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Gorenstein flat and projective (pre)covers

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Abstract. We consider a left GF-closed ring R. We prove that the class of Gorenstein flat complexes is covering in the category of complexes of left R-modules $\mathrm{Ch}(R)$. When R is right coherent and left n-perfect, we prove that the class of Gorenstein projective complexes is special precovering in $\mathrm{Ch}(R)$.

1. Introduction

Gorenstein homological algebra is the relative version of homological algebra that uses Gorenstein projective, Gorenstein flat and Gorenstein injective modules instead of the classical projective, flat and injective modules. But while the classical projective and injective resolutions are known to exist over arbitrary rings, things are different when it comes to the existence of the Gorenstein projective and Gorenstein injective resolutions. Their existence is well known over Gorenstein rings. But for arbitrary rings this is still an open question.

We focus on the existence of the Gorenstein projective precovers. As we already mentioned, their existence over Gorenstein rings is known (it was proved by Enochs and Jenda in 2000, in [8]). Then, in 2007, Jørgensen proved ([17]) that the class of Gorenstein projective modules is precovering over any commutative Noetherian ring with a dualizing complex. More recently (2011), Murfet and Salarian showed the existence of the Gorenstein projective precovers over

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any commutative Noetherian ring of finite Krull dimension. In a recent preprint, Bravo, Gillespie and Hovey proved [3, Proposition 8.10] that, over any ring R, the class of Gorenstein AC-projectives is the left half of a cotorsion pair cogenerated by a set, and so, it is precovering in R-Mod. In particular, if R is right coherent and left n-perfect for some integer $n \geq 0$, then the class of Gorenstein AC-projectives coincides with that of the Gorenstein projectives, $\mathcal{GP}$. Thus $\mathcal{GP}$ is precovering over such rings.

Our main result (Theorem 1) shows that if R is a right coherent ring that is also left n-perfect, then the class of Gorenstein projective complexes is special precovering in the category of complexes of left R-modules, $\mathrm{Ch}(R)$. In particular, this implies that every left R-module M has a special Gorenstein projective precover in this case (Corollary 2). Examples of such rings include but are not limited to: Gorenstein rings, commutative Noetherian rings of finite Krull dimension, as well as two-sided Noetherian rings R such that $\mathrm{id}_R R < \infty$.

Our Theorem 1 recovers a result of Gillespie [15, Corollary 8.3], but our methods are different from those of [3] and [15]. Also, the new proofs that we provide are more direct and much shorter.

We also consider the question of the existence of the Gorenstein flat covers for complexes. It has been proved recently ([23]) that the class of Gorenstein flat modules is precovering over any associative ring with unity. However, when it comes to complexes of modules, the best result is the following: if R is a two-sided Noetherian ring, then every complex of R-modules has a Gorenstein flat cover ([6]). We show here (Proposition 4) that the class of Gorenstein flat complexes is covering over any left GF-closed ring R. The proof of our main result (Theorem 1) relies on Proposition 4. In fact, Theorem 1 is essentially an application of Proposition 4.

2. Preliminaries

Throughout this section, R denotes an associative ring with unity.

Definition 1 ([8, Definition 10.2.1]). An R-module M is Gorenstein projective if there exists an exact sequence of projective R-modules

$$\cdots \to P_1 \to P_0 \to P_{-1} \to \cdots$$

such that the sequence

$$\cdots \to \mathrm{Hom}(P_{-1}, P) \to \mathrm{Hom}(P_0, P) \to \mathrm{Hom}(P_1, P) \to \cdots$$

is still exact for any projective R-module P and such that $M = \mathrm{Ker}(P_0 \to P_{-1})$.

We will use the notation $\mathcal{GP}$ for the class of Gorenstein projective modules.

By replacing "projective module" with "projective complex" in the above definition, we obtain the definition of a Gorenstein projective complex in the category of complexes of left R-modules, $\mathrm{Ch}(R)$. We recall that a complex P is projective if P is exact and if each $Z_n(P)$ is a projective module.

It is known ([20]) that over any ring R a complex C is Gorenstein projective if and only if each C_n is a Gorenstein projective R-module.

The Gorenstein flat modules are defined in terms of the tensor product.

Definition 2 ([8, Definition 10.3.1]). A left R-module G is Gorenstein flat if there exists an exact sequence of flat left R-modules

$$\cdots \to F_1 \to F_0 \to F_{-1} \to \cdots$$

such that the sequence

$$\cdots \to I \otimes F_1 \to I \otimes F_0 \to I \otimes F_{-1} \to \cdots$$

is still exact for any injective right R-module I and such that $G = \mathrm{Ker}(F_0 \to F_{-1})$.

We will use the notation $\mathcal{GF}$ for the class of Gorenstein flat modules.

The Gorenstein flat complexes were defined in [14]. We recall that if C is a complex of right R-modules and D is a complex of left R-modules, then the usual tensor product complex of C and D is the complex of Z-modules $C \otimes^{\cdot} D$ with $(C \otimes^{\cdot} D)_n = \oplus_{t \in Z}(C_t \otimes_R D_{n-t})$ and differentials

$$\delta(x \otimes y) = \delta_t^C(x) \otimes y + (-1)^t x \otimes \delta_{n-t}^D(y),$$

for $x \in C_t$ and $y \in D_{n-t}$.

In [14], García Rozas introduced another tensor product: if C is again a complex of right R-modules and D is a complex of left R-modules, then $C \otimes D$ is defined to be $\frac{C \otimes^{\cdot} D}{B(C \otimes^{\cdot} D)}$. Then with the maps

$$\frac{(C \otimes^{\cdot} D)_n}{B_n(C \otimes^{\cdot} D)} \to \frac{(C \otimes^{\cdot} D)_{n-1}}{B_{n-1}(C \otimes^{\cdot} D)}$$

$x \otimes y \to \delta_C(x) \otimes y$, where $x \otimes y$ is used to denote the coset in $\frac{C \otimes^{\cdot} D}{B(C \otimes^{\cdot} D)}$, we get a complex. This is the tensor product used to define Gorenstein flat complexes.

We recall that a complex F is flat if F is exact and $Z_n(F)$ is a flat module for any integer n.

Definition 3 ([14, Definition 5.4.1]). A complex G of left R-modules is Gorenstein flat if there exists an exact and Inj $\otimes$-exact sequence of flat complexes (of left R-modules)

$$\cdots \to F_1 \to F_0 \to F_{-1} \to \cdots$$

such that $G = \mathrm{Ker}(F_0 \to F_{-1})$.

It is known [22, Theorem 3.11] that over a left GF-closed ring R, a complex G is Gorenstein flat if and only if each module G_n is Gorenstein flat.

3. Gorenstein flat covers for complexes

We prove first the existence of the Gorenstein flat covers in the category of complexes, $\mathrm{Ch}(R)$, over any left GF-closed ring R.

We will use the following results:

Proposition 1 ([10, Proposition 2.10]). *For any ring R, the class of Gorenstein flat complexes is a Kaplansky class.*

Proposition 2 ([19, Lemma 3.1]). *If R is a left GF-closed ring, then the class of Gorenstein flat left R-modules is closed under direct limits.*

We will use the notation $\mathrm{Ch}(\mathcal{GF})$ for the class of complexes of Gorenstein flat left R-modules.

Our first result is:

Proposition 3. *If R is a left GF-closed ring, then the inclusion functor $\boldsymbol{K}(\mathcal{GF}) \to \boldsymbol{K}(R)$ has a right adjoint.*

Proof. By [19, Theorem 3.4], $(\mathcal{GF}, \mathcal{GF}^{\perp})$ is a perfect cotorsion pair. By Propositions 1 and 2, the cotorsion pair $(\mathcal{GF}, \mathcal{GF}^{\perp})$ is cogenerated by a set. Then, by [9, Theorem 7.2.14], the pair $(\mathrm{Ch}(\mathcal{GF}), \mathrm{Ch}(\mathcal{GF})^{\perp})$ is a cotorsion pair cogenerated by a set. Then, again by [9, Theorem 5.1.7], the inclusion functor $\mathbf{K}(\mathcal{GF}) \to \mathbf{K}(R)$ has a right adjoint. $\square$

We show that if R is a left GF-closed ring, then the class of Gorenstein flat complexes (of left R-modules) is covering in $\mathrm{Ch}(R)$.

Proposition 4. *Let R be a left GF-closed ring. Then every complex of left R-modules has a Gorenstein flat cover.*

PROOF. Since the cotorsion pair $(\mathrm{Ch}(\mathcal{GF}), \mathrm{Ch}(\mathcal{GF})^{\perp})$ is cogenerated by a set, it is a complete one. So the class $\mathrm{Ch}(\mathcal{GF})$ is special precovering. By Proposition 2, this class is also closed under direct limits, and therefore it is a covering class in $\mathrm{Ch}(R)$, see [8, Theorem 7.2.6]. By [22, Theorem 3.11], this is the class of Gorenstein flat complexes. □

Since every right coherent ring is a left GF-closed ring, we obtain, in particular, the existence of the Gorenstein flat covers in $\mathrm{Ch}(R)$ for any right coherent ring R.

4. Gorenstein projective (pre)covers

We recall that a ring R is left n-perfect if for any flat left R-module F, $p.d._R F \leq n$.

Left perfect rings, commutative Noetherian rings of finite Krull dimension, the universal enveloping algebra $\mathcal{U}(g)$ of a Lie algebra of dimension n are all examples of left n-perfect rings.

We also recall that the character module of a left R-module M is the right R-module $M^+ = \mathrm{Hom}_Z(M, Q/Z)$. Then $M^{++} = (M^+)^+$, for any ${}_RM$. In the following, we use the notation Flat^{++} for the class of all left R-modules of the form C^{++}, where C is any flat left R-module.

We prove that the class of Gorenstein projective modules is special precovering over a right coherent ring R that is left n-perfect.

We begin with the following:

Lemma 1. *Let R be a right coherent ring that is left n-perfect, and let F be a flat left R-module. Then there exists an exact sequence*

$$0 \to F \to S^0 \to S^1 \to \cdots \to S^n \to C \to 0$$

with all S^i in Flat^{++} and with C pure injective and flat.

PROOF. Since R is right coherent, we have that a module F is flat if and only if F^{++} is flat [1, Theorem 1].

The sequence $0 \to F \to F^{++} \to \frac{F^{++}}{F} \to 0$ is pure exact with F^{++} flat. Therefore, the module $\frac{F^{++}}{F}$ is flat. Repeating, we obtain an exact complex

$$0 \to F \to S^0 \to S^1 \to S^2 \cdots$$

with each S^i in Flat^{++} and with each $C^i = \mathrm{Im}(S^i \to S^{i+1})$ flat.

Let K be any flat R-module. The exact sequence $0 \to F \to S^0 \to C^0 \to 0$ gives a long exact sequence $\mathrm{Ext}^1(K, S^0) \to \mathrm{Ext}^1(K, C^0) \to \mathrm{Ext}^2(K, F) \to \mathrm{Ext}^2(K, S^0) \to \cdots$.

Since K is flat and S^0 is in Flat^{++} hence pure injective by [8, Proposition 5.3.7], we have $\mathrm{Ext}^i(K, S^0) = 0$ for all $i \geq 1$. Therefore, $\mathrm{Ext}^j(K, C^0) \simeq \mathrm{Ext}^{j+1}(K, F)$ for all $j \geq 1$.

Similarly, $\mathrm{Ext}^j(K, C^{n-1}) \simeq \mathrm{Ext}^{j+n}(K, F)$ for all $j \geq 1$. But the ring R is left n-perfect and K is flat, so $p.d._R(K) \leq n$. Then $\mathrm{Ext}^{j+n}(K, F) = 0$ for all $j \geq 1$.

So $\mathrm{Ext}^j(K, C^{n-1}) = 0$ for all $j \geq 1$ and for any flat module K.

In particular, we have that $\mathrm{Ext}^1(C^n, C^{n-1}) = 0$. This means that the exact sequence $0 \to C^{n-1} \to S^n \to C^n \to 0$ is split exact. Thus C^n is a direct summand of $S^n \in \mathrm{Flat}^{++}$, and so $C = C^n$ is pure injective. □

We will also use the following result:

Lemma 2 ([16, Proposition 3.22])**.** *Assume that R is right coherent. If T is a (left) Gorenstein flat R-module, then* $\mathrm{Ext}^i(T, K) = 0$ *for all integers $i > 0$, and for all cotorsion R-modules K with finite flat dimension.*

We recall that, by [18], an exact complex C of flat R-modules is *F-totally acyclic* if $E \otimes C$ is still exact for any injective R-module E. In particular, if C is F-totally acyclic, then for each integer n, $Z_n(C)$ is Gorenstein flat.

Other authors call such a complex a *complete flat resolution* (see, for example, [5, Definition 5.1.1].

Lemma 3. *Let R be a right coherent ring that is left n-perfect. If C is an F-totally acyclic complex of projective modules, then C is* $\mathrm{Hom}(-, Q)$*-exact for any flat R-module Q.*

Proof. Let Q be any flat R-module. By Lemma 1, there is an exact complex $0 \to Q \to S^0 \to S^1 \to \cdots \to S^{n+1} \to 0$ with each S^i pure injective and flat.

Let $M = Z_0(C)$. Since M is Gorenstein flat, we have that $\mathrm{Ext}^j(M, K) = 0$ for any flat and cotorsion module K, and for any $j \geq 1$ (by Lemma 2). In particular, $\mathrm{Ext}^j(M, S^i) = 0$ for all $j \geq 1$, for all $0 \leq i \leq n+1$. So by shifting dimensions with the above exact sequence, we have that $\mathrm{Ext}^i(M, Q) = 0$ for all $i \geq n+2$.

But if $T = Z_{n+1}(C)$, then since each module C_i is projective, we have that $\mathrm{Ext}^j(M, Q) \simeq \mathrm{Ext}^{j+n+1}(T, Q)$ for all $j \geq 1$. So we have that $\mathrm{Ext}^i(T, Q) = 0$ for all $i \geq 1$. But if we replace M with $Z_{-n-1}(C)$, then a similar argument gives that $\mathrm{Ext}^i(M, Q) = 0$ for all $i \geq 1$. Similarly, $\mathrm{Ext}^i(Z_j(C), Q) = 0$, for all j and for all $i \geq 1$. So $\mathrm{Hom}(C, Q)$ is an exact complex for any flat module Q. □

We recall that an exact complex C of projective modules is called *totally acyclic* if $\mathrm{Hom}(C, P)$ is exact for any projective module P.

Corollary 1. *Let R be a right coherent ring that is left n-perfect. Then any F-totally acyclic complex of projective modules is totally acyclic.*

PROOF. By Lemma 3, for any F-totally acyclic complex of projective modules, C, the complex $\mathrm{Hom}(C, P)$ is still exact, for any projective module P. It follows that C is totally acyclic. □

Proposition 5. *Let R be a right coherent ring. If R is left n-perfect, then every Gorenstein flat R-module M has Gorenstein projective dimension less than or equal to n.*

PROOF. Since M is Gorenstein flat, there is an exact and $\mathrm{Inj}\otimes$-exact complex $\overline{F} = \cdots \to F_1 \to F_0 \to F_{-1} \to \cdots$ such that $M = \mathrm{Ker}(F_0 \to F_{-1})$.

Consider a partial projective resolution of $\overline{F} : 0 \to C \to P_{n-1} \to \cdots \to P_0 \to \overline{F} \to 0$. Since $\overline{F}$ is exact and $\mathrm{Inj}\otimes$-exact, and each P_k is a projective complex, it follows that C is exact and $\mathrm{Inj}\otimes$-exact.

For each i, we have an exact complex $0 \to C_i \to P_{i,n-1} \to P_{i,n-2} \to \cdots \to P_{i,0} \to F_i \to 0$ with $P_{i,k} \in \mathrm{Proj}$. Since the projective dimension of F_i is less than or equal to n, it follows that each C_i is projective. So C is an exact and $\mathrm{Inj}\otimes$-exact complex of projective modules. By Corollary 1, C is totally acyclic. Then $Z_j(C)$ is Gorenstein projective for each j. The exact sequence of exact complexes $0 \to C \to P_{n-1} \to \cdots \to P_0 \to \overline{F} \to 0$ gives an exact sequence of modules $0 \to Z_j(C) \to Z_j(P_{n-1}) \to \cdots \to Z_j(\overline{F}) \to 0$. By the above, each $Z_j(C)$ is Gorenstein projective. Since each P_i is a projective complex, it follows that $Z_j(P_i)$ is a projective module for all j. So for each j, $Z_j(\overline{F})$ has Gorenstein projective dimension less than or equal to n. In particular, $G.p.d.M \leq n$. □

We can prove now the existence of special Gorenstein projective precovers over a right coherent and left perfect ring R.

We recall that, by [4, Proposition 3.7], if R is right coherent and any flat R-module has finite projective dimension, then any Gorenstein projective module is also Gorenstein flat. YANG and LIU proved in [20, Theorem 3.1] that over a right coherent ring R, a complex C is Gorenstein flat if and only if each C_n is a Gorenstein flat left R-module. They also proved in [20, Theorem 2.2] that over any ring R, a complex D is Gorenstein projective if and only if it is a complex of Gorenstein projective modules. Their results imply that over a right coherent ring that is also left n-perfect, every Gorenstein projective complex is also a Gorenstein flat complex.

We use the notation GorProj for the class of Gorenstein projective complexes, and we denote by GorFlat the class of Gorenstein flat complexes.

Theorem 1. *Let R be a right coherent ring. If R is left n-perfect, then the class of Gorenstein projective complexes is special precovering.*

Proof. (i) We show first that every Gorenstein flat complex G has a special Gorenstein projective precover.

Let

$$\overline{P} : 0 \to \overline{G} \to P_{n-1} \to \cdots \to P_0 \to G \to 0$$

be a partial projective resolution of G. Then, for each j, we have an exact sequence of modules

$$0 \to \overline{G}_j \to P_{n-1,j} \to \cdots \to P_{0,j} \to G_j \to 0.$$

Since $Gpd\ G_j \leq n$ (by Proposition 5), it follows that each $\overline{G}_j$ is Gorenstein projective. Thus $\overline{G}$ is a Gorenstein projective complex by [20, Theorem 2.2]. So there exists an exact and $\mathrm{Hom}(-,\mathrm{Proj})$-exact complex of projective complexes

$$\overline{T} : 0 \to \overline{G} \to T_{n-1} \to \cdots \to T_0 \to \cdots .$$

Let $T = \mathrm{Ker}(T_{-1} \to T_{-2})$. Then T is a Gorenstein projective complex, and we have a commutative diagram:

$$\begin{tikzcd}
\overline{T} : 0 \arrow[r] \arrow[d, "u"] & \overline{G} \arrow[r] \arrow[d, equal] & T_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & T_0 \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & 0 \\
\overline{P} : 0 \arrow[r] & \overline{G} \arrow[r] & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & G \arrow[r] & 0
\end{tikzcd}$$

So we have an exact sequence of complexes: $0 \to \overline{P} \to M(u) \to \overline{T}[1] \to 0$, where $M(u)$ is the mapping cone. Since both $\overline{P}$ and $\overline{T}$ are exact complexes, so is $M(u) : 0 \to \overline{G} \to \overline{G} \oplus T_{n-1} \to P_{n-1} \oplus T_{n-2} \to \cdots \to P_0 \oplus T \xrightarrow{\delta} G \to 0$. After factoring out the exact subcomplex $0 \to \overline{G} = \overline{G} \to 0$, we obtain the exact complex:

$$0 \to T_{n-1} \to P_{n-1} \oplus T_{n-2} \to \cdots \to P_1 \oplus T_0 \to P_0 \oplus T \xrightarrow{\delta} G \to 0.$$

Let $V = \mathrm{Ker}\,\delta$. Then V has finite projective dimension, so $\mathrm{Ext}^1(W, V) = 0$ for any Gorenstein projective complex W.

We have an exact sequence $0 \to V \to P_0 \oplus T \to G \to 0$ with $P_0 \oplus T$ Gorenstein projective and with V of finite projective dimension. Thus $P_0 \oplus T \to G$ is a special Gorenstein projective precover.

(ii) We prove now that every complex X has a special Gorenstein projective precover.

Let X be any complex of R-modules. By Proposition 4, every complex over a right coherent ring has a Gorenstein flat cover. So there exists an exact sequence

$$0 \to Y \to G \to X \to 0,$$

with G Gorenstein flat and with $\mathrm{Ext}^1(U, Y) = 0$ for any Gorenstein flat complex U.

By the above, there is an exact sequence

$$0 \to L \to P \to G \to 0,$$

with P Gorenstein projective and with L complex of finite projective dimension.

Form the pullback diagram:

$$\begin{tikzcd}
 & L \arrow[r, equal] \arrow[d] & L \arrow[d] & & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] & G \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

Since $L \in \mathrm{GorProj}^{\perp}$, $Y \in \mathrm{GorFlat}^{\perp}$, $\mathrm{GorProj} \subseteq \mathrm{GorFlat}$, and the sequence $0 \to L \to M \to Y \to 0$ is exact, it follows that $M \in \mathrm{GorProj}^{\perp}$. So $0 \to M \to P \to X \to 0$ is exact with P Gorenstein projective and with $M \in \mathrm{GorProj}^{\perp}$. $\square$

Corollary 2. *If R is a right coherent ring that is left n-perfect, then every module has a special Gorenstein projective precover.*

Proof. Consider a left R-module M, and let $\underline{M}$ denote the complex with M in the zeroth place and zeros everywhere else. By Theorem 1, there exists an exact sequence $0 \to L \to P \to \underline{M} \to 0$ with P a Gorenstein projective complex, and with $L \in \mathrm{GorProj}^{\perp}$. In particular, this gives an exact sequence of modules $0 \to L_0 \to P_0 \to M \to 0$ with P_0 a Gorenstein projective module. We show that $L_0 \in \mathcal{GP}^{\perp}$. Let G be any Gorenstein projective left R-module, and let $\overline{G} = \cdots \to$

$0 \to G \to G \to 0 \to \cdots$ be the complex where the two G's are in the first and zeroth place and with the map $G \to G$ the identity map. By [20, Theorem 2.2], $\overline{G}$ is a Gorenstein projective complex; so we have that $\mathrm{Ext}^1(\overline{G}, L) = 0$. By [9, Proposition 2.1.3], $\mathrm{Ext}^1(G, L_0) = 0$.

Thus $P_0 \to M$ is a special Gorenstein projective precover. □

Proposition 6. *Let R be a right coherent ring. If R is left n-perfect, then $(\mathcal{GP}, \mathcal{GP}^{\perp})$ is a complete hereditary cotorsion pair.*

Proof. Let $X \in {}^{\perp}(\mathcal{GP}^{\perp})$. By Corollary 2, there exists an exact sequence $0 \to M \to P \to X \to 0$ with P Gorenstein projective and with $M \in \mathcal{GP}^{\perp}$. But then $\mathrm{Ext}^1(X, M) = 0$, so $P \simeq M \oplus X$, and therefore X is Gorenstein projective. So $(\mathcal{GP}, \mathcal{GP}^{\perp})$ is a cotorsion pair.

By Corollary 2, the pair $(\mathcal{GP}, \mathcal{GP}^{\perp})$ is complete.

The pair $(\mathcal{GP}, \mathcal{GP}^{\perp})$ is hereditary because the class of Gorenstein projective modules is closed under kernels of epimorphisms. □

We recall that if R is a left Noetherian ring such that $\mathrm{id}_R R \leq n$, then, by [8, Proposition 9.1.2], R is left n-perfect.

By the above we obtain:

Corollary 3. *Let R be a right coherent and left Noetherian ring such that $\mathrm{id}_R R \leq n$. Then:*

(1) *the class of Gorenstein projective modules is special precovering in R-Mod;*

(2) *$(\mathcal{GP}, \mathcal{GP}^{\perp})$ is a complete hereditary cotorsion pair in R-Mod.*

(3) *the class of Gorenstein projective complexes is special precovering in $\mathrm{Ch}(R)$.*

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