Publ. Math. Debrecen
91/1-2 (2017), 123–131
DOI: 10.5486/PMD.2017.7690

A note on $(-\beta)$-shifts with the specification property

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Abstract. We consider expansions with negative real bases and associated dynamical systems. It is proved that the set of β for which the associated $(-\beta)$-shift has the specification property is of full Hausdorff dimension.

1. Introduction

The β-expansions of real numbers induced by the β-transformation T_β were introduced by Rényi [15]. Many combinatorial properties of β-expansions were subsequently obtained by Parry [14]. Since then, the link between the β-expansion of 1 and the associated β-shift S_β has been well investigated. Parry [14] proved that S_β is a subshift of finite type if and only if the β-expansion of 1 is finite. Bertrand-Mathis [2] obtained the necessary and sufficient conditions under which S_β is sofic and has the specification property, respectively (see [3] for more details on the classification of β-shifts). The size of the set of β for which the β-shift belongs to some classes was determined by Schmeling [16].

In this note, we consider expansions with negative real bases and associated dynamical systems, i.e., $(-\beta)$-expansions and $(-\beta)$-transformations, $\beta > 1$. The expansions with negative non-integer bases were first introduced by Ito and

Mathematics Subject Classification: 11K55, 11A63, 28A78.
Key words and phrases: $(-\beta)$-expansion, $(-\beta)$-shift, specification property, Hausdorff dimension.
The first author is supported by NSF of Jiangxi Province (No. 20161BAB211004) and NCHU (No. EA201507010). The third author is supported by NSFC (No. 11401445).

Sadahiro [12], with the map $f_{-\beta}$ defined on the interval $[-\frac{\beta}{\beta+1}, \frac{1}{\beta+1})$ as follows:

$$f_{-\beta}(x) = -\beta x - \left\lfloor -\beta x + \frac{\beta}{\beta+1} \right\rfloor,$$

where $\lfloor \xi \rfloor$ denotes the largest integer no more than ξ. For the sake of comparing with the β-transformation, we define the $(-\beta)$-transformation $T_{-\beta}$ on $(0, 1]$ by

$$T_{-\beta}(x) = -\beta x + \lfloor \beta x \rfloor + 1, \tag{1.1}$$

following [13]. It was pointed out in [13] that $T_{-\beta}$ is conjugate to $f_{-\beta}$ by a linear map. Similar to the β-shift for β-expansions, the $(-\beta)$-shift was defined in [12] (see details in Section 2 below). Frougny and Lai in [10], and Ito and Sadahiro in [12] obtained the necessary and sufficient conditions under which the $(-\beta)$-shift is a subshift of finite type and sofic, respectively. By [5, Theorem 1.5], the set of β for which the $(-\beta)$-shift has the specification property is of Lebesgue measure 0. Schmeling [16] obtained the Hausdorff dimension of the set of β for which the β-shift has the specification property (see Section 2 for the definition of the specification property). A natural question is to determine the Hausdorff dimension of the similar set of β for the $(-\beta)$-shift. It was proved by Liao and Steiner [13] that the $(-\beta)$-shift is not transitive, and hence does not have the specification property for any $1 < \beta < \frac{\sqrt{5}+1}{2}$, which is different from the β-shift. The main result of this note is the following.

Theorem 1.1. *The set of $\beta > 1$ for which the $(-\beta)$-shift has the specification property is of full Hausdorff dimension.*

For more dynamical properties of the $(-\beta)$-shift, the reader is referred to [10], [11], [12], [13] and the references therein. For more results on the classification of β-shifts and the size of related sets, see [3] and [16], respectively. See [1], [17] for combinatorial properties of the $(-\beta)$-expansions of the critical point. See [6], [7], [8] for other kinds of dynamical systems about expansions with negative bases. The paper is organized as follows. We shall introduce some definitions and properties of $(-\beta)$-expansions and $(-\beta)$-shifts in the next section. The main theorem will be proved by constructing a suitable Cantor set in the last section.

2. Preliminary

Let us recall the definitions and some properties of $(-\beta)$-expansions and $(-\beta)$-shifts. Let $\beta > 1$ be a real number, and $T_{-\beta}$ be the $(-\beta)$-transformation defined on $(0,1]$ by (1.1). Then each $x\in(0,1]$ can be expressed as the following series induced by $T_{-\beta}$:

$$x=\frac{\varepsilon_1(x,-\beta)}{\beta}-\frac{\varepsilon_2(x,-\beta)}{\beta^2}+\cdots+(-1)^{n-1}\frac{\varepsilon_n(x,-\beta)}{\beta^n}+\cdots,$$

where $\varepsilon_n(x,-\beta)=\lfloor\beta T_{-\beta}^{n-1}(x)\rfloor+1$ is called the n-th digit of x with base $(-\beta)$. The infinite word

$$\varepsilon_1(x,-\beta)\varepsilon_2(x,-\beta)\cdots\varepsilon_n(x,-\beta)\cdots\in\{1,2,\ldots,\lfloor\beta\rfloor+1\}^{\mathbb{N}}$$

is called the $(-\beta)$-expansion of x. Let

$$\Sigma_\beta^{\mathbb{N}}=\{1,2,\ldots,\lfloor\beta\rfloor+1\}^{\mathbb{N}},$$

and let $\Sigma_\beta^{\mathbb{N}}$ be endowed with the usual product topology, and σ be the shift on $\Sigma_\beta^{\mathbb{N}}$, i.e.,

$$\sigma(\epsilon_1\epsilon_2\cdots)=\epsilon_2\epsilon_3\cdots$$

for any $\epsilon_1\epsilon_2\cdots\in\Sigma_\beta^{\mathbb{N}}$. Then the closure of the set of $(-\beta)$-expansions of all $x\in(0,1]$ is defined to be the $(-\beta)$-shift $S_{-\beta}$, which is a subshift of $\Sigma_\beta^{\mathbb{N}}$. The alternating lexicographic order $\prec$ on $\Sigma_\beta^{\mathbb{N}}$ is defined as follows:

$$\epsilon_1\epsilon_2\cdots\epsilon_n\cdots\prec\epsilon_1'\epsilon_2'\cdots\epsilon_n'\cdots$$

if there exists an integer $k\geq 1$ such that $\epsilon_j=\epsilon_j'$ for all $1\leq j<k$ and $(-1)^k(\epsilon_k-\epsilon_k')>0$. It was proved by Ito and Sadahiro [12] that

$$S_{-\beta}=\{\epsilon_1\epsilon_2\cdots\in\Sigma_\beta^{\mathbb{N}}:\sigma^n(\epsilon_1\epsilon_2\cdots)\preceq\varepsilon_1(1,-\beta)\varepsilon_2(1,-\beta)\cdots\text{ for all }n\geq 0\}\quad(2.1)$$

if the $(-\beta)$-expansion of 1 is not periodic with odd period. The subshift $S_{-\beta}$ was also characterized when the $(-\beta)$-expansion of 1 is periodic with odd period in [12], but we shall not use it in this paper. By [12], when the $(-\beta)$-expansion of 1 is not periodic with odd period, an infinite word $\epsilon_1\epsilon_2\cdots\in\Sigma_\beta^{\mathbb{N}}$ is the $(-\beta)$-expansion of some $x\in(0,1]$ if and only if

$$1\varepsilon_1(1,-\beta)\varepsilon_2(1,-\beta)\cdots\prec\sigma^n(\epsilon_1\epsilon_2\cdots)\preceq\varepsilon_1(1,-\beta)\varepsilon_2(1,-\beta)\cdots\quad(2.2)$$

for all $n\geq 0$.

Let $F(S_{-\beta})$ be the set of factors of elements in $S_{-\beta}$, i.e., the set of finite words which appear in elements in $S_{-\beta}$. The $(-\beta)$-shift $S_{-\beta}$ has the *specification property* if there exists an integer $k\geq 1$ such that for any $u,v\in F(S_{-\beta})$, we have

$$uwv\in F(S_{-\beta})$$

for some word $w \in F(S_{-\beta})$ of length k, following [3]. For the definition of the specification property on general dynamical systems, see Bowen [4].

Let ϕ be the morphism of words with the alphabet set $\{1, 2\}$ defined by $\phi(2) = 211,\ \phi(1) = 2$. For two finite words u and v, we denote by $\{u, v\}^\infty$ the set of all infinite words which are composed by all the possible concatenations of u and v. For a finite word u, we denote by $\overline{u}$ the infinite word $uu\cdots$. The characterization of the $(-\beta)$-expansion of 1 was obtained by Steiner [17] as follows.

Lemma 2.1 ([17, Theorem 2])**.** *Let $\epsilon_1, \epsilon_2, \ldots$ be a sequence of non-negative integers. Then $\epsilon_1\epsilon_2\cdots$ is the $(-\beta)$-expansion of 1 for some (unique) $\beta > 1$ if and only if the following hold:*

(1) $\epsilon_k\epsilon_{k+1}\cdots \preceq \epsilon_1\epsilon_2\cdots$ *for all* $k \geqslant 2$*;*

(2) $\epsilon_1\epsilon_2\cdots \succ w_1w_2\cdots := \lim_{n\to\infty} \phi^n(2) = 211222112112112221122\cdots$*;*

(3) $\epsilon_1\epsilon_2\cdots \notin \{\epsilon_1\cdots\epsilon_k,\ \epsilon_1\cdots\epsilon_{k-1}(\epsilon_k - 1)1\}^\infty \setminus \{\overline{\epsilon_1\cdots\epsilon_k}\}$ *for all* $k \geq 1$ *with* $\overline{\epsilon_1\cdots\epsilon_k} \succ w_1w_2\cdots$*;*

(4) $\epsilon_1\epsilon_2\cdots \notin \{\epsilon_1\cdots\epsilon_k1,\ \epsilon_1\cdots\epsilon_{k-1}(\epsilon_k+1)\}^\infty$ *for all* $k \geq 1$ *with* $\overline{\epsilon_1\cdots\epsilon_{k-1}(\epsilon_k + 1)} \succ w_1w_2\cdots$.

Similar to β-expansions of 1, the following relation about the alternating lexicographic order of words and the usual order of real numbers was obtained by Steiner [17].

Lemma 2.2 ([17, Theorem 3])**.** *Let $\beta, \beta' > 1$ be two real numbers. Then*

$$\varepsilon_1(1, -\beta)\varepsilon_2(1, -\beta)\cdots \prec \varepsilon_1(1, -\beta')\varepsilon_2(1, -\beta')\cdots$$

if and only if $\beta < \beta'$.

3. Proofs

In the rest part of this paper, we always let N be an integer with $N \geq 4$. First, we shall show that a family of infinite words are $(-\beta)$-expansions of 1.

Lemma 3.1. *Let $\epsilon_1\epsilon_2\cdots \in \{1, 2, \ldots, N\}^{\mathbb{N}}$ be an infinite word with $\epsilon_1 = N$ and $1 \leq \epsilon_i \leq N - 2$ for $i \geq 2$. Then $\epsilon_1\epsilon_2\cdots$ is the $(-\beta)$-expansion of 1 for some (unique) $\beta > 1$.*

Proof. Since $\epsilon_1 = N \geq 4$ and $\epsilon_i \leq N - 2$ for all $i \geq 2$, the four conditions in Lemma 2.1 are satisfied immediately. $\square$

Let

$$E_N = \{\beta > 1 : \varepsilon_1(1, -\beta) = N,\ 1 \le \varepsilon_i(1, -\beta) \le N - 2 \text{ for all } i \ge 2\}. \qquad (3.1)$$

Note that by the definition of the $(-\beta)$-expansion, the $(-\beta)$-expansions of 1 for $\beta = N - 1$ and $\beta = N$ are $\overline{N}$ and $\overline{N+1}$, respectively. Thus, by Lemma 2.2, $E_N \subseteq [N - 1, N]$. Define the map $\varphi : E_N \to (0, 1)$ by

$$\varphi(\beta) = \sum_{k=1}^{\infty} \frac{\varepsilon_{k+1}(1, -\beta)}{N^k}.$$

By Lemma 3.1, $\varphi(E_N)$ consists of the points whose N-ary expansions contain only the digits $1, 2, \ldots, N - 2$. By [9, Theorem 9.3],

$$\dim_H \varphi(E_N) = \log(N - 2)/\log N, \qquad (3.2)$$

where $\dim_H$ denotes the Hausdorff dimension.

Next, we shall prove that for any $\beta \in E_N$, the $(-\beta)$-shift has the specification property. Buzzi [5, Proposition 2.1] established a criterion for the specification property of a class of piecewise monotonic maps. By this criterion, one can deduce the specification property of $S_{-\beta}$ for any $\beta \in E_N$. However, we give a direct proof here.

Lemma 3.2. *Let $\beta > 1$ be a real number. If $\varepsilon_1(1, -\beta) = N$ and $1 \le \varepsilon_i(1, -\beta) \le N - 2$ for all $i \ge 2$, then the $(-\beta)$-shift has the specification property.*

Proof. For any $u, v \in F(S_{-\beta})$, we shall prove that there exists $w \in F(S_{-\beta})$ of length 2 such that $uwv \in F(S_{-\beta})$. Since $S_{-\beta}$ is a subshift and $v \in F(S_{-\beta})$, there exist integers $\delta_1, \delta_2, \ldots$ such that $v\delta_1\delta_2 \cdots \in S_{-\beta}$. By (2.1), it follows that

$$\sigma^k(v\delta_1\delta_2 \cdots) \preceq \varepsilon_1(1, -\beta)\varepsilon_2(1, -\beta) \cdots \qquad (3.3)$$

for all $k \ge 0$. Now, we distinguish two cases.

Case 1. Any suffix of u is not a prefix of $\varepsilon_1(1, -\beta)\varepsilon_2(1, -\beta) \cdots$. Let $w = 11$. Combing the facts that $\varepsilon_1(1, -\beta) = N$ and (3.3), it follows that for all $k \ge 0$,

$$\sigma^k(u11v\delta_1\delta_2 \cdots) \preceq \varepsilon_1(1, -\beta)\varepsilon_2(1, -\beta) \cdots$$

in this case. Thus, $uwv\delta_1\delta_2 \cdots \in S_{-\beta}$ and $uwv \in F(S_{-\beta})$.

Case 2. There exists an integer $m \ge 1$ such that the suffix of u with length m is a prefix of $\varepsilon_1(1, -\beta)\varepsilon_2(1, -\beta) \cdots$. If m is even, then we let $w = \varepsilon_{m+1}(1, -\beta)(N - 1)$. Combing the facts $\varepsilon_1(1, -\beta) = N$, $\varepsilon_i(1, -\beta) \le N - 2$ for $i \ge 2$, we have $uwv\delta_1\delta_2 \cdots \in S_{-\beta}$ and $uwv \in F(S_{-\beta})$. If m is odd, then we let $w = (N - 1)1$. Similarly, we have $uwv \in F(S_{-\beta})$. □

We should point out that in the proof of Lemma 3.2, w is defined independent from v. As a result, for any $\beta \in E_N$, the subshift $S_{-\beta}$ actually has the specification property defined by Bowen [4].

In order to estimate the Hausdorff dimension of E_N, we shall prove that φ is locally Lipschitz.

Lemma 3.3. *There exists a constant C_N depending only on N such that*

$$|\varphi(\beta) - \varphi(\beta')| \le C_N |\beta - \beta'|$$

for any $\beta, \beta' \in E_N$.

Then, we can prove Theorem 1.1 immediately by the following lemma, and we shall postpone the proof of Lemma 3.3 to the last part of the note.

Lemma 3.4 ([9, Proposition 2.3])**.** *Let $F \subset \mathbb{R}^n$, and suppose that $f : F \to \mathbb{R}^m$ satisfies a Hölder condition*

$$|f(x) - f(y)| \le c|x - y|^\alpha \quad (x, y \in F).$$

Then $\dim_H f(F) \le 1/\alpha \dim_H F$.

Proof of Theorem 1.1. By Lemmas 3.3, 3.4 and (3.2), we have

$$\dim_H E_N \ge \dim_H \varphi(E_N) = \log(N-2)/\log N.$$

Since $N \ge 4$ is arbitrary, the conclusion follows. □

It remains to prove Lemma 3.3. From now on, we always let $\beta, \beta' \in E_N \subset [N-1, N]$. Then, there exists an integer $n \ge 1$ such that

$$\varepsilon_i(1, -\beta) = \varepsilon_i(1, -\beta'), \quad 1 \le i \le n, \varepsilon_{n+1}(1, -\beta) \ne \varepsilon_{n+1}(1, -\beta'). \tag{3.4}$$

Now, we give the following upper bound estimate of $|\varphi(\beta) - \varphi(\beta')|$. In the rest part of the note, the integer n is always defined by (3.4).

Lemma 3.5. *Let $\beta, \beta' \in E_N$ and n be defined by* (3.4)*. Then*

$$|\varphi(\beta) - \varphi(\beta')| \le N^{-n+1}.$$

Proof. Since $\varepsilon_i(1, -\beta) = \varepsilon_i(1, -\beta')$ for $1 \le i \le n$ and $1 \le \varepsilon_i(1, -\beta)$, $\varepsilon_i(1, -\beta') \le N - 2$ for all $i > n$, we have

$$
\begin{aligned}
&|\varphi(\beta)-\varphi(\beta')| \\
&= \left|\left(\frac{\varepsilon_{n+1}(1,-\beta)}{N^n}+\frac{\varepsilon_{n+2}(1,-\beta)}{N^{n+1}}+\cdots\right)-\left(\frac{\varepsilon_{n+1}(1,-\beta')}{N^n}+\frac{\varepsilon_{n+2}(1,-\beta')}{N^{n+1}}+\cdots\right)\right| \\
&\le \sum_{k\ge n}\frac{N-2}{N^k}-\sum_{k\ge n}\frac{1}{N^k}\le N^{-n+1}. \qquad \square
\end{aligned}
$$

Now, we are in a position to prove Lemma 3.3.

Proof of Lemma 3.3. Assume that $\beta,\beta'\in E_N$, $\beta<\beta'$ and (3.4) holds. Let $\epsilon_i=\varepsilon_i(1,-\beta)$ for $1\le i\le n$. We define the polynomials

$$P_1(x)=-x+\epsilon_1, P_k(x)=-xP_{k-1}(x)+\epsilon_k$$

for any $2\le k\le n$. Then

$$T^k_{-\beta}(1)=P_k(\beta),\quad T^k_{-\beta'}(1)=P_k(\beta')$$

for any $1\le k\le n$. It follows from [17, Remark 1] that

$$(-1)^nP'_n(x)=x^{n-1}\left(1+\sum_{j=1}^{n-1}\frac{P_j(x)}{(-x)^j}\right),$$

and $P_j(x)\in[0,1]$ for any $1\le j\le n-1$ and any $x\in(\beta,\beta')$. So,

$$0<(-1)^nP'_n(x)<\frac{x^{n+1}}{x^2-1}$$

for any $x\in(\beta,\beta')\subset[N-1,N]$. Thus,

$$|T^n_{-\beta}(1)-T^n_{-\beta'}(1)|<|\beta-\beta'|\frac{\zeta^{n+1}}{\zeta^2-1}$$

for some $\zeta\in(\beta,\beta')$, hence

$$|\beta-\beta'|>|T^n_{-\beta}(1)-T^n_{-\beta'}(1)|\frac{N^2-1}{N^{n+1}} \tag{3.5}$$

for $n\ge 1$. Let

$$
\begin{aligned}
&\Delta_N(x,y) \\
&= \min\left\{\left(\frac{2}{x}-\sum_{k=1}^{\infty}\frac{N-2}{x^{2k}}+\sum_{k=1}^{\infty}\frac{1}{x^{2k+1}}\right)-\left(\frac{1}{y}-\sum_{k=1}^{\infty}\frac{1}{y^{2k}}+\sum_{k=1}^{\infty}\frac{N-2}{y^{2k+1}}\right),\right. \\
&\quad\left.\left(\frac{N-2}{y}-\sum_{k=1}^{\infty}\frac{N-2}{y^{2k}}+\sum_{k=1}^{\infty}\frac{1}{y^{2k+1}}\right)-\left(\frac{N-3}{x}-\sum_{k=1}^{\infty}\frac{1}{x^{2k}}+\sum_{k=1}^{\infty}\frac{N-2}{x^{2k+1}}\right)\right\}
\end{aligned}
$$

for $x, y \in [N-1, N]$. Since

$$\Delta_N(x,x) = \frac{1}{x} - \sum_{k=2}^{\infty} \frac{N-3}{x^k} = \frac{1}{x}(1 - \frac{N-3}{x-1}) \geq \frac{1}{N}(1 - \frac{N-3}{N-2}) = \frac{1}{N(N-2)}$$

for $x \in [N-1, N]$ and Δ_N is continuous, there exist $C(N) > 0$ and $\delta(N) > 0$ such that

$$\Delta_N(\beta, \beta') \geq C(N) \tag{3.6}$$

if $0 < |\beta - \beta'| \leq \delta(N)$. By the definition of $(-\beta)$-transformation, we have

$$T^n_{-\beta}(1) = \sum_{i=1}^{\infty} \frac{\varepsilon_{n+i}(1, -\beta)}{\beta^i}, \quad T^n_{-\beta'}(1) = \sum_{i=1}^{\infty} \frac{\varepsilon_{n+i}(1, -\beta')}{\beta'^i}.$$

Note that

$$|T^n_{-\beta}(1) - T^n_{-\beta'}(1)| \geq \Delta_N(\beta, \beta') \tag{3.7}$$

for any $\beta, \beta' \in E_N$ with $0 < \beta' - \beta < \delta(N)$. By (3.5), (3.6) and (3.7), we have

$$|\beta - \beta'| \geq C_1(N) N^{-n}$$

for a constant $C_1(N)$ and any $\beta, \beta' \in E_N$ with $0 < |\beta' - \beta| \leq \delta(N)$. Together with Lemma 3.5, this proves the lemma. $\square$

Acknowledgements. The authors thank the referee for the detailed comments which improved the presentation and the proof of the paper.

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(Received April 25, 2016; revised November 2, 2016)