Publ. Math. Debrecen
91/1-2 (2017), 143–151
DOI: 10.5486/PMD.2017.7707

On Leibniz differences

By BRUCE EBANKS (Louisville)

Abstract. Cauchy differences, which are two-place functions of the form $F(x,y) = f(x) + f(y) - f(x+y)$, are characterized on abelian groups by means of the cocycle functional equation together with symmetry. Here we introduce an analogous result for functions of the form $L(x,y) = yf(x) + xf(y) - f(xy)$ for functions $L : K^2 \to K$ where K is a field of characteristic 0. Such functions are called Leibniz differences.

1. Introduction

In this article, we consider the question of how to characterize Leibniz differences on an integral domain R of characteristic 0. A *Leibniz difference* is a two-place function L of the form

$$L(x,y) = yf(x) + xf(y) - f(xy)$$

for some function $f : R \to R$. The reason for this terminology is that $L = 0$ if and only if f satisfies the Leibniz rule for the derivative of a product. That is, L measures how much f differs from being a solution of the Leibniz functional equation $f(xy) = xf(y) + yf(x)$.

A function $L : R \times R \to R$ is *symmetric* if $L(x,y) = L(y,x)$ for all $x, y \in R$.

The functional equation we use to characterize Leibniz differences is

$$L(xy,z) + zL(x,y) = L(x,yz) + xL(y,z), \quad x,y,z \in R. \tag{1}$$

Mathematics Subject Classification: 39B52, 39B22.
Key words and phrases: cocycle functional equation, Cauchy difference, Leibniz difference, field, integral domain, coboundary.

Our motivation for using this functional equation comes from [4] (see also [3]), where it was involved in a simplified proof of Sydler's Theorem on polyhedra.

The main result of our paper is that the functional equation (1) paired with symmetry essentially characterizes Leibniz differences on an integral domain R of characteristic 0. To be precise, the solutions are Leibniz differences, although the generating function f may take some of its values in the fraction field of R. We give an example to show that the values of f need not all lie in R. As a consequence, Leibniz differences on a field of characteristic 0 are characterized. We also solve (1) without the symmetry condition.

Finally, we solve a Pexiderized version of (1) with four unknown functions.

The key to our results is the cocycle equation and its use in characterizing Cauchy differences on commutative semigroups. If S is a commutative semigroup and H is an abelian group, the *Cauchy difference* of a function $\phi : S \to H$ is the mapping $F : S \times S \to H$ defined by

$$F(x,y) := \phi(x) + \phi(y) - \phi(x+y), \quad x, y \in S.$$

(For discussions of these objects and their significance, see [2] and its references. Cauchy differences are also known as coboundaries in homological algebra.) It is easy to check that every Cauchy difference satisfies the *cocycle functional equation*

$$F(x,y) + F(x+y,z) = F(x,y+z) + F(y,z), \quad x, y, z \in S. \tag{2}$$

Under certain conditions on the semigroup S and group H, the converse is also true for symmetric F. Let us assume first that our semigroup S is *cancellative*, that is, if $ab = ac$ for some $a, b, c \in S$ with $a \neq 0$, then $b = c$. Second, we assume that our group H is *divisible*, meaning that for any positive integer n and any element $y \in H$ there exists an element $x \in H$ such that $nx = y$. A group has 2-torsion if $2x = 0$ for some $x \neq 0$ in the group.

The main tool we use is the following result from [1] (cf. also [2]).

Proposition 1. *Let S be a cancellative commutative semigroup, and H be a divisible abelian group with no 2-torsion. The general solution $F : S \times S \to H$ of the cocycle equation (2) is*

$$F(x,y) = \phi(x) + \phi(y) - \phi(x+y) + \Psi(x,y), \quad x, y \in S,$$

for a map $f : S \to H$ and a map $\Psi : S \times S \to H$ satisfying the system of equations

$$\Psi(x,y) = -\Psi(y,x),$$
$$\Psi(x+y,z) = \Psi(x,z) + \Psi(y,z)$$

for all $x, y, z \in S$.

It follows that F is a symmetric solution of the cocycle equation (2) *if and only if F is a Cauchy difference.*

The last displayed equation states that Ψ is a morphism in its first variable, and by skew-symmetry (the first condition on Ψ) it is also a morphism in its second variable hence a bimorphism. Thus the proposition states that, under the given conditions on S and H, every cocycle is the sum of a Cauchy difference plus a skew-symmetric bimorphism, and therefore every symmetric cocycle is a Cauchy difference.

2. Characterization of Leibniz differences

For a ring R, let $R^* = R \setminus \{0\}$. Our main result is the following.

Theorem 2. *Let R be an integral domain of characteristic* 0*, and let K be the fraction field of R. A function $L : R \times R \to R$ is a symmetric solution of equation* (1) *if and only if there exists a function $f : R \to K$ such that*

$$L(x, y) = yf(x) + xf(y) - f(xy), \quad x, y \in R. \tag{3}$$

Proof. The "if" part is straightforward. For the "only if" part, we begin by restricting x, y, z to R^* in (1) and, working in K, divide equation (1) by xyz. Defining $F : R^* \times R^* \to K$ by

$$F(x, y) := \frac{L(x, y)}{xy}, \quad x, y \in R^*, \tag{4}$$

the result is that F is a symmetric solution of the cocycle equation (2) on R^* (into K).

Since R is an integral domain, the structure $(R^*, \cdot)$ is a cancellative commutative semigroup. Moreover, the additive group $(K, +)$ is divisible, abelian, and has no 2-torsion (in fact no torsion at all) since R has characteristic 0. By Proposition 1, therefore, there exists a map $\phi : R^* \to K$ such that

$$F(x, y) = \phi(x) + \phi(y) - \phi(xy), \quad x, y \in R^*.$$

Referring to the definition of F, we have

$$L(x, y) = xy\phi(x) + xy\phi(y) - xy\phi(xy), \quad x, y \in R^*.$$

Now, defining $g : R^* \to K$ by $g(x) = x\phi(x)$, we arrive at

$$L(x, y) = yg(x) + xg(y) - g(xy), \quad x, y \in R^*. \tag{5}$$

Finally, let us consider what happens when some of the variables in (1) are 0. If $x = 0$, the equation reduces to

$$L(0, z) + zL(0, y) = L(0, yz), \quad y, z \in R.$$

Since the right hand side is symmetric in y and z, we conclude that

$$L(0, z) + zL(0, y) = L(0, yz) = L(0, zy) = L(0, y) + yL(0, z), \quad y, z \in R,$$

or with $z = 0$,

$$L(0, 0)(1 - y) = L(0, y), \quad y \in R.$$

Putting $\lambda = L(0, 0)$, we have therefore

$$L(0, y) = \lambda(1 - y), \quad y \in R. \tag{6}$$

Now put $y = z = 0$ in (1) and use (6) to get

$$\lambda(1 - x) = L(x, 0), \quad x \in R. \tag{7}$$

By equations (5), (6) and (7), we have the desired form (8) for L, where the function $f : R \to K$ is the extension of g defined by $f(x) := g(x)$ for $x \in R^*$ and $f(0) := -\lambda$. □

This theorem has the following immediate consequence.

Corollary 3. *Let K be a field of characteristic 0. A function $L : K \times K \to K$ is a symmetric solution of equation (1) if and only if there exists a function $f : K \to K$ such that*

$$L(x, y) = yf(x) + xf(y) - f(xy), \quad x, y \in K. \tag{8}$$

The question remains whether the conclusion of Theorem 2 can be strengthened to state that the generator f takes its values in R rather than the field of fractions of R. The next example shows that such a strengthening is not possible in general.

Example. Define $f : \mathbb{Z} \to \frac{1}{2}\mathbb{Z}$ by

$$f(2k+1) := 1, \quad k \in \mathbb{Z}, \qquad f(4k) := 0, \quad k \in \mathbb{Z},$$

$$f(4k+2) := \frac{2k+1}{2}, \quad k \in \mathbb{Z},$$

and define L on $\mathbb{Z} \times \mathbb{Z}$ by

$$L(x,y) = yf(x) + xf(y) - f(xy), \quad x, y \in \mathbb{Z}.$$

Clearly, the range of f is not contained in $\mathbb{Z}$, since, for example, $f(6) = \frac{3}{2}$. We show nevertheless that $L(x,y)$ always lies in the integral domain $\mathbb{Z}$, by considering three cases.

Case 1. Suppose x and y are odd. Then

$$\begin{aligned} L(x,y) &= L(2k+1, 2n+1) \\ &= (2n+1)f(2k+1) + (2k+1)f(2n+1) - f((2k+1)(2n+1)) \\ &= (2n+1) + (2k+1) - 1 \in \mathbb{Z}. \end{aligned}$$

Case 2. Suppose one of x, y is even and the other is odd, say $x = 2k, y = 2n+1$. Then

$$L(x,y) = L(2k, 2n+1) = (2n+1)f(2k) + (2k)f(2n+1) - f((2k)(2n+1)).$$

We consider two sub-cases. If $k = 2p$, then we have

$$L(x,y) = L(4p, 2n+1) = 4p \in \mathbb{Z}.$$

On the other hand, if $k = 2p+1$, then

$$\begin{aligned} L(x,y) &= L(4p+2, 2n+1) \\ &= (2n+1)f(4p+2) + (4p+2)f(2n+1) - f((4p+2)(2n+1)) \\ &= (2n+1)\frac{2p+1}{2} + (4p+2) - \frac{(2p+1)(2n+1)}{2} = 4p+2 \in \mathbb{Z}. \end{aligned}$$

Case 3. Suppose x and y are even. Then

$$\begin{aligned} L(x,y) = L(2k, 2n) &= (2n)f(2k) + (2k)f(2n) - f(4kn) \\ &= (2n)f(2k) + (2k)f(2n). \end{aligned}$$

We consider three sub-cases. If $k = 2p, n = 2q$, then we have

$$L(x, y) = L(4p, 4q) = 0 \in \mathbb{Z}.$$

If $k = 2p + 1, n = 2q$, then

$$L(x, y) = L(4p + 2, 4q) = (4q)f(4p + 2) + (4p + 2)f(4q) = 2q(2p + 1) \in \mathbb{Z},$$

and a similar calculation works if k is even and n is odd. Finally, if $k = 2p+1, n = 2q + 1$, then

$$\begin{aligned} L(x, y) = L(4p + 2, 4q + 2) &= (4q + 2)f(4p + 2) + (4p + 2)f(4q + 2) \\ &= 2(2q + 1)(2p + 1) \in \mathbb{Z}. \end{aligned}$$

Therefore, $L : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$.

3. Extensions

Throughout this section, when referring to a bimorphism we always mean a function that is in each variable a morphism with respect to multiplication in the domain and addition in the range. That is, Ψ is a bimorphism if

$$\Psi(xy, z) = \Psi(x, z) + \Psi(y, z) \quad \text{and} \quad \Psi(x, yz) = \Psi(x, y) + \Psi(x, z)$$

for all specified values of x, y, z.

Our first extension deals with the solutions of equation (1) without assuming symmetry.

Theorem 4. *Let R be an integral domain of characteristic 0, and let K be the fraction field of R. A function $L : R \times R \to R$ satisfies equation (1) if and only if there exists a function $f : R \to K$ and a function $\Psi : R \times R \to K$ such that*

$$L(x, y) = yf(x) + xf(y) - f(xy) + xy\Psi(x, y), \quad x, y \in R, \tag{9}$$

where the restriction of Ψ to $R^* \times R^*$ is a skew-symmetric bimorphism.

Proof. The proof follows the same outline as that of Theorem 2. The first (and essentially the only) change is that the function F defined by (4) is not necessarily symmetric now. Applying Proposition 1, we have now a map

$\phi : R^* \to K$ and a skew-symmetric bimorphism (with respect to multiplication in R^* and addition in K) $\Psi : R^* \times R^* \to K$ such that

$$F(x,y) = \phi(x) + \phi(y) - \phi(xy) + \Psi(x,y), \quad x, y \in R^*.$$

Retracing the steps in the proof of Theorem 2, we find that L has the form (9), where Ψ is extended (arbitrarily) to $(R \times \{0\}) \cup (\{0\} \times R)$.

We observe that the function Ψ cannot be extended to $R \times R$ in such a way as to preserve the bimorphism property, unless Ψ is identically 0. □

Finally, we consider the Pexiderized version of (1) with four unknown functions.

Theorem 5. *Let R be an integral domain of characteristic 0, and let K be the fraction field of R. Functions $L_1, L_2, L_3, L_4 : R \times R \to R$ satisfy the functional equation*

$$L_1(xy,z) + zL_2(x,y) = L_3(x,yz) + xL_4(y,z), \quad x, y, z \in R, \tag{10}$$

if and only if there exist functions $f_1, f_2, f_3, f_4, f_5, f_6 : R \to K$ and $\Psi : R \times R \to K$ such that

$$\begin{aligned} L_1(x,y) &= yf_2(x) + xf_4(y) - f_1(xy) + xy\Psi(x,y), \\ L_2(x,y) &= yf_5(x) + xf_6(y) - f_2(xy) + xy\Psi(x,y), \\ L_3(x,y) &= yf_5(x) + xf_3(y) - f_1(xy) + xy\Psi(x,y), \\ L_4(x,y) &= yf_6(x) + xf_4(y) - f_3(xy) + xy\Psi(x,y), \end{aligned}$$

for all $x, y \in R$, where the restriction of Ψ to $R^* \times R^*$ is a skew-symmetric bimorphism.

Proof. The "if" part is a simple verification that we omit. For the "only if" part, we begin by putting $x = 1$ in (10) to get

$$L_4(y,z) = L_1(y,z) + zL_2(1,y) - L_3(1,yz), \quad y, z \in R. \tag{11}$$

Substituting this into (10) we have

$$\begin{aligned} &L_1(xy,z) + zL_2(x,y) \\ &\quad = L_3(x,yz) + x[L_1(y,z) + zL_2(1,y) - L_3(1,yz)], \quad x, y, z \in R. \end{aligned} \tag{12}$$

Putting $z = 1$ in this equation yields

$$L_2(x,y) = L_3(x,y) + x[L_1(y,1) + L_2(1,y) - L_3(1,y)] - L_1(xy,1), \quad x,y \in R, \quad (13)$$

which reduces (12) to

$$\begin{aligned} &L_1(xy,z) + z[L_3(x,y) - xL_3(1,y)] + xzL_1(y,1) - zL_1(xy,1) \\ &\quad = L_3(x,yz) - xL_3(1,yz) + xL_1(y,z). \end{aligned} \quad (14)$$

Next, setting $y = 1$ in (14) we get

$$\begin{aligned} &L_3(x,z) - xL_3(1,z) \\ &\quad = L_1(x,z) + z[L_3(x,1) - xL_3(1,1)] + xzL_1(1,1) - zL_1(x,1) - xL_1(1,z), \end{aligned} \quad (15)$$

and with this (14) simplifies to

$$\begin{aligned} &L_1(xy,z) + z[L_1(x,y) - xL_1(1,y)] + xzL_1(y,1) - zL_1(xy,1) \\ &\quad = L_1(x,yz) + x[L_1(y,z) - L_1(1,yz)], \end{aligned}$$

for all $x,y,z \in R$.

Defining $L : R \times R \to R$ by

$$L(x,y) := L_1(x,y) - xL_1(1,y) - yL_1(x,1), \quad x,y \in R,$$

the previous equation is exactly (1). By Theorem 4, therefore, L has the form

$$L(x,y) = yf_1(x) + xf_1(y) - f_1(xy) + xy\Psi(x,y), \quad x,y \in R,$$

for some function $f_1 : R \to K$, with Ψ as described above.

Referring to the definition of L, we find that L_1 has the form asserted in the theorem, where

$$f_2(x) := f_1(x) + L_1(x,1), \qquad f_4(x) := f_1(x) + L_1(1,x), \quad x \in R.$$

Next, returning to (15), we see that L_3 has the asserted form with

$$f_3(x) := f_1(x) + L_3(1,x) - xL_3(1,1), \qquad f_5(x) := f_1(x) - xf_1(1) + L_3(x,1).$$

With this, (13) yields the desired form for L_2 if we define

$$f_6(x) := f_2(x) + L_2(1,x) - xf_5(1).$$

Finally, the stated form for L_4 is obtained from (11), and this completes the proof. □

Acknowledgements. I want to thank Árpád Száz for suggesting (in a personal communication) the problem of characterizing Leibniz differences.

References

[1] T. M. K. Davison and B. Ebanks, Cocycles on cancellative semigroups, *Publ. Math. Debrecen* **46** (1995), 137–147.

[2] B. Ebanks, The cocycle equation on commutative semigroups, *Results Math.* **67** (2015), 253–264.

[3] B. Ebanks, On some functional equations of Jessen, Karpf, and Thorup, *Math. Scand.* **44** (1979), 231–234.

[4] B. Jessen, A. Karpf and A. Thorup, Some functional equations in groups and rings, *Math. Scand.* **22** (1968), 257–265.

BRUCE EBANKS
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LOUISVILLE
LOUISVILLE, KY 40292
USA

E-mail: ebanks1950@gmail.com

(Received May 11, 2016)