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An application of the transversality theorem

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Introduction: Let H be an infinite dimensional real Hilbert space and $L(H)$ the Banach space of the linear continuous maps $A : H \to H$.

We consider the following set:

$$GL(H) = \{A \in L(H) \mid A \text{ is bijective and } A^{-1} \in L(H)\}.$$

It is well-known that $GL(H)$ is an open set in $L(H)$ (see ([3], vol. 1, pp. 26), and the mapping $GL(X) \times GL(H) \to GL(H)$, $(A, B) \to AB$, respectively $GL(H) \to GL(H)$, $A \to A^{-1}$ are of the C^{∞} -class ([2], vol. 2, pp. 27). Consequently, $GL(H)$ is an infinite dimensional Lie group.

Now, we consider the following subsets of $L(H)$:

$$S(H) = \{A \in L(H) \mid A = A^*\};$$

$$AS(H) = \{A \in L(H) \mid A = -A^*\};$$

$$GS(H) = \{A \in L(H) \mid A = A^*,\ A^2 = 1_H\};$$

$$U(H) = \{A \in L(H) \mid AA^* = A^*A = 1_H\}.$$

where $A^* \in L(H)$ is the adjoint operator of $A \in L(H)$, $S(H)$ and $AS(H)$ represent the sets of symmetric respectively skew-symmetric operators in $L(H)$ and they are closed linear subspaces of $L(H)$. Moreover, $L(H) = S(H) \oplus AS(H)$, where $\oplus$ denotes the algebraic-topological sum. The set $U(H)$ is an infinite dimensional Lie group, called the unitary group of H.

Let M, N be two Banach manifolds, and $N_1 \subset N$ a submanifold of N. Let $F : M \to N$ be a differentiable function. The function F is called transverval to N_1 at $p \in M$ if either $q = F(p) \notin N_1$ or else the following two conditions hold:

i) $T_pF^{-1}(T_qN_1)$ splits T_pM;
ii) the image $T_pF(T_pM)$ contains a closed complement of T_qN_1 in T_qN.

Notation: $F \pitchfork_p N_1$. If $F \pitchfork_p N_1$ for all $p \in M$, then F is called transversal to N_1. Notation: $F \pitchfork N_1$.

In what follows we need the following

Transversality Theorem. *Let N_1 be a submanifold of N and $F : M \to N$ be transversal to N_1. If $M_1 = F^{-1}(N_1) \neq \emptyset$, then M_1 is a submanifold of M with* $\operatorname{codim} M_1 = \operatorname{codim} N_1$ *and $T_p(M_1) = T_pF^{-1}(T_qN_1)$ for $q = F(p)$.*

For the proof and diverse applications and extensions of this result see ([1], §3.5) or ([4], Capitulos VI, VII).

Lemma 1. *Let $A \in GL(H) \cap S(H)$. The following assertions are equivalent:*

1. $\langle Ax, x\rangle > 0$ for each $x \in H \setminus \{0\}$.

2. There is a unique $B \in GL(H) \cap S(H)$ such that $\langle Bx, x\rangle > 0$ for each $x \in H \setminus \{0\}$ and $B^2 = A$.

3. There exists a $C \in GL(H)$ such that $C^*C = A$.

Proof. 1) $\Longrightarrow$ 2): We consider the operator $B = A^{\frac{1}{2}}$, see ([2], vol.1, pp. 119) which satisfies 2).

2) $\Longrightarrow$ 3): We set $C = B$ and observe that $C^* = C = B$.

The implication 3) $\Longrightarrow$ 1) is obvious.

Lemma 2. *For $A \in GL(H)$ consider the function $\psi_A : L(H) \to L(H)$, $\psi_A(X) = AX$. Then we have:*

1) $\psi_A \in GL(L(H))$.

2) $L(H) = \psi_A(S(H)) \oplus \psi_A(AS(H))$.

Proof. 1) Evidently ψ_A is linear and bijective. If $X, Y \in L(H)$, then we have $\|\psi_A(X) - \psi_A(Y)\| = \|AX - AY\| = \|A(X - Y)\| \le \|A\|\|X - Y\|$, i.e. ψ_A is continuous. By the open mapping theorem we have that $\psi_A \in GL(L(H))$.

2) The relation $L(H) = \psi_A(S(H)) \oplus \psi_A(AS(H))$ is a direct consequence of 1) and of the decomposition $L(H) = S(H) \oplus AS(H)$.

Lemma 3. *Let $B \in GS(H)$ and $A \in L(H)$ be such that $A^*A = B$. Consider the function $\varphi_A : L(H) \to L(H)$, $\varphi_A(X) = A^*X + X^*A$. Then we have:*

$$\operatorname{Ker} \varphi_A = \varphi_{AB}(AS(H)). \tag{1}$$

Proof. By Lemma 2, we have $L(H) = \psi_A(S(H)) \oplus \psi_A(AS(H))$, therefore for every $X \in L(H)$ there exists $C \in S(H)$, $D \in AS(H)$, uniquely determined, such that $X = \psi_{AB}(C) + \psi_{AB}(D) = ABC + ABD$.

If $x \in \operatorname{Ker} \varphi_A$, we have $A^*X + X^*A = 0$, and consequently $A^*(ABC + ABD) + (C^*B^*A^* + D^*B^*A^*)A = 0$. We obtain $C + D + C^* + D^* = 0$. Since $C \in S(H)$, $D \in AS(H)$, it follows that $C = 0$, which proves the inclusion $\operatorname{Ker} \varphi_A \subset \psi_{AB}(AS(H))$. If $X \in \psi_{AB}(AS(H))$, there exists $Y \in AS(H)$ such that $X = ABY$. Then $\varphi_A(X) = A^*X + X^*A = A^*ABY + Y^*B^*A^*A = Y + Y^* = 0$ and, in conclusion, $\psi_{AB}(AS(H)) = \operatorname{Ker} \varphi_A$.

Now we consider the mapping $F : L(H) \to S(H)$, $F(X) = X^*X$.

This mapping is of C^∞-class, and for each $A \in L(H)$ we have the relation:

$$(T_AF)(X) = A^*X + X^*A. \tag{2}$$

Consider now the submanifold $GS(H)$ of $S(H)$. If $B \in GS(H)$, then the tangent space to $GS(H)$ at B is the real subspace $T_B(GS(H))$ of $L(H)$ defined by

$$T_B(GS(H)) = \{X \in L(H) \mid X^* = X \text{ and } BX = -XB\}.$$

Each element $B \in GS(H)$ determines a splitting of $S(H)$ as a direct sum $S(H) = T_B(GS(H)) \oplus N_B(GS(H))$, where $N_B(GS(H)) = \{X \in S(H)) \mid XB = BX\}$, see [5].

Theorem. *The mapping $F : L(H) \to S(H)$, $F(X) = X^*X$ is transversal to $GS(H)$, i.e. $F \pitchfork GS(H)$.*

Proof. We have to verify the conditions i) and ii) from the definition of transversality.

i) We prove that $T_AF^{-1}(T_B(GS(H)))$ splits in $L(H)$ for every $A \in L(H)$ and $B \in GS(H)$ such that $A^*A = B$. We have $T_AF(X) = A^*X + X^*A$ for every $X \in L(H)$. But using Lemma 3 it follows that $\operatorname{Ker} T_AF = \psi_{AB}(AS(H))$ and hence

$$L(H) = \operatorname{Ker} T_AF \oplus \psi_{AB}(S(H)). \tag{3}$$

Therefore, for every $X \in L(H)$, there exist $X' \in \operatorname{Ker} T_AF$ and $S \in S(H)$ such that $X = X' + ABS$, implying $(T_AF)(X) = T_AF(ABS) = A^*ABS + S^*B^*A^*A = 2S$.

Hence

$T_AF^{-1}(T_B(GS(H))) = \{X = X' + ABS | BT_AF(X) + T_AF(X)B\} = 0 =$

$\{X = X' + ABS | BS + SB = 0\} = \operatorname{Ker} T_AF \oplus \psi_{AB}(T_B(GS(H)).$

But $S(H) = T_B(GS(H) \oplus \{X \in S(H) | BX = XB\}$, i.e.

$L(H) = \operatorname{Ker} T_AF \oplus \psi_{AB}(T_B(GS(H)) \oplus \psi_{AB}(\{X \in S(H) \mid BX = XB\}) =$

$T_AF^{-1}(T_B(GS(H))) \oplus \psi_{AB}(\{X \in S(H) \mid BX = XB\})$, and

$T_AF^{-1}(T_B(GS(H))$ splits in $L(H)$.

ii) The second condition is obvious, because $T_A F(L(H)) = S(H)$.

Indeed, for every $X \in L(H)$ there exist $X' \in \operatorname{Ker} T_A F$, $S \in S(H)$ such that $X = X' + ABS$ and $(T_A F)(X) = 2S$.

Remark. Let $B \in GS(H)$ be a positively defined operator and consider the mapping $F : GL(H) \to S(H)$, $F(X) = X^* X$. Then the fiber $f^{-1}(B)$ is a nonvoid submanifold of $GL(H)$, which is diffeomorphic to $U(H)$.

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