L}$ *if and only if there is an index set* I *where* $L = \Pi_d\text{-}(\langle a_i, b_i, L_i\rangle)_{i\in I}$ *for some family* $(]a_i, b_i[)_{i\in I}$ *of non-empty, pairwise disjoint open subintervals of* $[0, 1]$ *and for some subfamily* $(L_i)_{i\in I}$ *of* $\mathfrak{L}$.

Theorems 3.3 and 4.5 provide us with a construction method for proper quasi-copulas by means of Π-vertical, Π-horizontal or Π-diagonal ordinal sums if not all the summands are elements of $\mathfrak{C}$.

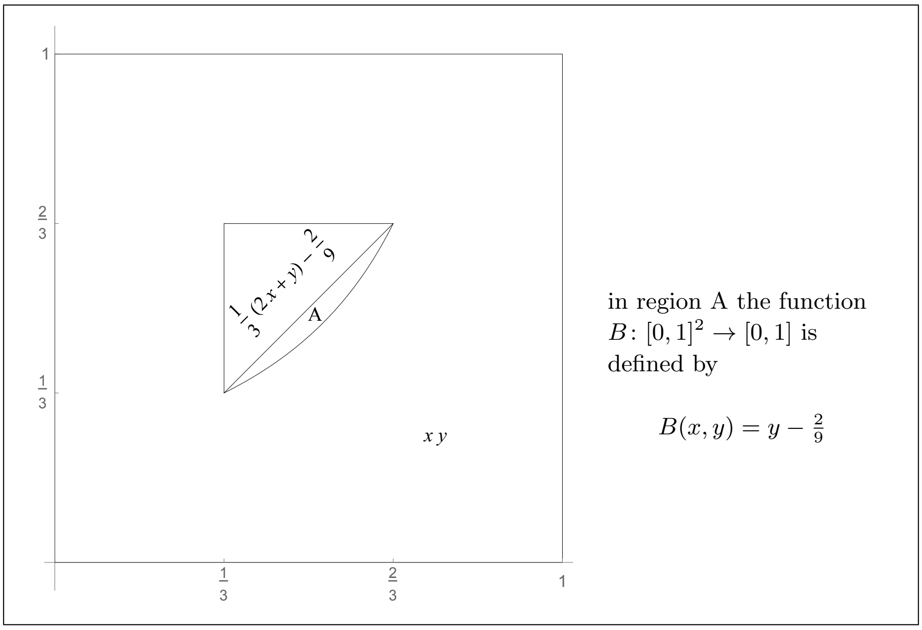

Figure 2. The function B in Example 4.6 is a proper quasi-copula.

Example 4.6. Consider the conjunctive operation $B_4 \in \mathfrak{L}_1 \setminus \mathfrak{S}$ from Example 2.4, given by $B_4(x,y) = \mathrm{med}(x \cdot y, 2y - x, x)$, and put $B = \Pi_d\text{-}\big(\big\langle \frac{1}{3}, \frac{2}{3}, B_4 \big\rangle\big)$, i.e.,

$$B(x,y) = \begin{cases} \mathrm{med}\left(x \cdot y, y - \frac{2}{9}, \frac{2x+y}{3} - \frac{2}{9}\right) & \text{if } (x,y) \in \left[\frac{1}{3}, \frac{2}{3}\right]^2, \\ \Pi(x,y) & \text{otherwise.} \end{cases}$$

Then $B \in \mathfrak{Q} \setminus \mathfrak{C}$, i.e., B is a quasi-copula but not a copula (for instance, we have $V_B\left(\left[\frac{1}{2}, \frac{5}{9}\right] \times \left[\frac{4}{9}, \frac{1}{2}\right] = -\frac{2}{81} < 0\right)$.

Concluding remarks

We have recalled several sets of conjunctive functions and operations which can be seen as extensions of the classical Boolean conjunction. We have recalled M- and W-ordinal sums, as well as three types of ordinal sums based on Π. Note that all of them were introduced first of all as construction methods, and they can be applied to construct new t-norms (only in the case of M-ordinal sums), copulas, quasi-copulas, semicopulas, etc. On the other hand, M- and W-ordinal sums can be seen also as representation methods on these sets of conjunction operations. However, this is no more the case for the Π-vertical, the Π-horizontal and the Π-diagonal ordinal sum when considering quasi-copulas or semicopulas (in the case of copulas, all these three product-based ordinal sums are also representations).

Due to an increasing interest in quasi-copulas, we have focused on this class and have looked for the smallest set $\mathfrak{Q}_v$ of conjunctive operations containing all quasi-copulas and such that the Π-vertical ordinal sums can be seen both as a construction method and a representation on $\mathfrak{Q}_v$. We have shown the prominent role of Lipschitz conjunctive operations. Denote this set as $\mathfrak{L}$, and the set of conjunctive operations which are 1-Lipschitz and monotone non-decreasing in the first coordinate by $\mathfrak{Q}^{(1)}$ (i.e., they satisfy the conditions of a quasi-copula in their first coordinate), and similarly, by $\mathfrak{Q}^{(2)}$ the set of conjunctive operations satisfying the conditions of a quasi-copula in the second coordinate. Then, it is obvious that $\mathfrak{Q} = \mathfrak{Q}^{(1)} \cap \mathfrak{Q}^{(2)}$ and $\mathfrak{Q}_v = \mathfrak{L} \cap \mathfrak{Q}^{(1)}$. If we denote by $\mathfrak{Q}_h$ and $\mathfrak{Q}_d$ the smallest sets of conjunctive operations containing all quasi-copulas and such that the Π-horizontal and the Π-diagonal ordinal sum, respectively, can be seen both as a construction method and a representation, then we also have $\mathfrak{Q}_h = \mathfrak{L} \cap \mathfrak{Q}^{(2)}$ and $\mathfrak{Q}_d = \mathfrak{L}$.

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SUSANNE SAMINGER-PLATZ
DEPARTMENT OF KNOWLEDGE-BASED
MATHEMATICAL SYSTEMS
JOHANNES KEPLER UNIVERSITY
ALTENBERGER STRASSE 69
4040 LINZ
AUSTRIA

E-mail: susanne.saminger-platz@jku.at

ERICH PETER KLEMENT
DEPARTMENT OF KNOWLEDGE-BASED
MATHEMATICAL SYSTEMS
JOHANNES KEPLER UNIVERSITY
ALTENBERGER STRASSE 69
4040 LINZ
AUSTRIA

E-mail: ep.klement@jku.at

MICHAL DIBALA
DEPARTMENT OF MATHEMATICS
AND DESCRIPTIVE GEOMETRY
FACULTY OF CIVIL ENGINEERING
SLOVAK UNIVERSITY OF TECHNOLOGY
RADLINSKÉHO 11
810 05 BRATISLAVA
SLOVAKIA

E-mail: michal.dibala@stuba.sk

RADKO MESIAR
DEPARTMENT OF MATHEMATICS
AND DESCRIPTIVE GEOMETRY
FACULTY OF CIVIL ENGINEERING
SLOVAK UNIVERSITY OF TECHNOLOGY
RADLINSKÉHO 11
810 05 BRATISLAVA
SLOVAKIA

E-mail: radko.mesiar@stuba.sk

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