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Finite 2-groups of rank 2

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Abstract. Let G be a 2-group. In this paper, we investigate the 2-group G in which $r(G) = 2$ and G has more than three involutions. We prove that if $\Omega_1(G) \cong D_{2^n}$ or $D_{2^n} * C_4$ with $n \geq 3$, then G' is abelian and there exists a maximal subgroup M of G such that M is metacyclic. If $\Omega_1(G) \cong D_{2^n} * Q_{2^m}$ with $n, m \geq 3$, then either $\Phi(G) \leq \Omega_1(G)$ or $|\Phi(G)| = |\Omega_1(G)|$ and $G' \cap \Omega_1(G)$ is a maximal subgroup of $\Omega_1(G)$.

1. Introduction

All groups considered in this paper are finite.

Let G be a p-group. Then $r(G) = \max\{\log_p |E| \bigm| E$ is an elementary abelian subgroup in $G\}$ is called the rank of G. A well-known result is that G is cyclic or G is generalized quaternion if $r(G) = 1$. So it is natural to investigate p-groups with $r(G) = 2$. For the case $p > 2$, Blackburn in [3] has given the classification of p-groups with $r(G) = 2$. For the case $p = 2$, many authors have investigated 2-groups in which there are exactly three involutions. For example, Ustjužaninov [8] proves that if a 2-group G has exactly three involutions and $Z(G)$ is non-cyclic, then G has a normal metacyclic subgroup M of index at most 4 and G/M is elementary abelian. Konvisser [7] goes one step further and proves that if a 2-group G has exactly three involutions and $Z(G)$ is cyclic, then G has a metacyclic subgroup M of index at most 4 and "M is normal in G in most of the cases". Janko [5] clears up this remaining, very difficult situation

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and determines completely the structure of G in terms of two generators and relations. Now, it is natural to ask the following question:

How about the structure of a 2-group G with $r(G) = 2$ in which there are more than three involutions?

For convenience, we call a 2-group G a $\mathcal{R}_2\mathcal{I}^{>3}$-group if $r(G) = 2$ and there are more than three involutions in G. In this paper, we hope to investigate the structure of a 2-group G satisfying the condition $\mathcal{R}_2\mathcal{I}^{>3}$. According to a result of Johnsen [6], we see that a 2-group G is a $\mathcal{R}_2\mathcal{I}^{>3}$-group if and only if $\Omega_1(G) \cong D_{2^n}$ or $D_{2^n} * C_4$ or $D_{2^n} * Q_{2^m}$ with $n, m \geq 3$. Hence, we investigate 2-groups satisfying the condition $\mathcal{R}_2\mathcal{I}^{>3}$ in terms of the structure of $\Omega_1(G)$.

2. Preliminaries

For convenience, we use D_{2^n} and Q_{2^n} to denote the dihedral group and the generalized quaternion group of order 2^n, respectively. We use C_{p^m} to denote a cyclic group of order p^m, $C_{p^m}^n$ the direct product of n cyclic groups of order p^m. If H and K are groups, then $H * K$ means a central product of H and K. For other notation and terminology, the reader is referred to [4].

Now, we list some results which will be used later.

Lemma 2.1 ([6, Theorem 3.1]). *Let G be a 2-group. Then $r(G) = 2$ if and only if $\Omega_1(G) \cong C_2 \times C_2$ or D_{2^n}, or $D_{2^n} * C_4$, or $D_{2^n} * Q_{2^m}$, with $n, m \geq 3$.*

Lemma 2.2 ([5, Theorem 2.2]). *Let G be a non-metacyclic 2-group with exactly three involutions. If W is a maximal normal abelian non-cyclic subgroup of exponent ≤ 4 in G, then $C_G(W)$ is metacyclic.*

Lemma 2.3 ([1, Section 1, Lemma 1.1]). *If a non-abelian p-group G has an abelian maximal subgroup, then $|G| = p|G'||Z(G)|$.*

Lemma 2.4 ([2, Section 50, Lemma 50.3]). *Let G be a 2-group which has no normal elementary abelian subgroups of order 8. Then every subgroup U of G is generated by four elements.*

Lemma 2.5 ([1, Section 41, Remark 2]). *Let G be a p-group. Then G is metacyclic if and only if $\Omega_2(G)$ is metacyclic.*

Lemma 2.6 ([1, Section 1, Exercise 85]). *Let a non-cyclic p-group G be metacyclic. If G is not a 2-group of maximal class, then $\Omega_1(G) \cong C_p \times C_p$.*

Lemma 2.7 ([1, Section 1, Proposition 1.13]). *Let G be a p-group, and let $N \leq \Phi(G)$ be G-invariant. If $Z(N)$ is cyclic, then N is also cyclic.*

Theorem 2.8. *If a 2-group G is a $\mathcal{R}_2\mathcal{I}^{>3}$-group, then $Z(G)$ is cyclic.*

Proof. If $Z(G)$ is not cyclic, then $r(Z(G))=2$. It follows that $\Omega_1(G)\leq Z(G)$, which implies G has exactly three involutions, a contradiction. $\square$

Theorem 2.9. *If G is a 2-group such that $\Omega_1(G) \cong D_{2^n}$ or $D_{2^n} * Q_{2^m}$ with $n, m \geq 3$, then $|Z(G)| = |Z(\Omega_1(G))| = 2$.*

Proof. It is clear that $|Z(\Omega_1(G))| = 2$, and that there exists an element $x \in \Omega_1(G)$ such that $Z(\Omega_1(G)) = \langle x^2 \rangle$. If $Z(G) > Z(\Omega_1(G))$, then there exists an element $g \in Z(G)$ such that $o(g) = 2^s \geq 4$ and $g^{2^{s-1}} \in Z(G) \cap \Omega_1(G) = \langle x^2 \rangle$. In this case, $o(xg^{2^{s-2}}) = 2$. Thus $xg^{2^{s-2}} \in \Omega_1(G)$, and so $g^{2^{s-2}} \in Z(\Omega_1(G))$, in contradiction to $|Z(\Omega_1(G))| = 2$. Hence, $|Z(G)| = |Z(\Omega_1(G))| = 2$. $\square$

Corollary 2.10. *If G is a 2-group such that $\Omega_1(G) \cong D_{2^n}$ or $D_{2^n} * Q_{2^m}$ with $n, m \geq 3$, then $C_G(\Omega_1(G)) = Z(\Omega_1(G))$.*

Proof. If there exists an element $g \in C_G(\Omega_1(G))$ such that $g \notin Z(\Omega_1(G))$, then $H = \Omega_1(G)\langle g \rangle > \Omega_1(G)$ and $g \in Z(H)$. Noticing that $\Omega_1(H) = \Omega_1(G)$, we see that $Z(H) = Z(\Omega_1(H))$ by Theorem 2.9. Thus $g \in Z(\Omega_1(H)) = Z(\Omega_1(G))$, a contradiction. So $C_G(\Omega_1(G)) = Z(\Omega_1(G))$. $\square$

Lemma 2.11. *Let a 2-group G be a $\mathcal{R}_2\mathcal{I}^{>3}$-group, and let N be a normal subgroup of G with $N \leq \Phi(G)$. If N is not cyclic, then $\Omega_1(N) \cong C_2 \times C_2$.*

Proof. Lemma 2.7 implies that $Z(N)$ is not cyclic. Thus $r(N) = 2$. It follows from Lemma 2.1 and Theorem 2.8 that $\Omega_1(N) \cong C_2 \times C_2$. $\square$

Lemma 2.12. *Let a 2-group G be a $\mathcal{R}_2\mathcal{I}^{>3}$-group. If $\Phi(G)$ is not cyclic, then G has the unique normal subgroup N such that $N \cong C_2 \times C_2$.*

Proof. By Lemma 2.11, $\Omega_1(\Phi(G)) \cong C_2 \times C_2$. If there exists $N \trianglelefteq G$ such that $N \cong C_2 \times C_2$ and $N \neq \Omega_1(\Phi(G))$, then $C_G(N)$ is a maximal subgroup in G by Theorem 2.8. It follows that $\Omega_1(\Phi(G))\leq\Omega_1(C_G(N))=N$, a contradiction. $\square$

Lemma 2.13. *Let G be a group of order 2^n and $\Omega_1(G) \cong D_{2^m}$ with $m \geq 3$. If G has a maximal subgroup M such that $\Omega_1(G) \leq M$ and M is of maximal class, then G is of maximal class.*

Proof. By the hypotheses of the lemma, we see that M is dihedral or semidihedral of order 2^{n-1}. Then, we may assume $M = \langle a, b \mid a^{2^{n-2}} = b^2 = 1, [a, b] =$

$a^{i2^{n-3}-2}\rangle$, with $n \geq 4$ if $i = 0$, and $n \geq 5$ if $i = 1$. Thus $|M'| = 2^{n-3}$, and therefore $2^{n-3} \leq |G'| \leq 2^{n-2}$. If $|G'| = 2^{n-2}$, then G is of maximal class. Now, we assume $G' = M' = \langle a^2 \rangle$. Take $x \in G \setminus M$. It follows from $[a, x] \in \langle a^2 \rangle$ that $[a, x^2] \in \langle a^4 \rangle$, which implies $x^2 \in \langle a \rangle$. Assume $[b, x] = a^{2j}$. If $2 \mid j$ or $i = 0$, then $[b, xa^{-j}] = 1$. If $2 \nmid j$ and $i = 1$, then $[b, xa^{2^{n-4}-j}] = 1$. Thus, without loss of generality, we may assume $[b, x] = 1$. Then $x^2 \in Z(M) = \langle a^{2^{n-3}} \rangle$, and so $x^2 = a^{2^{n-3}}$. Clearly, $[a^{2^{n-4}}, x] = 1$ or $a^{2^{n-3}}$. Thus $o(a^{2^{n-4}}x) = 2$ if $[a^{2^{n-4}}, x] = 1$, and $o(a^{2^{n-4}}bx) = 2$ if $[a^{2^{n-4}}, x] = a^{2^{n-3}}$. It follows that $x \in M$ in both cases, a contradiction. □

Theorem 2.14. *Let G be a 2-group and $\Omega_1(G) \cong D_{2^n}$ with $n \geq 3$. Then $G = \Omega_1(G)$ is dihedral or G is semi-dihedral with $|G : \Omega_1(G)| = 2$.*

Proof. If $\Omega_1(G) = G$, then the result is clear. Now, we assume $\Omega_1(G) < G$ and H is a subgroup of G such that $\Omega_1(G)$ is a maximal subgroup of H. It follows from Lemma 2.13 that H is of maximal class. Thus H is dihedral or semi-dihedral. If H is dihedral, then $\Omega_1(G) < H = \Omega_1(H)$, a contradiction. So H is semi-dihedral. If $H < G$, then there exists $K \leq G$ such that H is a maximal subgroup in K. By Lemma 2.13 again, K is of maximal class. Then, for any $L < K$, we see that L is not a semi-dihedral group, a contradiction. So $H = G$ and G is semi-dihedral. □

3. 2-groups with $\Omega_1(G) \cong D_{2^n} * C_4$

Lemma 3.1. *If $\Omega_1(G) \cong D_{2^n} * C_4 = \langle a, b, c \mid a^{2^{n-1}} = b^2 = 1, a^{2^{n-2}} = c^2, [a, b] = a^{-2}, [c, a] = [c, b] = 1 \rangle$ with $n \geq 4$, then $[\Omega_1(G), G] \leq \langle a \rangle$.*

Proof. It is clear that $Z(\Omega_1(G)) = \langle c \rangle$ and $\Omega_1(G)' = \langle a^2 \rangle$. Then $[\langle c \rangle, G] \leq \langle c^2 \rangle \leq \langle a \rangle$ and $\langle a^{2^{n-3}} \rangle \operatorname{char} \langle a^2 \rangle \trianglelefteq G$. Thus $C_G(a^{2^{n-3}})$ is a maximal subgroup of G, and so $[\Omega_1(G), G] \leq \Omega_1(G) \cap C_G(a^{2^{n-3}}) = \langle a, c \rangle$. For any $g \in G$, it follows from $[b^2, g] = [(ab)^2, g] = 1$ that $[b, g] \in \langle a \rangle$ and $[ab, g] \in \langle a \rangle$, which implies $[a, g] \in \langle a \rangle$. Hence, $[\Omega_1(G), G] \leq \langle a \rangle$. □

Lemma 3.2. *If $\Omega_1(G) \cong D_8 * C_4 = \langle a, b \rangle * \langle c \rangle$, then there exists an involution $g \notin \langle c \rangle$ such that $[\Omega_1(G), G] \leq \langle cg \rangle$.*

Proof. It is easy to see that there exists $N \trianglelefteq G$ such that $N \cong C_2 \times C_2$. Without loss of generality, we may assume $o(a) = o(b) = 2$ and $N = \langle b, c^2 \rangle$. Then $[\langle b \rangle, G], [\langle c \rangle, G] \leq \langle c^2 \rangle \leq \langle bc \rangle$, and so $\langle bc \rangle \trianglelefteq G$. Thus $[\Omega_1(G), G] \leq \Omega_1(G) \cap C_G(bc) = \langle b, c \rangle$. For any $g \in G$, it follows from $[a^2, g] = 1$ that $[a, g] \in \langle bc \rangle$, which implies $[\Omega_1(G), G] \leq \langle bc \rangle$. □

Theorem 3.3. *Let G be a 2-group and $\Omega_1(G) \cong D_{2^n} * C_4$ with $n \geq 3$. Then there exists a maximal subgroup M of G such that M is metacyclic and G' is abelian.*

Proof. We consider the following two cases: $n \geq 4$ and $n = 3$.

Case 1. $n \geq 4$.

We may assume $\Omega_1(G) = \langle a, b, c \mid a^{2^{n-1}} = b^2 = 1, a^{2^{n-2}} = c^2, [a, b] = a^{-2}, [c, a] = [c, b] = 1\rangle$. Then $[\Omega_1(G), G] \leq \langle a\rangle$ and $\langle a^{2^{n-3}}\rangle \trianglelefteq G$ by Lemma 3.1. Since $b \notin C_G(a^{2^{n-3}})$, we see that $C_G(a^{2^{n-3}})$ is a maximal subgroup of G.

Let $M = C_G(a^{2^{n-3}})$. Then $\langle a^{2^{n-3}}c, a^{2^{n-2}}\rangle \leq \Omega_1(M) \leq \Omega_1(G) \cap M = \langle a, c\rangle$. Thus $\Omega_1(M) = \langle a^{2^{n-3}}c, a^{2^{n-2}}\rangle \cong C_2 \times C_2$. Let $K = \langle a^{2^{n-3}}, c\rangle$. Then K is a normal abelian subgroup of G with $\exp(K) = 4$ and $K \leq \Omega_2(M)$. For any $g \in \Omega_2(M)$ and $o(g) = 4$, we see $g^2 \in \Omega_1(M)$. If $g^2 = a^{2^{n-2}}$, then $o(a^{2^{n-3}}g) = 2$, and so $g \in K$. If $g^2 = a^{\pm 2^{n-3}}c$, then $[a^{2^{n-3}}, g] = [c, g] = 1$. Hence, $K \leq Z(\Omega_2(M))$ in both cases.

We claim $\Omega_2(M)$ is metacyclic. Let L be a maximal normal abelian subgroup of M such that $\exp(L) = 4$ and $K \leq L$. If $K = L$, then $C_{\Omega_2(M)}(L) = \Omega_2(M)$ is metacyclic by Lemma 2.2. So, we may assume $K < L < \Omega_2(M)$. For any $h \in \Omega_2(M) \setminus K$ and $o(h) = 4$, we have $h^2 = a^{\pm 2^{n-3}}c$ by the above. Since $[a, h^2] = [a^{2^{n-3}}, h] = 1$, we see $[a^2, h] = 1$. It follows from $[b, h^2] = a^{2^{n-2}}$ that $[b, h] = a^{\pm 2^{n-3}}$. Take $x \in \Omega_2(M) \setminus L$ and $y \in L \setminus K$ such that $o(x) = o(y) = 4$. Then $[b, x] = a^{\pm 2^{n-3}}$ and $[b, y] = a^{\pm 2^{n-3}}$. Thus $[b, xy] \in \langle a^{2^{n-2}}\rangle$. Noticing that $o(xy) \leq 4$, we see $xy \in K$, and so $x \in \langle K, y\rangle \leq L$, a contradiction. Thus $\Omega_2(M)$ is metacyclic. By Lemma 2.5, M is metacyclic. Then M' is cyclic and $G' \leq C_G(M')$. Since $G = M\langle b\rangle$, we see $G' = \langle M', [b, M]\rangle \leq \langle M', a\rangle$, which implies G' is abelian.

Case 2. $n = 3$.

In this case, we may assume $\Omega_1(G) = \langle a, b, c \mid a^2 = b^2 = c^4 = 1, [b, a] = c^2, [b, c] = [a, c] = 1\rangle$ and $[\Omega_1(G), G] \leq \langle bc\rangle$ by Lemma 3.2. Then $\langle bc\rangle \trianglelefteq G$ and $C_G(bc)$ is a maximal subgroup of G.

Let $M = C_G(bc)$. Then $\langle b, c^2\rangle \leq \Omega_1(M) \leq \Omega_1(G) \cap M = \langle b, c\rangle$. Thus $\Omega_1(M) = \langle b, c^2\rangle$. Let $K = \langle b, c\rangle$. Then K is a normal abelian subgroup of G and $\exp(K) = 4$. For any $g \in \Omega_2(M)$ and $o(g) = 4$, we see $g^2 \in \Omega_1(M)$. If $g^2 = c^2 = (bc)^2$, then $o(bcg) = 2$, and so $g \in K$. If $g^2 = bc^2$ or b, then $[b, g] = [c, g] = 1$. So $K \leq Z(\Omega_2(M))$.

We claim $\Omega_2(M)$ is metacyclic. Let L be a maximal normal abelian subgroup of M such that $\exp(L) = 4$ and $K \leq L$. If $K = L$, then $C_{\Omega_2(M)}(L) = \Omega_2(M)$ is metacyclic. We assume $K < L < \Omega_2(M)$. For any $h \in \Omega_2(M) \setminus K$ and $o(h) = 4$, we have $h^2 = bc^2$ or b, which implies $[a, h] = bc^{\pm 1}$. Take $x \in \Omega_2(M) \setminus L$ and

$y \in L \setminus K$ such that $o(x) = o(y) = 4$. Then $[a, x] = bc^{\pm 1}$ and $[a, y] = bc^{\pm 1}$. Since $[a, xy] \in \langle c^2 \rangle$ and $o(xy) \leq 4$, we see $xy \in K$, and so $x \in \langle K, y \rangle \leq L$, a contradiction. Hence $\Omega_2(M)$ is metacyclic, and therefore M is metacyclic. Then $G' \leq C_G(M')$. It follows from $G = M\langle a \rangle$ that $G' = \langle M', [a, M] \rangle \leq \langle M', bc \rangle$. So G' is abelian. □

Corollary 3.4. *Let G be a $\mathcal{R}_2\mathcal{I}^{>3}$-group. Then $d(\Omega_1(G)) \leq 3$ if and only if there exists a maximal subgroup M of G such that M is metacyclic.*

Proof. If $d(\Omega_1(G)) \leq 3$, then, by Theorems 2.14 and 3.3, there exists a maximal subgroup M of G such that M is metacyclic.

Conversely, if there exists a maximal subgroup M of G such that M is metacyclic, then $|\Omega_1(G) : M \cap \Omega_1(G)| \leq 2$ and $d(M \cap \Omega_1(G)) \leq 2$. It follows that $d(\Omega_1(G)) \leq 3$. □

Corollary 3.5. *Let G be a 2-group and $\Omega_1(G) \cong D_{2^n} * C_4$ with $n \geq 3$. If $d(G) = 2$, then $G' = [\Omega_1(G), G]$ is cyclic and $\Phi(G) \cap \Omega_1(G)$ is a maximal subgroup of $\Omega_1(G)$.*

Proof. By Theorem 3.3, there exists a maximal subgroup M of G such that M is metacyclic and $\Omega_1(G) \not\leq M$. Then $\Omega_1(G) \not\leq \Phi(G)$. We assume $Z(\Omega_1(G)) = \langle a \rangle$. Since G is not metacyclic and $d(G) = 2$, we see $a \in \Phi(G)$. It follows from Lemmas 3.1 and 3.2 that there exists an element $b \in \Omega_1(G)$ such that $[\Omega_1(G), G] \leq \langle b \rangle$, and $\langle a, b \rangle$ is a maximal subgroup of $\Omega_1(G)$. Then $\langle b \rangle \trianglelefteq G$, and so $b \in \Phi(G)$. Thus $\Phi(G) \cap \Omega_1(G) = \langle a, b \rangle$. So there exist $c \in \Omega_1(G) \setminus \langle a, b \rangle$ and $g \in G$ such that $G = \langle c, g \rangle$, which implies $G' = [\Omega_1(G), G]$ is cyclic. □

Corollary 3.6. *Let G be a 2-group and $\Omega_1(G) \cong D_{2^n} * C_4$ with $n \geq 3$. Then either G' is cyclic or $\Omega_1(G) \cap G'$ is a maximal subgroup of $\Omega_1(G)$.*

Proof. We may assume G' is not cyclic. Then Corollary 3.5 implies $d(G) = 3$ and $\Omega_1(G) < G$. By Theorem 3.3, there exists a maximal subgroup M of G such that M is metacyclic. Then $G = \Omega_1(G)M$. It follows from G' being not cyclic that M is not of maximal class. Then $\Omega_1(M) \cong C_2 \times C_2$ by Lemma 2.6. If $\Omega_1(G) \cap M \not\leq \Phi(M)$, then, by Lemmas 3.1 and 3.2, we see $G' = [\Omega_1(G), G]$ is cyclic. Thus $\Omega_1(G) \cap M \leq \Phi(M)$. We may assume $M = \langle x, y \rangle$, $\langle x \rangle \trianglelefteq M$ and $\Omega_1(G) = \langle a, b, c \mid a^{2^{n-1}} = b^2 = 1, a^{2^{n-2}} = c^2, [a, b] = a^{-2}, [c, a] = [c, b] = 1 \rangle$. Now, we consider the following two cases: $n \geq 4$ and $n = 3$.

Case 1. $n \geq 4$.

Since $a^2 \in M$ and $\Omega_1(M) \cong C_2 \times C_2$, we see $\Omega_1(G) \cap M = \Omega_1(G) \cap \Phi(G) = \langle a, c \rangle$. Then $\Omega_1(M) = \langle a^{2^{n-2}}, a^{2^{n-3}}c \rangle$. By Lemma 3.1, $[\Omega_1(G), G] \leq \langle a \rangle$. If $a \notin G'$, then, for any $g \in G$, we see $[b,g] \in \langle a^2\rangle$. Since $\Phi(G) = \Phi(M) = \langle x^2, y^2\rangle$, $[\langle b\rangle, \Phi(G)] \le \langle a^4\rangle$, which implies $[b,a] \in \langle a^4\rangle$, a contradiction. So $a \in G'$. It follows from G' being not cyclic that $\Omega_1(M) \le G'$. Thus, $\Omega_1(G) \cap G' = \Omega_1(G) \cap M$ is a maximal subgroup of $\Omega_1(G)$.

Case 2. $n = 3$.

In this case, $|\Omega_1(G)| = 16$. If $b \in M$, then, since $\Omega_1(M) \cong C_2 \times C_2$, we see $ab \notin M$. Without loss of generality, we may assume $b \notin M$. Since $|\Omega_1(G) \cap M| = 8$, we see $\Omega_1(G) \cap M$ is abelian. If $\Omega_1(G) \cap G' < \Omega_1(G) \cap M$, then $\Omega_1(G) \cap G' \cong C_2 \times C_2$. It follows from Lemma 3.2 that $[\Omega_1(G), G] \cong C_2$, which implies $[\Omega_1(G), \Phi(G)] = 1$. Thus $\langle b, \Omega_1(G) \cap M\rangle$ is abelian, and so $r(G) = 3$, in contradiction to the hypothesis. So $\Omega_1(G) \cap G' = \Omega_1(G) \cap M$ is a maximal subgroup of $\Omega_1(G)$. □

4. 2-groups with $\Omega_1(G) \cong D_{2^n} * Q_{2^m}$

Lemma 4.1. *Let G be a $\mathcal{R}_2\mathcal{I}^{>3}$-group of order 2^n. If G' is cyclic, then $d(\Omega_1(G)) \le 3$ or $G = \Omega_1(G) \cong D_{2^{n-2}} * Q_8$ with $n \ge 5$.*

Proof. If $d(\Omega_1(G)) > 3$, then $\Omega_1(G) \cong D_{2^m} * Q_8 \cong D_8 * Q_{2^m}$ with $m \ge 3$ by Lemma 2.1. If $\Omega_1(G) < G$, then, without loss of generality, we may assume $\Omega_1(G)$ is a maximal subgroup of G. Thus, we may assume $\Omega_1(G) \cong D_{2^{n-3}} * Q_8 = \langle a, b, c, d \mid a^{2^{n-4}} = b^2 = 1, c^2 = d^2 = a^{2^{n-5}}, [a,b] = a^{-2}, [c,d] = c^2, [a,c] = [a,d] = [b,c] = [b,d] = 1\rangle$. Now, we consider the following two cases: $n \ge 7$ and $n = 6$.

Case 1. $n \ge 7$.

In this case, $\langle a^{2^{n-6}}\rangle \operatorname{char} \langle a^2\rangle \trianglelefteq G$. Then $C_G(a^{2^{n-6}})$ is a maximal subgroup of G and $[\Omega_1(G), G] \le C_{\Omega_1(G)}(a^{2^{n-6}}) = \langle a, c, d\rangle$. By calculation, we see $[\langle a\rangle, G], [\langle b\rangle, G] \le \langle a\rangle$ and $[\langle c\rangle, G]$, $[\langle d\rangle, G] \le \langle c^2\rangle$. Noticing that $[c,d] = c^2$, we may take a suitable element $x \in G \setminus \Omega_1(G)$ such that $[c,x] = [d,x] = 1$. Then $x^2 \in C_G(a^{2^{n-6}}) \cap C_G(\langle c, d\rangle) \cap \Omega_1(G) = \langle a\rangle$. If $x^2 = a^{2i}$, then $[a^i, x] = 1$ or $a^{2^{n-5}}$. Thus $o(a^{-i}x) = 2$ if $[a^i, x] = 1$, and $o(a^{-i}cx) = 2$ if $[a^i, x] = a^{2^{n-5}}$, which implies $x \in \Omega_1(G)$, a contradiction. So we may assume $x^2 = a$. It follows from $[b, x^2] = a^2$ that $[b,x] = x^2$ or $x^{2+2^{n-4}}$. Then $o(bx) = 2$ if $[b,x] = x^2$, and $o(bcx) = 2$ if $[b,x] = x^{2+2^{n-4}}$, another contradiction.

Case 2. $n = 6$.

If $|G'| = 2$, then $\Phi(G) = \mho_1(G) \le Z(G)$. Thus $\Phi(G) = Z(G) \cap \Omega_1(G) = Z(\Omega_1(G)) = \langle a^2\rangle$, which implies $d(G) = 5$, in contradiction to Lemma 2.4. Thus

$|G'| = 4$. Assume $G' = \langle x\rangle$. Then $C_G(x)$ is a maximal subgroup of G by Theorem 2.9. It follows that $\Omega_1(G) \cap C_G(x) \cong D_8 * C_4$. We may assume $N = \Omega_1(G) \cap C_G(x) = \langle y, z\rangle * \langle x\rangle$, $w \in \Omega_1(G) \setminus C_G(x)$ and $g \in C_G(x) \setminus \Omega_1(G)$ with $o(y) = o(z) = o(w) = 2$. Then $G = \langle N, g\rangle\langle w\rangle$. Noticing that $[N, G] \leq \langle x^2\rangle$ and $[y, z] = x^2$, we may assume $[N, \langle g\rangle] = 1$. Thus $[w, g] = x^{\pm 1}$, and $g^2 \in Z(N) = \langle x\rangle$. Since $[w, g^2] \neq 1$, we may assume $g^2 = x$. If $[w, g] = x$, then $o(wg) = 2$, which implies $g \in \Omega_1(G)$, a contradiction. So $[w, g] = x^{-1}$. Then $o(wyg) = 2$ if $[w, y] = x^2$, $o(wzg) = 2$ if $[w, z] = x^2$, and $o(wyzg) = 2$ if $[w, y] = [w, z] = 1$, another contradiction. □

Theorem 4.2. *Let G be a 2-group and $\Omega_1(G) \cong D_{2^n} * Q_{2^m}$ with $n, m \geq 3$. Then one of the following holds:*

(1) *If G has more than one normal subgroup of order 4, then there exists a maximal subgroup M of G such that $d(\Omega_1(M)) = 3$.*

(2) *If G has the unique normal subgroup of order 4, then for any maximal subgroup M of G, $d(\Omega_1(M)) \neq 3$ and $n = m$.*

Proof. By Lemma 4.1, we may assume G' is not cyclic. Then $\Omega_1(G') \cong C_2 \times C_2$ by Lemma 2.11. We assume $\Omega_1(G) = \langle a, b, c, d \mid a^{2^{n-1}} = b^2 = 1, a^{2^{n-2}} = c^{2^{m-2}} = d^2, [a, b] = a^{-2}, [c, d] = c^{-2}, [a, c] = [a, d] = [b, c] = [b, d] = 1\rangle$.

If there exists $N \trianglelefteq G$ such that $N \neq \Omega_1(G')$ and $|N| = 4$, then, by Lemma 2.12, $N \cong C_4$. Thus $C_G(N)$ is a maximal subgroup of G by Theorem 2.9. If $d(\Omega_1(C_G(N))) \neq 3$, then $\Omega_1(C_G(N)) \cong C_2 \times C_2$ by Lemma 2.1 and Theorem 2.9. Assume $N = \langle x\rangle$. Then $\langle x^2\rangle \leq Z(G) \cap \Omega_1(G)$ and $x^2 = a^{2^{n-2}} = c^{2^{m-2}} = d^2$. Thus there exists an element $g \in \langle c^{2^{m-3}}, d\rangle \setminus \langle d^2\rangle$ such that $[x, g] = 1$. Without loss of generality, we may assume $[c^{2^{m-3}}, x] = 1$, which implies $o(c^{2^{m-3}}x) = 2$. Since $\Omega_1(C_G(N)) \cong C_2 \times C_2$ and $\Omega_1(\langle a, b\rangle) = \langle a, b\rangle \leq C_G(\langle c, d\rangle)$, we see $x \notin \langle c, d\rangle$. Then $\Omega_1(C_G(N)) = \langle x^2, c^{2^{m-3}}x\rangle$. It follows that $[d, x] = x^2$. If $[a^{2^{n-3}}, x] = x^2$, then $a^{2^{n-3}}d \in \Omega_1(C_G(N))$, a contradiction. If $[a^{2^{n-3}}, x] = 1$, then $a^{2^{n-3}}x \in \Omega_1(C_G(N))$, and so $\langle x\rangle = \langle a^{2^{n-3}}\rangle$, in a contradiction to $[d, x] = x^2$. Thus $d(\Omega_1(C_G(N))) = 3$.

If G has the unique normal subgroup N such that $|N| = 4$, then $N = \Omega_1(G') \cong C_2 \times C_2$. If there exists a maximal subgroup M of G such that $d(\Omega_1(M)) = 3$, then, by Lemma 2.1, we see $Z(\Omega_1(M)) \cong C_4$ and $Z(\Omega_1(M)) \trianglelefteq G$, a contradiction. If $n \neq m$, then we may assume $n > m$, and so $\mho_{n-3}(\Omega_1(G)) \cong C_4$, a contradiction. Hence, for any maximal subgroup M of G, $d(\Omega_1(M)) \neq 3$ and $n = m$. □

Theorem 4.3. *Let G be a 2-group and $\Omega_1(G) \cong D_{2^n} * Q_{2^m}$ with $n, m \geq 3$.*

If G has more than one normal subgroup of order 4, then G' is abelian and $\Phi(G) < \Omega_1(G)$.

Proof. If G' is cyclic, then the result is clear by Lemma 4.1. We assume G' is not cyclic, and so $\Omega_1(\Phi(G)) \cong C_2 \times C_2$ by Lemma 2.11. It follows from Lemma 2.12 that there exists $N \trianglelefteq G$ such that $N \cong C_4$. Then $C_G(N)$ is a maximal subgroup of G by Theorem 2.9 and $\Omega_1(\Phi(G))N \cong C_4 \times C_2$. By Theorem 4.2, there exists a maximal subgroup M of G such that $d(\Omega_1(M)) = 3$. Then $G = M\Omega_1(G)$ and $G' = \langle M', [G, \Omega_1(G)]\rangle$. Now, we may assume $N = \langle x\rangle$ and $\Omega_1(G) = \langle a, b, c, d \mid a^{2^{n-1}} = b^2 = 1, a^{2^{n-2}} = c^{2^{m-2}} = d^2, [a, b] = a^{-2}, [c, d] = c^{-2}, [a, c] = [b, c] = [a, d] = [b, d] = 1\rangle$. We consider the two cases: $n \geq 4$ and $n = 3$.

Case 1. $n \geq 4$.

If $m \geq 4$, then $\Omega_1(\Phi(G)) = \langle a^{2^{n-2}}, a^{2^{n-3}}c^{2^{m-3}}\rangle$. Since $x^2 = a^{2^{n-2}}$ and $a^{2^{n-3}} \in C_G(x)$, we see $o(a^{2^{n-3}}x) = 2$, and so $x \in \Omega_1(G)$. Thus $\langle x\rangle = \langle a^{2^{n-3}}\rangle$ or $\langle c^{2^{m-3}}\rangle$. It follows that $\langle c^{2^{m-3}}\rangle \trianglelefteq G$ and $\langle a^{2^{n-3}}\rangle \trianglelefteq G$. Then $C_G(a^{2^{n-3}})$ and $C_G(c^{2^{m-3}})$ are maximal subgroups of G.

If $m = 3$, then $\langle a^{2^{n-3}}\rangle \operatorname{char} \langle a^2\rangle \trianglelefteq G$. Thus $C_G(a^{2^{n-3}})$ is a maximal subgroup of G. Without loss of generality, we may assume $\Omega_1(\Phi(G)) = \langle a^{2^{n-2}}, a^{2^{n-3}}c\rangle$. It follows that $\langle c\rangle \trianglelefteq G$ and $C_G(c)$ is a maximal subgroup of G.

In either case, we see $C_G(a^{2^{n-3}})$ and $C_G(c^{2^{m-3}})$ are maximal subgroups of G. Then $[\Omega_1(G), G] \leq \langle a, c\rangle$. Thus $G' = \langle M', [G, \Omega_1(G)]\rangle \leq \langle M', a, c\rangle$. By calculation, we see $[\langle a\rangle, G], [\langle b\rangle, G] \leq \langle a\rangle$ and $[\langle c\rangle, G], [\langle d\rangle, G] \leq \langle c\rangle$. So $\langle a\rangle \trianglelefteq G$ and $\langle c\rangle \trianglelefteq G$, which implies $G' \leq C_G(a) \cap C_G(c)$. Noticing that $d(\Omega_1(M)) = 3$, we see $r(M) = 2$, and therefore there is a positive integer t with $t \geq 3$ such that $\Omega_1(M) \cong D_{2^t} * C_4$ by Lemma 2.1. Now using Theorem 3.3, we see M' is abelian, and so G' is abelian.

If there exists an element $g \in \Phi(G) \setminus \Omega_1(G)$, then $[b, g] \in \langle a^2\rangle$. Assume $[b, g] = a^{2i}$. Then $[b, ga^{-i}] = 1$ and $ga^{-i} \in \Phi(G) \setminus \Omega_1(G)$. So we may assume $[b, g] = 1$. Similarly, we may assume $[d, g] = 1$. Then $g \in C_G(\langle a^{2^{n-3}}, c^{2^{m-3}}\rangle) \cap C_G(\langle b, d\rangle) \leq C_G(\Omega_1(G))$. By Corollary 2.10, $g \in Z(\Omega_1(G))$, a contradiction. So $\Phi(G) \leq \Omega_1(G)$. It follows from $\Omega_1(G) \nleq M$ that $\Phi(G) < \Omega_1(G)$.

Case 2. $n = 3$.

Since $D_{2^m} * Q_8 \cong D_8 * Q_{2^m}$, we assume $m = 3$, and so $x^2 = c^2 = d^2$. Then there exists an element $y \in \langle c, d\rangle \setminus \langle c^2\rangle$ such that $x^2 = y^2$ and $[x, y] = 1$. Thus $o(xy) = 2$, and so $x \in \Omega_1(G)$. By Lemma 2.3, we see $\Omega_1(G)$ has no abelian maximal subgroup. Then $C_{\Omega_1(G)}(\Omega_1(\Phi(G))N) = \Omega_1(\Phi(G))N \cong C_4 \times C_2$.

It follows from $G' \le C_G(N) \cap C_G(\Omega_1(\Phi(G)))$ that $[\Omega_1(G), G] \le \Omega_1(\Phi(G))N$. Thus $G' = \langle M', [G, \Omega_1(G)]\rangle \le \langle M', \Omega_1(\Phi(G)), N\rangle$, and therefore G' is abelian.

It follows from $[\Omega_1(G), G] \le \Omega_1(\Phi(G))N \cong C_4 \times C_2$ and $[\Omega_1(\Phi(G))N, G] \le \langle x^2\rangle$ that $[\Omega_1(G), \Phi(G)] \le \langle x^2\rangle$. If there exists an element $g \in \Phi(G) \setminus \Omega_1(G)$, then $g^2 \in C_G(\Omega_1(G)) = Z(\Omega_1(G))$ and $g^2 = c^2 = d^2$. Thus there exists an element $h \in \langle c, d\rangle \setminus \langle c^2\rangle$ such that $[g, h] = 1$ and $g^2 = h^2$. Then $o(gh) = 2$ and $g \in \Omega_1(G)$, a contradiction. So $\Phi(G) < \Omega_1(G)$. □

Theorem 4.4. *Let G be a 2-group and $\Omega_1(G) \cong D_{2^n} * Q_{2^m}$ with $n, m \ge 3$. If G has more than one normal subgroup of order 4, then $|G : \Omega_1(G)| \le 4$.*

Proof. If G' is cyclic, then the conclusion holds by Lemma 4.1. Thus we may assume G' is not cyclic. By Theorem 4.2, there exists a maximal subgroup M of G such that $d(\Omega_1(M)) = 3$. Then $\Omega_1(M) \le \Omega_1(G) \cap M < \Omega_1(G)$. It follows from Theorem 2.8 that $Z(\Omega_1(G) \cap M)$ is cyclic. So $\Phi(G) < \Omega_1(G) \cap M < \Omega_1(G)$ by Lemma 2.7 and Theorem 4.3. Hence $|G : \Omega_1(G)| \le 4$ by Lemma 2.4. □

Theorem 4.5. *Let G be a 2-group and $\Omega_1(G) \cong D_{2^n} * Q_{2^m}$ with $n, m \ge 3$. If G has the unique normal subgroup of order 4, then*

(1) *G' is not abelian;*

(2) *$|G : \Omega_1(G)| \le 8$;*

(3) *either $\Phi(G) \le \Omega_1(G)$ or $|\Omega_1(G)| = |\Phi(G)|$ and $\Omega_1(G) \cap G'$ is a maximal subgroup of $\Omega_1(G)$.*

Proof. By Lemma 4.1 and Lemma 2.11, G' is not cyclic and $\Omega_1(\Phi(G)) \cong C_2 \times C_2$. Then $C_G(\Omega_1(\Phi(G)))$ is a maximal subgroup of G by Theorem 2.8 and $[\Omega_1(G), G] \le C_{\Omega_1(G)}(\Omega_1(\Phi(G)))$. By Theorem 4.2, we may assume $\Omega_1(G) = \langle a, b, c, d \mid a^{2^{n-1}} = b^2 = 1, a^{2^{n-2}} = c^{2^{n-2}} = d^2, [a, b] = a^{-2}, [c, d] = c^{-2}, [a, c] = [b, c] = [a, d] = [b, d] = 1\rangle$. We consider the two cases: $n \ge 4$ and $n = 3$.

Case 1. $n \ge 4$.

In this case, $\Omega_1(\Phi(G)) = \langle a^{2^{n-2}}, a^{2^{n-3}} c^{2^{n-3}}\rangle$ and $[\Omega_1(G), G] \le \langle a, c, bd\rangle$. By the hypotheses of the theorem, $\langle a^{2^{n-3}}\rangle \ntrianglelefteq G$ and $\langle c^{2^{n-3}}\rangle \ntrianglelefteq G$. For any $x \in G \setminus N_G(a^{2^{n-3}})$, by calculation, we see $[a^{2^{n-3}}, x] = a^{\pm 2^{n-3}} c^{2^{n-3}}$, $[c^{2^{n-3}}, x] = a^{\pm 2^{n-3}} c^{2^{n-3}}$, $[b, x] = a^{\pm 2^{n-3}} c^i bd$ and $[d, x] = c^{\pm 2^{n-3}} a^j bd$. It follows that $ac \in G'$, which implies G' is not abelian. For any $y \in N_G(a^{2^{n-3}})$, by calculation, we see $[b, y] \in \langle a\rangle$ and $[c, y] \in \langle c\rangle$.

Clearly, $\Phi(G) \le N_G(a^{2^{n-3}}) \cap N_G(c^{2^{n-3}})$. It follows that $[\langle b\rangle, \Phi(G)], [\langle ab\rangle, \Phi(G)] \le \langle a\rangle$ and $[\langle d\rangle, \Phi(G)], [\langle cd\rangle, \Phi(G)] \le \langle c\rangle$. Then $\Phi(G) \le N_G(a) \cap N_G(c)$.

Take $g \in G \setminus N_G(a^{2^{n-3}})$. Then $[a^{2^{n-3}}, g] = a^{\pm 2^{n-3}} c^{2^{n-3}}$. If there exists an element $z \in G \setminus N_G(a^{2^{n-3}})\langle g\rangle$, then $[a^{2^{n-3}}, gz] \in \langle a^{2^{n-2}}\rangle$, and so $gz \in N_G(a^{2^{n-3}})$, a contradiction. Thus $G = N_G(a^{2^{n-3}})\langle g\rangle$. It follows from $g^2 \in N_G(a^{2^{n-3}})$ that $N_G(a^{2^{n-3}})$ is a maximal subgroup of G. Since $\Omega_1(G) \le N_G(a^{2^{n-3}})$, we see $|N_G(a^{2^{n-3}}) : \Omega_1(N_G(a^{2^{n-3}}))| = |N_G(a^{2^{n-3}}) : \Omega_1(G)| \le 4$ by Theorem 4.4, which implies $|G : \Omega_1(G)| \le 8$.

If $\Phi(G) \nleq \Omega_1(G)$, then there exists an element $w \in \Phi(G)$ such that $w \notin \Omega_1(G)$. If $[a^{2^{n-3}}, w] = a^{2^{n-2}}$, then $[a^{2^{n-3}}, a^j bdw] = 1$. Since $a^j bdw \in \Phi(G) \setminus \Omega_1(G)$, we may assume $[a^{2^{n-3}}, w] = 1$. Since $w \in C_G(\Omega_1(\Phi(G)))$, $[c^{2^{n-3}}, w] = 1$. It follows that $\langle a, c\rangle \cap \langle w\rangle \ne 1$. Now, we may assume $\langle a, c\rangle \cap \langle w\rangle = w^{2^s}$.

If $[b, w] = 1$, then $w^{2^s} \in \langle c\rangle$. If $w^{2^s} = c^{2k}$, then $[c^k, w] = 1$ or $c^{2^{n-2}}$. Thus $o(c^{-k}w^{2^{s-1}}) = 2$ or $o(c^{2^{n-3}-k}w^{2^{s-1}}) = 2$, which implies $w^{2^{s-1}} \in C_{\Omega_1(G)}(\langle a^{2^{n-3}}, c^{2^{n-3}}\rangle) = \langle a, c\rangle$, a contradiction. So we assume $w^2 = c$. It follows from $[d, w^2] = c^2$ that $[d, w] = c$ or $c^{1+2^{n-2}}$. Then $o(dw) = 2$ or $o(da^{2^{n-3}}w) = 2$, another contradiction. So $[b, w] \ne 1$. Similarly, $[d, w] \ne 1$. If $[b, w] = a^{2l}$, then $[b, w(ac)^{-l}] = 1$, a contradiction. So $[b, w] = a^t$ with $2 \nmid t$ and $[d, w] = c^r$ with $2 \nmid r$. It follows that $\langle a, c, bd\rangle \le G'$ and $w^2 = a^u c^v$, where $2 \nmid u$ and $2 \nmid v$. If there exists an element $w_1 \in \Phi(G) \setminus \langle w, c, bd\rangle$, then $[b, w_1] = a^{t_1}$ with $2 \nmid t_1$ and $[b, ww_1] \in \langle a^2\rangle$. By the above, we see that $ww_1 \in \Phi(G) \cap \Omega_1(G) = \langle a, c, d\rangle$, a contradiction. Thus $\Phi(G) = \langle w, c, bd\rangle$. So $|\Omega_1(G)| = |\Phi(G)|$, and $\langle a, c, bd\rangle = \Omega_1(G) \cap \Phi(G) = \Omega_1(G) \cap G'$ is a maximal subgroup of $\Omega_1(G)$.

Case 2. $n = 3$.

Let $H = [\Omega_1(G), G]$ and $N = \Omega_1(G) \cap C_G(\Omega_1(\Phi(G)))$. Then N is a maximal subgroup of $\Omega_1(G)$. Without loss of generality, we may assume $\Omega_1(\Phi(G)) = \langle a^2, ac\rangle$ or $\langle a^2, b\rangle$.

Subcase 1. $\Omega_1(\Phi(G)) = \langle a^2, ac\rangle$.

In this case, $N = \langle a, c, bd\rangle$. If $H < N$, then H is abelian by Lemma 2.7. Thus $H \le \langle a, c\rangle$ or $\langle ac, bd\rangle$. If $H \le \langle a, c\rangle$, then $[\langle b\rangle, G] \le \langle a\rangle$ and $[\langle ab\rangle, G] \le \langle a\rangle$, which implies $\langle a\rangle \trianglelefteq G$, a contradiction. If $H \le \langle ac, bd\rangle$, then $[\langle b\rangle, G], [\langle d\rangle, G] \le \langle acbd\rangle$. Since $(bd)^2 \in Z(G)$, we see $[\langle bd\rangle, G] \le \langle d^2\rangle$, and so $\langle bd\rangle \trianglelefteq G$, another contradiction. So $H = N$, which implies $\Omega_1(G) \cap G'$ is a maximal subgroup of $\Omega_1(G)$, and G' is not abelian.

It is easy to see that $[\Omega_1(G), \Phi(G)] \le [\Omega_1(G), G, G] < H$. Thus $[\Omega_1(G), G, G]$ is abelian by Lemma 2.7, and so $[\Omega_1(G), G, G] \le \langle a, c\rangle$ or $\langle ac, bd\rangle$.

If $[\Omega_1(G), G, G] \le \langle a, c\rangle$, then $\Phi(G) \le N_G(a) \cap N_G(c) \cap C_G(ac)$, $[\langle b\rangle, \Phi(G)] \le \langle a\rangle$, and $[\langle d\rangle, \Phi(G)] \le \langle c\rangle$. Since $a \in H$, we see $[\langle a\rangle, G] \le \langle a, c\rangle$, and so $[\langle a\rangle, G] \le \langle a^2, ac\rangle$. It follows that $N_G(a)$ is a maximal subgroup of G. Thus $|G : \Omega_1(G)| \le 8$ by Theorem 4.4. If there exists an element $x \in \Phi(G)$ such that $x \notin \Omega_1(G)$, then we may assume $[a, x] = [c, x] = 1$, $[b, x] = 1$ or a, and $[d, x] = 1$ or c. Thus $x^4 \in C_G(\Omega_1(G))$. By Corollary 2.10, $x^4 = a^2$. It follows that $x^2 \in \Omega_1(G) \cap \Phi(G) \setminus \langle a^2\rangle$.

If $[b,x]=1$, then $x^2=c^{\pm 1}$ and $[d,x]=c$. Thus $o(dax)=2$ if $x^2=c$, and $o(dx)=2$ if $x^2=c^{-1}$, a contradiction. So $[b,x]=a$. If there exists an element $y\in\Phi(G)\setminus\langle x,H\rangle$, then we may assume $[b,y]=a$, and so $[b,xy]\in\langle a^2\rangle$, which implies $xy\in H$, a contradiction. So $\Phi(G)=\langle x,H\rangle$ and $|\Omega_1(G)|=|\Phi(G)|$.

If $[\Omega_1(G),G,G]\leq\langle ac,bd\rangle$, then $[\langle b\rangle,\Phi(G)]\leq\langle acbd\rangle$ and $[N,G]\leq\langle ac,bd\rangle$. Thus $[\langle bd\rangle,G]\leq\langle a^2,ac\rangle$, and so $N_G(bd)$ is a maximal subgroup of G. Then $|G:\Omega_1(G)|\leq 8$ by Theorem 4.4. If there exists an element $x\in\Phi(G)\setminus\Omega_1(G)$, then, since $N\leq\Phi(G)$, we assume $[bd,x]=1$. It is easy to see that $\mho_2(\Phi(G))\leq C_G(\Omega_1(G))$. Then $x^4\in\mho_2(\Phi(G))\leq\langle(bd)^2\rangle$ by Corollary 2.10. It follows that $x^2\in N\setminus Z(\Omega_1(G))$. If $[b,x]=1$, then $[d,x]=1$ and $x^2=bd^{\pm1}$. It follows from $[(ab)^2,x]=1$ that $[a,x]=bd^{\pm1}$, and we may assume $[a,x]=bd$. Then $o(abx)=2$ if $x^2=bd$, and $o(ax)=2$ if $x^2=bd^{-1}$, a contradiction. It follows that $[b,x]=abcd^{\pm1}$. If there exists an element $y\in\Phi(G)\setminus\langle x,N\rangle$, then $[b,y]=abcd^{\pm1}$. Since $[b,xy]\in\langle a^2\rangle$, we see $xy\in\Phi(G)\cap\Omega_1(G)=N$, a contradiction. So $\Phi(G)=\langle x,N\rangle$. It follows from $x^2\in N$ that $|\Omega_1(G)|=|\Phi(G)|$.

Subcase 2. $\Omega_1(\Phi(G))=\langle a^2,b\rangle$.

In this case, $N=\langle b,c,d\rangle$ and $\Phi(G)\leq C_G(b)$. If $H<N$, then H is abelian, and we may assume $H\leq\langle b,c\rangle$. By calculation, we see $[\langle a\rangle,G]\leq\langle bc\rangle$ and $[\langle c\rangle,G]\leq\langle b,c^2\rangle$. Since $(ac)^2\in Z(G)$, we see $[\langle c\rangle,G]\leq\langle c^2\rangle$, which implies $\langle c\rangle\trianglelefteq G$, a contradiction. So $H=N=\Omega_1(G)\cap G'$, and G' is not abelian.

Since $[\Omega_1(G),\Phi(G)]\leq[\Omega_1(G),G,G]<H$, we see $[\Omega_1(G),G,G]$ is abelian, and we may assume $[\Omega_1(G),G,G]\leq\langle b,c\rangle$. Then $[\langle a\rangle,\Phi(G)]\leq\langle bc\rangle$ and $[N,G]\leq\langle b,c\rangle$. Thus $[\langle b\rangle,G]\leq\langle c^2\rangle$, $[\langle c\rangle,G]\leq\langle b,c^2\rangle$, and so $N_G(c)$ is a maximal subgroup of G. Then $|G:\Omega_1(G)|\leq 8$ by Theorem 4.4. It is easy to see that $[\langle d\rangle,\Phi(G)]\leq\langle b,c^2\rangle$. Then $\mho_2(\Phi(G))\leq C_G(\Omega_1(G))=Z(\Omega_1(G))$ and $\mho_1(\Phi(G))\leq C_G(\langle b,c,d\rangle)$. If there exists an element $x\in\Phi(G)\setminus\Omega_1(G)$, then we may assume $[c,x]=1$. Since $x^4\in Z(\Omega_1(G))=\langle c^2\rangle$, we see $x^2\in N\setminus Z(\Omega_1(G))$, which implies $[a,x^2]\neq 1$, and so $[a,x]=bc^{\pm1}$. It follows that $\Phi(G)=\langle x,N\rangle$ and $|\Omega_1(G)|=|\Phi(G)|$. □

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