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On products of cyclic and elementary abelian p-groups

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Abstract. We investigate the structure of a finite p-group $G = AB$ that is the product of a cyclic subgroup A and an elementary abelian subgroup B. Examples of factorised groups are presented and information on the derived subgroup and exponent of such products is also provided.

1. Introduction

The study of groups that are the product of a cyclic subgroup and an abelian subgroup has been largely confined to products of two cyclic subgroups. Work in this area goes back to Rédei [17], who investigated the derived subgroup of a group $G = AB$ that is the product of two cyclic subgroups A and B, at least one of which is infinite. Rédei showed, in particular, that where A is infinite and B is finite, then $G^{'}$ can be generated by two elements. There followed Wielandt [18], who showed that if p is the largest prime divisor of $|G|$, where G is a finite group that is the product of two cyclic subgroups, then G has a normal factorised Sylow p-subgroup, while Douglas [7]–[10] examined conjugacy in products of two finite cyclic groups. Huppert [12] investigated products of pairwise permutable cyclic subgroups and showed, in particular, that if p is an odd prime and G is the product of two cyclic p-groups, then G is metacyclic. Huppert also showed that the derived subgroup of a product of two cyclic 2-groups is not always cyclic. Following Douglas, Yacoub [19] determined permutation representations for products of two finite cyclic groups, while Itô [14] and Itô and Ôhara [15]–[16] examined the structure of $G^{'}$ and $G/G^{'}$, where G is the product of two cyclic

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2-groups. Blackburn [4] built on this to determine the structure of $G^{'}$ in the case where $G^{'}$ is non-cyclic and G is the product of two cyclic 2-groups. Rédei's result was extended by Cohn [5] to products of two infinite cyclic subgroups, and Heineken and Lennox [11] later showed that if G is the product of the infinite cyclic subgroups $\langle a\rangle$ and $\langle b\rangle$, then there exists an integer m such that $\langle a^m, b^m\rangle$ is a torsion-free abelian normal subgroup of G.

Given the extensive literature on factorised groups, it is perhaps surprising that, apart from the above cases, the structure of products of finite cyclic and abelian groups (not to mention products of finite abelian groups in general) has remained largely unexplored. Of course, such groups will be metabelian by the Theorem of Itô ([13, Satz 1] or [2, Theorem 3.1.7]). Aside from this, the only general results applicable to products of finite cyclic and abelian groups are those of Conder and Isaacs [6], who showed, in particular, that if $G = AB$ for abelian subgroups A and B such that B is finite and either A or B is cyclic, then $G^{'}/(G^{'} \cap A)$ is isomorphic to a subgroup of B. As an initial attempt to provide more detail on such products, the present paper investigates the structure of finite p-groups that can be expressed as the product of a cyclic subgroup and an elementary abelian subgroup. For the odd prime p, Lemma 2.5 shows that the elementary abelian factor has index at most p in its normal closure. The remainder of Section 2 consists of applications of Lemma 2.5 and some examples, leading to Theorems 2.13, 2.14 and 2.15, which clarify how such factorised groups can be constructed. For $p = 2$, the key results are Lemmas 3.3 and 3.7, which ultimately lead to the classification given by Theorems 3.6, 3.15 and 3.16.

The following notation is used. For the subgroup U of a group G, U_G denotes the core of U in G. Thus $U_G = \bigcap_{g\in G} U^g$. The normal closure of U in G is denoted by U^G, so that $U^G = \langle U^g \mid g \in G\rangle$. The rank of the finite abelian group A, that is the size of a minimal generating set of A, is denoted by $r(A)$. C_{p^k} denotes the cyclic group of order p^k, while $o(g)$ denotes the order of the element g of a finite group G. If G is a finite p-group, then $\Omega_k(G) = \langle g \in G \mid g^{p^k} = 1\rangle$ denotes the characteristic subgroup of G generated by those elements of G whose order is a divisor of p^k. In particular, if G is non-trivial, then $\Omega_1(G)$ is the subgroup generated by all elements of order p in G. Moreover, if A is cyclic of order p^k, where $k > 0$, then for $0 < s \le k$, $\Omega_s(A)$ is the unique subgroup of order p^s in A. Finally, $\exp(G)$ denotes the exponent of the finite group G.

2. The case p odd

We first present some elementary results that will be used in the remainder of this paper.

Lemma 2.1. *Let A be an abelian subgroup of a finite group G. If A^G is abelian, then $\exp(A^G) = \exp(A)$. In particular, if A is elementary abelian and A^G is abelian, then A^G is elementary abelian.*

Lemma 2.2. *Let G be a finite p-group, and let N be a normal subgroup of G such that G/N is a non-trivial cyclic factor group of G. Let U/N be the unique subgroup of order p^k in G/N, where k is such that $p^k \leq |G/N|$. Then $\Omega_k(G) \leqslant U$.*

Lemma 2.3. *Let $G = AB$ be a finite p-group for subgroups A and B such that A is cyclic of order p^k, where $k \geq 2$. Then*

(i) *if $B \leqslant U \leqslant G$, then $U = \Omega_s(A)B$ for a suitable value s such that $0 \leq s \leq k$;*

(ii) *if $A \cap B = 1$, then $B = N_0 \lhd \Omega_1(A)B = N_1 \lhd \Omega_2(A)B = N_2 \lhd \cdots \lhd \Omega_k(A)B = N_k = AB = G$ is the unique series of subgroups of G that contain B, and which satisfy $N_i/N_{i-1} \cong C_p$ for $i = 1, \ldots, k$;*

(iii) *if B has exponent p and $A \cap B \neq 1$, then $B = \Omega_1(A)B = N_0 \lhd \Omega_2(A)B = N_1 \lhd \cdots \lhd \Omega_k(A)B = N_{k-1} = AB = G$ is the unique series of subgroups of G that contain B, and which satisfy $N_i/N_{i-1} \cong C_p$ for $i = 1, \ldots, k-1$.*

Lemma 2.4. *Let p be a prime, and let G be a finite p-group such that $G = AB$, where A and B are elementary abelian, normal subgroups of G. Then $G^{'} \leqslant Z(G)$ (so G has class at most 2). If, in addition, p is odd, then $\exp(G) = p$.*

Proof. We have $G/(A \cap B) = A/(A \cap B) \times B/(A \cap B)$, which is abelian, so $G^{'} \leqslant A \cap B \leqslant Z(AB) = Z(G)$. If p is odd, then, for $g \in G$, we let $g = ab$, where $a \in A$ and $b \in B$. Since G has class at most 2, we have $a^b = az$, where $z \in A \cap B \leqslant Z(G)$. By induction, we have $g^p = b^p a^p z^{\frac{(p+1)p}{2}}$. But $a^p = b^p = z^p = 1$ and p is odd, so $z^{\frac{(p+1)p}{2}} = 1$. We conclude that $\exp(G) = p$. $\square$

We note that the dihedral group of order 8 provides an example of a finite 2-group that is the product of two elementary abelian, normal subgroups, but which does not have exponent 2. Our next result provides some key information concerning products of elementary abelian and cyclic p-groups in the case where p is odd.

Lemma 2.5. *Let p be an odd prime, and let $G = AB$ be a finite p-group for subgroups A and B such that A is cyclic and B is elementary abelian. Then $\Omega_1(A)B \trianglelefteq G$.*

Proof. We let $|A| = p^k$, and let $A = \langle x\rangle$. If $k = 2$, then the result follows from Lemma 2.3. We thus assume that $k \geq 3$. We first consider the case where $A \cap B = \Omega_1(A) = \langle x^{p^{k-1}}\rangle$ (or, equivalently, where $A \cap B \neq 1$). We let $n = r(B)$ and use induction on n. For $n = 1$, we have $B = \Omega_1(A) \trianglelefteq A = G$. For $n > 1$, B is non-cyclic, so $A \neq G$. We let M be a maximal proper subgroup of G that contains A. Then $|G : M| = p$ and $M \trianglelefteq G$. Since $A \leqslant M$, we have $M = A(B \cap M)$. Now $|G| = \dfrac{|A||B|}{|A \cap B|} = p^{k+n-1}$, so $|M| = p^{k+n-2}$. Since $\Omega_1(A) = A \cap B = A \cap B \cap M$, we see, by comparison of orders, that $|B \cap M| = p^{n-1}$. We let $B_1 = B \cap M$. Then B_1 is elementary abelian of rank $n-1$ and $M = AB_1$, where $A \cap B_1 = \Omega_1(A)$. By induction, $B_1 \trianglelefteq M$. Since B_1 is trivially normalised by B, we have $B_1 \trianglelefteq G = MB$. We let $y \in B \setminus B_1$. Then $B = B_1\langle y\rangle$ and $G = AB_1\langle y\rangle = M\langle y\rangle$. We note that $o(y) = p$.

We have $M/B_1 = AB_1/B_1 \trianglelefteq G/B_1$. But $AB_1/B_1 \cong A/(A \cap B_1) = A/\Omega_1(A)$. Hence $M/B_1 \cong C_{p^{k-1}}$. In addition, $\Omega_2(A)B_1/B_1$ is the unique subgroup of order p in M/B_1. Since p is odd and y is an element of order p that acts by conjugation on M/B_1, we have $[x, y] \in \Omega_2(A)B_1$. Hence $[x, y] = ub_1$, where $u \in \Omega_2(A)$ and $b_1 \in B_1$. Now $\Omega_2(A)B_1/B_1$ is characteristic in M/B_1, so $\Omega_2(A)B_1/B_1 \trianglelefteq G/B_1$. Moreover, $\Omega_2(A)B_1/B_1 \cong C_p$, so $\Omega_2(A)B_1/B_1 \leqslant Z(G/B_1)$. Thus $[u, y] = b_2$, where $b_2 \in B_1$. Now $y \in B \leqslant C_G(B_1) \trianglelefteq G$, so $u = ub_1b_1^{-1} = [x, y]b_1^{-1} \in C_G(B_1)$. Hence b_1 and b_2 are centralised by both u and y. Since $o(y) = p$, we see by induction that $x = x^{y^p} = xu^pb_1^pb_2^{\frac{p(p-1)}{2}}$. Now $b_1^p = 1$, since B is elementary abelian. In addition, since p is odd, we have $b_2^{\frac{p(p-1)}{2}} = 1$. Thus $x = xu^p$, so $u^p = 1$. Since $u \in \Omega_2(A)$ and $u^p = 1$, we have $u \in \Omega_1(A) \leqslant B_1$. Therefore, $[x, y] \in B_1$, so $B = B_1\langle y\rangle = \Omega_1(A)B \trianglelefteq G$.

If $A \cap B = 1$, then $AB_G/B_G \cong A \cong C_{p^k}$. In addition, B/B_G is elementary abelian. Since B/B_G has a trivial core and G is a p-group, we see, by a result of Itô ([13, Satz 2] or [1, Lemma 2.1.4]), that $(AB_G/B_G) \cap Z(G/B_G) \neq 1$. By minimality, we have $1 \neq \Omega_1(A)B_G/B_G \leqslant Z(G/B_G)$. Hence $\Omega_1(A)B/B_G$ is elementary abelian and $AB_G/B_G \cap \Omega_1(A)B/B_G \neq 1_{G/B_G}$. From the above, we have $\Omega_1(A)B/B_G \trianglelefteq G/B_G$, and it follows that $\Omega_1(A)B \trianglelefteq G$. □

For the odd prime p, we use Lemma 2.5 to shed some light on the structure of finite p-groups that are products of elementary abelian and cyclic subgroups.

We first consider the case where the normal closure of the elementary abelian factor is non-abelian.

Theorem 2.6. *Let p be an odd prime, and let $G = AB$ be a finite p-group for subgroups A and B such that A is cyclic of order p^k and B is elementary abelian. If B^G is not abelian, then*

(i) $A_G = 1$;

(ii) $B^G = \Omega_1(A)B$;

(iii) $B_G = Z(B^G)$ *and* $|B^G : B_G| = p^2$;

(iv) B^G *has class* 2 *and exponent* p;

(v) $k \geq 2$ *and* $G/B_G \cong \langle x, y \mid x^{p^k} = y^p = 1,\ x^y = x^{1+p^{k-1}} \rangle$;

(vi) $B^G = \Omega_1(G)$;

(vii) $\Omega_1(A)B_G \trianglelefteq G$ *and* $\Omega_1(A)B_G = \Omega_1(A) \times B_G$, *so, in particular,* $\Omega_1(A)B_G$ *is an elementary abelian, normal subgroup of* G;

(viii) $G/\Omega_1(A)B_G$ *is abelian of type* (p, p^{k-1});

(ix) *if N is an elementary abelian, normal subgroup of G, then* $N \leqslant \Omega_1(A)B_G$ *(so $\Omega_1(A)B_G$ is the unique maximal elementary abelian, normal subgroup in G, and, in particular, $\Omega_1(A)B_G$ is characteristic in G)*;

(x) G *has exponent* p^k;

(xi) G' *is elementary abelian and* $r(G') \leq r(B)$.

Proof. If $A_G \neq 1$, then $\Omega_1(A)$ is the unique subgroup of order p in A_G, so $\Omega_1(A) \leqslant Z(G)$. By Lemma 2.5, $\Omega_1(A)B$ is then a normal elementary abelian subgroup of G. It follows that B^G is abelian. But this is ruled out, so $A_G = 1$ and (i) follows.

For (ii) we have $\Omega_1(A)B \trianglelefteq G$ by Lemma 2.5, so $B^G \leqslant \Omega_1(A)B$. By comparison of orders, either $B^G = B$ or $B^G = \Omega_1(A)B$. Hence, if B^G is not abelian, we have $B^G = \Omega_1(A)B$.

Since B^G is non-abelian, we see for (iii), that B is a proper subgroup of B^G. Thus $|B^G : B| = |\Omega_1(A)B : B| = |\Omega_1(A)| = p$. Letting $A = \langle x \rangle$, we have $B^x \neq B$, as otherwise $B \trianglelefteq G$. Hence, by comparison of orders, $B^G = BB^x$ and $|B : B \cap B^x| = |B^G : B| = p$, so $|B^G : B \cap B^x| = p^2$. Since B is abelian, we have $B \cap B^x \leqslant Z(\langle B, B^x \rangle) = Z(B^G)$, so $|B^G : Z(B^G)| \leq p^2$. Then $|B^G : Z(B^G)| = p^2$, as otherwise $B^G/Z(B^G)$ is cyclic and B^G is abelian. It follows that $B \cap B^x = Z(B^G)$. Since $Z(B^G) \trianglelefteq G$ and $Z(B^G) \leqslant B$, we have $Z(B^G) \leqslant B_G$. But $|B : Z(B^G)| = |B : B \cap B^x| = p$, so $Z(B^G) = B_G$ and $|B^G : B_G| = |B^G : Z(B^G)| = p^2$.

From the preceding paragraph, we have $B^G = BB^x$, where $|B^G : B| = |B^G : B^x| = p$. Thus B^G is the product of two elementary abelian, normal subgroups, so (iv) follows from Lemma 2.4.

We observe that $k \geq 2$, as otherwise $B \trianglelefteq G$ and B^G is abelian. Letting $y \in B \setminus B_G$, we note that $B = B_G\langle y\rangle$. Since $A \cap B \leqslant A_G = 1$ and $|G : AB_G| = |\langle y\rangle| = p$, we see that G/B_G is the extension of the normal subgroup $AB_G/B_G \cong C_{p^k}$ by $B/B_G \cong C_p$. In addition, G/B_G is non-abelian, as otherwise $B/B_G \trianglelefteq G/B_G$, which is excluded. Since $\mathrm{Aut}(C_{p^k})$ possesses precisely one subgroup of order p, we conclude that $G/B_G \cong \langle x, y \mid x^{p^k} = y^p = 1,\ x^y = x^{1+p^{k-1}}\rangle$, in accordance with (v).

For (vi), we note, from (v), that $\Omega_1(G/B_G) \cong C_p \times C_p$. By comparison of orders, we then have $\Omega_1(G)/B_G \leqslant \Omega_1(G/B_G) = \Omega_1(A)B_G/B_G \times \langle y\rangle B_G/B_G = \Omega_1(A)B/B_G \leqslant \Omega_1(G)/B_G$, so $\Omega_1(G) = \Omega_1(A)B = B^G$.

Since $(B/B_G)_{G/B_G} = 1_{G/B_G}$, we see, by [13, Satz 2] (or [1, Lemma 2.1.4]), that $AB_G/B_G \cap Z(G/B_G) \neq 1_{G/B_G}$. Now $\Omega_1(A) \nleqslant B$, as otherwise $B = \Omega_1(A)B \trianglelefteq G$, so $\Omega_1(A)B_G/B_G$ is the unique subgroup of order p in AB_G/B_G. Hence $\Omega_1(A)B_G/B_G \leqslant Z(G/B_G)$. In particular, $\Omega_1(A)B_G \trianglelefteq G$. We have $\Omega_1(A) \leqslant B^G \leqslant C_G(B_G)$, so $\Omega_1(A)B_G$ is elementary abelian. In addition, $\Omega_1(A) \cap B_G \leqslant A \cap B \leqslant A_G = 1$. Hence $\Omega_1(A)B_G = \Omega_1(A) \times B_G$, and (vii) follows.

For (viii), we have $A \cap \Omega_1(A)B_G = \Omega_1(A)$, so $AB_G/\Omega_1(A)B_G \cong A/\Omega_1(A) \cong C_{p^{k-1}}$. Now $|G : AB_G| = |B : B_G| = p$, so $AB_G \trianglelefteq G$. In addition, $B^G/\Omega_1(A)B_G \cong C_p$. We further have $AB_G/\Omega_1(A)B_G \cap B^G/\Omega_1(A)B_G = (A \cap B^G)B_G/\Omega_1(A)B_G = \Omega_1(A)B_G/\Omega_1(A)B_G = 1_{G/\Omega_1(A)B_G}$. Hence $G/\Omega_1(A)B_G$ is the direct product of $AB_G/\Omega_1(A)B_G$ and $B^G/\Omega_1(A)B_G$, so $G/\Omega_1(A)B_G \cong C_p \times C_{p^{k-1}}$. Therefore, $G/\Omega_1(A)B_G$ is abelian of type (p, p^{k-1}).

Now, for (ix), if N is an elementary abelian, normal subgroup of G, then $N \leqslant \Omega_1(G) = \Omega_1(A)B$. If $NB_G = \Omega_1(G)$, then, since $B_G = Z(B^G) = Z(\Omega_1(G))$ and N is abelian, we see that $B^G = \Omega_1(G)$ is abelian, which is excluded. Hence NB_G is a proper subgroup of $\Omega_1(G)$. If $N \nleqslant \Omega_1(A)B_G$, then NB_G/B_G and $\Omega_1(A)B_G/B_G$ are distinct proper subgroups of $\Omega_1(G)/B_G$. From (v), $\Omega_1(G)/B_G \cong C_p \times C_p$, so $\Omega_1(G)/B_G = NB_G/B_G \times \Omega_1(A)B_G/B_G$. Since NB_G/B_G and $\Omega_1(A)B_G/B_G$ are normal subgroups of order p in G/B_G, we then have $\Omega_1(G)/B_G \leqslant Z(G/B_G)$. It follows that $G/B_G = (AB_G/B_G)(\Omega_1(G)/B_G)$ is abelian, in contradiction to (v). We conclude that $N \leqslant \Omega_1(A)B_G$.

Next, we recall that $A \cong C_{p^k}$. From (vii) and (viii), we see that $G/\Omega_1(A)B_G$ has exponent p^{k-1} and that $\Omega_1(A)B_G$ is elementary abelian. Hence G has exponent p^k, in accordance with (x). Finally, by (viii), $G^{'} \leqslant \Omega_1(A)B_G$, so $G^{'}$ is elementary abelian. In addition, $r(G^{'}) \leq r(\Omega_1(A)B_G) = 1 + r(B) - 1 = r(B)$, so (xi) follows. $\square$

In the case where the normal closure of the elementary abelian factor is non-abelian, the decomposition $G = AB$, as in Theorem 2.6, displays some aspects of uniqueness, as our next result shows.

Corollary 2.7. *Let p be an odd prime, and let $G = AB$ be a finite p-group for subgroups A and B such that A is cyclic and B is elementary abelian. If B^G is not abelian and if G is also the product of the cyclic subgroup $\widetilde{A}$ and the elementary abelian subgroup $\widetilde{B}$, then*

(i) $\widetilde{B}^G = \Omega_1(\widetilde{A})\widetilde{B} = \Omega_1(G) = B^G$;

(ii) $(\widetilde{B})_G = Z(\Omega_1(G)) = B_G$;

(iii) $\widetilde{A} \cong A$ *and* $\widetilde{B} \cong B$.

Proof. If $\widetilde{B}^G$ is abelian (and hence elementary abelian), then $\widetilde{B}^G \leqslant \Omega_1(A)B$ by Theorem 2.6 (ix). Now $G/\widetilde{B}^G$ is isomorphic to a factor group of $\widetilde{A}$, so $G/\widetilde{B}^G$ is cyclic. But $G/\Omega_1(A)B_G$ is isomorphic to a factor group of $G/\widetilde{B}^G$, so $G/\Omega_1(A)B_G$ is also cyclic, in contradiction to Theorem 2.6 (viii). Hence $\widetilde{B}^G$ is non-abelian, so (i) and (ii) follow from Theorem 2.6 (vi) and (iii), respectively.

By Theorem 2.6 (x), we have $\exp(G) = |A| = |\widetilde{A}|$, so $A \cong \widetilde{A}$. Finally, both B and $\widetilde{B}$ have non-abelian normal closures, so we apply Theorem 2.6 (iii) to see that $|\widetilde{B}^G : \widetilde{B}| = |B^G : B| = p$. Since $\widetilde{B}^G = B^G$, it follows that $|\widetilde{B}| = |B|$, so $\widetilde{B} \cong B$ and (iii) is established. $\square$

We now develop an alternative characterisation of the groups in Theorem 2.6. For this, we require some preparatory results and examples.

Lemma 2.8. *Let G be a finite p-group, and let W and A be subgroups of G such that W is elementary abelian and $A = \langle x \rangle \cong C_{p^k}$, for $k \geq 2$. Suppose that*

(i) $[A, W] \leqslant W$;

(ii) $[\Omega_1(A), W] = 1$;

(iii) $\Omega_1(A) \cap W = 1$.

Then, if $w \in W$, we have $o(xw) = p^k$.

Proof. We note that $(xw)^{p^{k-1}} = w^{x^{-1}} \cdots w^{x^{-p^{k-1}}} x^{p^{k-1}} = w_1 x^{p^{k-1}}$, where $w_1 = w^{x^{-1}} \cdots w^{x^{-p^{k-1}}}$. Since $[A, W] \leqslant W$, we have $w_1 \in W$. We further have $1 \neq x^{p^{k-1}} \in \Omega_1(A)$. Now $[\Omega_1(A), W] = \Omega_1(A) \cap W = 1$, so $o(w_1 x^{p^{k-1}}) = p$. It follows that $o(xw) = p^k$. $\square$

Example 2.9. Let p be an odd prime, and let $k \geq 2$. Let $W = \langle w_1, \ldots, w_{p^{k-1}} \rangle$ be elementary abelian of rank p^{k-1}. Let $\langle x \rangle \cong C_{p^k}$, and let x act on W as follows:

$$w_i^x = w_{i+1}, \quad i = 1, \ldots,\ p^{k-1} - 1, \quad w_{p^{k-1}}^x = w_1.$$

Then x defines an automorphism of W such that $C_{\langle x \rangle}(W) = \langle x^{p^{k-1}} \rangle = \Omega_1(\langle x \rangle)$. We let H_k be the semi-direct product of W by $\langle x \rangle$, and see that H_k can be expressed as

$$H_k = \left\langle \begin{array}{c|l} w_1, \ldots, w_{p^{k-1}} & w_1^p = \cdots = w_{p^{k-1}}^p = x^{p^k} = 1;\ [w_i, w_j] = 1,\ i, j = 1, \ldots, p^{k-1} \\ x & w_i^x = w_{i+1},\ i = 1, \ldots, p^{k-1} - 1;\ w_{p^{k-1}}^x = w_1 \end{array} \right\rangle.$$

We let $z = w_1 \cdots w_{p^{k-1}}$. Then $Z(H_k) = \langle z, x^{p^{k-1}} \rangle \cong C_p \times C_p$.

Lemma 2.10. *Let H_k, W, x and z be as in Example 2.9, and let $w \in W$. Then $w^{x^{-1}} \cdots w^{x^{-p^{k-1}}} \in \langle z \rangle$.*

Proof. Since $x^{p^{k-1}}$ centralises W, we can see that $(w^{x^{-1}} \cdots w^{x^{-p^{k-1}}})^x = w w^{x^{-1}} \cdots w^{x^{-p^{k-1}+1}} = w^{x^{-1}} \cdots w^{x^{-p^{k-1}+1}} w = w^{x^{-1}} \cdots w^{x^{-p^{k-1}}}$. It then follows that $w^{x^{-1}} \cdots w^{x^{-p^{k-1}}} \in W \cap Z(H_k) = \langle z \rangle$. $\square$

Example 2.11. We let p be an odd prime, and let H_k be as in Example 2.9. We define a mapping, y, of H_k as follows:

$$w_i^y = w_i, \quad \text{for } i = 1, \ldots, p^{k-1}, \qquad x^y = x^{1+p^{k-1}} w_1.$$

Now $x^{p^{k-1}}$ centralises W, so we see that x^y acts on W in the same manner as x. In addition, $o(x^y) = p^k$ by Lemma 2.8. Hence y extends to an automorphism of H_k. We have $(x^{p^{k-1}})^y = (x x^{p^{k-1}} w_1)^{p^{k-1}} = (x w_1)^{p^{k-1}} x^{p^{2k-2}}$. Since $k \geq 2$, we have $x^{p^{2k-2}} = 1$, so

$$(x^{p^{k-1}})^y = (x w_1)^{p^{k-1}} = w_1^{x^{-1}} \cdots w_1^{x^{-p^{k-1}}} x^{p^{k-1}} = x^{p^{k-1}} z.$$

Then $[x^{p^{k-1}}, y] = z$. In addition, we have $x^{y^2} = (x x^{p^{k-1}} w_1)^y = x x^{p^{k-1}} w_1 x^{p^{k-1}} z w_1 = x (x^{p^{k-1}})^2 w_1^2 z$, and we see inductively that

$$x^{y^p} = x (x^{p^{k-1}})^p w_1^p z^{\frac{p(p-1)}{2}}.$$

Since p is odd, it follows that $x^{y^p} = x$, so y has order p in $\mathrm{Aut}(H_k)$. We then let G_k be the semi-direct product of H_k by $\langle y \rangle$, so that

$$G_k = \left\langle \begin{array}{c|l} w_1, \ldots, w_{p^{k-1}} & w_1^p = \cdots = w_{p^{k-1}}^p = x^{p^k} = y^p = 1;\ [w_i, w_j] = 1, i, j = 1, \ldots, p^{k-1} \\ x & w_i^x = w_{i+1},\ i = 1, \ldots, p^{k-1} - 1;\ w_{p^{k-1}}^x = w_1 \\ y & w_i^y = w_i,\ i = 1, \ldots, p^{k-1};\ x^y = x^{1+p^{k-1}} w_1 \end{array} \right\rangle.$$

Letting $A = \langle x \rangle$ and $B = \langle w_1, \ldots, w_{p^{k-1}}, y \rangle$, we see that $G_k = AB$, where A is cyclic of order p^k and B is elementary abelian. Since $[x, y] = x^{p^{k-1}} w_1$, we see that $B^{G_k} = \Omega_1(A)B$. But $[x^{p^{k-1}}, y] = z \neq 1$, so B^{G_k} is non-abelian. Thus G_k is a group that satisfies the hypotheses of Theorem 2.6. We note that G_k further provides us with an example of a group that is the product of a cyclic subgroup and an elementary abelian subgroup, but which is neither an extension of an elementary abelian group by a cyclic group, nor an extension of a cyclic group by an elementary abelian group. Our next result shows that certain groups are isomorphic to G_k.

Lemma 2.12. *Let p be an odd prime, and let $k \geq 2$. Let $G = \langle \widetilde{x} \rangle V \langle \widetilde{y} \rangle$ be a finite p-group such that*

(i) *V is an elementary abelian, normal subgroup of G;*
(ii) *$\langle \widetilde{x} \rangle \cong C_{p^k}$ and $\langle \widetilde{y} \rangle \cong C_p$;*
(iii) *$[V, \Omega_1(\langle \widetilde{x} \rangle)] = [V, \langle \widetilde{y} \rangle] = 1$;*
(iv) *there exists an element $v \in V$ such that $V = \langle v^{\widetilde{x}^i} \mid i = 1, \ldots, p^k \rangle$;*
(v) *$(V\langle \widetilde{y} \rangle)^G$ is non-abelian.*

Then G is isomorphic to G_k, for G_k as in Example 2.11.

Proof. Since $\Omega_1(\langle \widetilde{x} \rangle)$ and $\langle \widetilde{y} \rangle$ both centralise V, v has at most p^{k-1} conjugates in G. Hence $r(V) \leq p^{k-1}$. Now G is the product of the cyclic subgroup $\langle \widetilde{x} \rangle$ and the elementary abelian subgroup $V\langle \widetilde{y} \rangle$, where $(V\langle \widetilde{y} \rangle)^G$ is non-abelian. We have $V \trianglelefteq G$, so $V \leqslant (V\langle \widetilde{y} \rangle)_G$. Then $V = (V\langle \widetilde{y} \rangle)_G$, as otherwise $V\langle \widetilde{y} \rangle \trianglelefteq G$ and $(V\langle \widetilde{y} \rangle)^G$ is abelian. Hence, by Theorem 2.6 (v), we have $G/V \cong \langle x, y \mid x^{p^k} = y^p = 1,\ x^y = x^{1+p^{k-1}} \rangle$. In addition, $(V\langle \widetilde{y} \rangle)^G = \Omega_1(\langle \widetilde{x} \rangle)V\langle \widetilde{y} \rangle$ by Theorem 2.6 (ii).

We have $\langle \widetilde{x} \rangle \cap V \leqslant \langle \widetilde{x} \rangle \cap Z(G) \leqslant \langle \widetilde{x} \rangle_G = 1$ (by Theorem 2.6 (i)), so $\langle \widetilde{x} \rangle V/V \cong C_{p^k}$. Thus we may assume that $\widetilde{x}^{\widetilde{y}} = \widetilde{x}^{1+p^{k-1}} v_1$, where $v_1 \in V$. Since $k \geq 2$ and v_1 centralises $\Omega_1(\langle \widetilde{x} \rangle) = \langle \widetilde{x}^{p^{k-1}} \rangle$, we have

$$(\widetilde{x}^{p^{k-1}})^{\widetilde{y}} = (\widetilde{x}\widetilde{x}^{p^{k-1}} v_1)^{p^{k-1}} = (\widetilde{x} v_1)^{p^{k-1}} \widetilde{x}^{p^{2k-2}} = (\widetilde{x} v_1)^{p^{k-1}} = v_1^{\widetilde{x}^{-1}} \cdots v_1^{\widetilde{x}^{-p^{k-1}}} \widetilde{x}^{p^{k-1}}.$$

Hence the relation $(\widetilde{x}^{p^{k-1}})^{\widetilde{y}} = \widetilde{x}^{p^{k-1}} v_1^{\widetilde{x}^{-1}} \cdots v_1^{\widetilde{x}^{-p^{k-1}}}$ is satisfied.

We let $V_1 = \langle v_1, v_1^{\widetilde{x}}, \ldots, v_1^{\widetilde{x}^{p^{k-1}-1}} \rangle$. Then $V_1 = \langle v_1^{\widetilde{x}^i} \mid i = 1, \ldots, p^k \rangle$, so $\widetilde{x}$ normalises V_1. Hence the mapping ϕ defined by

$$\phi(w_1) = v_1, \phi(w_2) = v_1^{\widetilde{x}}, \ldots, \phi(w_{p^{k-1}}) = v_1^{\widetilde{x}^{p^{k-1}-1}}, \qquad \phi(x) = \widetilde{x},$$

extends to an epimorphism from H_k, as in Example 2.9, onto $\langle\widetilde{x}\rangle V_1$. We note that $\phi(W) = V_1$, and that $\phi(\langle x\rangle) = \langle\widetilde{x}\rangle \cong C_{p^k}$. Since $\langle\widetilde{x}\rangle \cap V_1 \leqslant \langle\widetilde{x}\rangle \cap V = 1$, we have $\ker(\phi) \leqslant W$. If ϕ is not an isomorphism, then, by minimality, $\langle z\rangle \leqslant \ker(\phi)$. Now $\phi(w_1) = v_1$, and we can apply Lemma 2.10 to see that $w_1^{x^{-1}} \cdots w_1^{x^{-p^{k-1}}} \in \langle z\rangle$, so $v_1^{\widetilde{x}^{-1}} \cdots v_1^{\widetilde{x}^{-p^{k-1}}} = 1$. It follows that $(\widetilde{x}^{p^{k-1}})^{\widetilde{y}} = \widetilde{x}^{p^{k-1}}$. Hence $\widetilde{y}$ centralises $\Omega_1(\widetilde{x})$, so $(V\langle\widetilde{y}\rangle)^G = \Omega_1(\widetilde{x})V\langle\widetilde{y}\rangle$ is abelian, which is ruled out. We conclude that ϕ defines an isomorphism from H_k to $\langle\widetilde{x}\rangle V_1$. In particular, $r(V_1) = r(W)$, so $V_1 = V$ by comparison of orders. We further extend ϕ by letting $\phi(y) = \widetilde{y}$. Since G/V is non-abelian, we have $\langle\widetilde{y}\rangle \cap \langle\widetilde{x}\rangle V = 1$. Hence ϕ, thus extended, defines an isomorphism from G_k to G. $\square$

Theorem 2.13. *Let p be an odd prime, let $k \geq 2$ and let G_k be as in Example 2.11. Then the following are equivalent for the finite p-group G:*

(i) *$G = AB$ for subgroups A and B, where A is cyclic of order p^k, B is elementary abelian and B^G is non-abelian;*

(ii) *$G = AB$ for subgroups A and B, where A is cyclic of order p^k, B is elementary abelian, $|B : B_G| = p$, and, for $y \in B \setminus B_G$, $\langle A, y\rangle \cong G_k$;*

(iii) *G is of the form $G = AW\langle y\rangle$, where*
   (a) *W is an elementary abelian, normal subgroup of G;*
   (b) *$A \cong C_{p^k}$ and $\langle y\rangle \cong C_p$;*
   (c) *$[\Omega_1(A), W] = [W, \langle y\rangle] = 1$;*
   (d) *$\langle A, y\rangle \cong G_k$.*

Proof. To show that (i) implies (ii), we note first that $|B^G : B_G| = p^2$ by Theorem 2.6 (iii). Since B^G is non-abelian, it follows that $|B : B_G| = p$. Letting $A = \langle x\rangle$ and $y \in B \setminus B_G$, we may assume, by Theorem 2.6 (v), that $x^y = xx^{p^{k-1}}v$, where $v \in B_G$. By Theorem 2.6 (ii) and (iii), we have $\langle x^{p^{k-1}}\rangle = \Omega_1(A) \leqslant B^G \leqslant C_G(B_G)$. We let $V = \langle v, v^x, \ldots, v^{x^{p^{k-1}-1}}\rangle$. Then $V \leqslant B_G$, so V is elementary abelian and $[V, \Omega_1(A)] = [V, \langle y\rangle] = 1$. In addition, x normalises V, so $V \trianglelefteq \langle A, V, y\rangle = \langle A, y\rangle = \langle x\rangle V\langle y\rangle$.

Now, $|\langle A, y\rangle/V| = |\langle x\rangle V\langle y\rangle/V| \leq |\langle x\rangle||\langle y\rangle| = p^{k+1}$. In addition, $G = AB = \langle A, y\rangle B$ and $B = \langle y\rangle B_G$, so $G/B_G = \langle A, y\rangle B_G/B_G \cong \langle A, y\rangle/(\langle A, y\rangle \cap B_G)$. But $V \leqslant \langle A, y\rangle \cap B_G$ and $|G/B_G| = p^{k+1}$ (by Theorem 2.6 (v)), so, by comparison of orders, we have $V = \langle A, y\rangle \cap B_G$ and $\langle A, y\rangle/V \cong G/B_G$. It follows from Theorem 2.6 (v) that $(V\langle y\rangle)^{\langle A,y\rangle}/V = \Omega_1(A)V\langle y\rangle/V$, so $(V\langle y\rangle)^{\langle A,y\rangle} = \Omega_1(A)V\langle y\rangle$. But $B^G = \Omega_1(A)B = \Omega_1(A)B_G\langle y\rangle$ and $B_G = Z(B^G)$ (by Theorem 2.6 (iii)). In addition, B^G is non-abelian, so $1 \neq (B^G)' = [\Omega_1(A), \langle y\rangle] = ((V\langle y\rangle)^{\langle A,y\rangle})'$.

Therefore, $(V\langle y\rangle)^{\langle A,y\rangle}$ is non-abelian, so we may apply Lemma 2.12 to see that $\langle A, y\rangle \cong G_k$.

Conversely, if $|B : B_G| = p$ and $\langle A, y\rangle \cong G_k$ for $y \in B \setminus B_G$, then, letting $y \in B\setminus B_G$, we have $B = B_G\langle y\rangle$ and $\langle A, y\rangle = A(\langle A, y\rangle \cap B)$. Letting $\widetilde{B} = \langle A, y\rangle \cap B$, we see that $\widetilde{B}$ is elementary abelian and that $\langle A, y\rangle = A\widetilde{B}$. Since $\langle A, y\rangle$ is isomorphic to G_k, which is the product of a cyclic subgroup and an elementary abelian subgroup whose normal closure is non-abelian, we apply Corollary 2.7 to see that $\widetilde{B}^{\langle A,y\rangle} \cong \Omega_1(G_k)$, which is non-abelian. But $\widetilde{B}^{\langle A,y\rangle} \leqslant B^G$, so B^G is non-abelian. Hence (i) and (ii) are equivalent.

To show that (i) and (ii) imply (iii), we let $y \in B \setminus B_G$. Then $G = AB_G\langle y\rangle$, where B_G is an elementary abelian, normal subgroup of G, $A \cong C_{p^k}$ and $\langle y\rangle \cong C_p$. Since B^G is non-abelian, we have $\Omega_1(A) \leqslant B^G \leqslant C_G(B_G)$, so $[\Omega_1(A), B_G] = [B_G, \langle y\rangle] = 1$. In addition, we have $\langle A, y\rangle \cong G_k$ from (ii). Thus, letting $W = B_G$, we have $G = AW\langle y\rangle$ and see that the conditions for (iii) are satisfied.

Finally, if (iii) holds, we let $B = W\langle y\rangle$. Then $G = AB$, where $A \cong C_{p^k}$ and B is elementary abelian. Since $G_k \cong \langle A, y\rangle = A(\langle A, y\rangle \cap B)$, we let $\widetilde{B} = (\langle A, y\rangle \cap B)$ and again apply Corollary 2.7 to see that $\widetilde{B}^{\langle A,y\rangle}$, and hence B^G, is non-abelian. We thus conclude that (iii) implies (i). $\square$

For the odd prime, p, Theorem 2.13 shows that finite p-groups which factorise as the product of a cyclic subgroup and an elementary abelian subgroup whose normal closure is non-abelian can always be realised as an extension of an elementary abelian group by a cyclic group, in turn extended by a suitable group of automorphisms of order p. Our next two results deal with the case where the normal closure of the elementary abelian factor is abelian (and hence elementary abelian by Lemma 2.1). The first gives conditions under which certain products can be realised as faithful split extensions of elementary abelian groups by cyclic groups.

Theorem 2.14. *Let p be an odd prime, and let $G = AB$ be a finite p-group for subgroups A and B such that A is cyclic and B is elementary abelian. Then the following are equivalent:*

(i) *B^G is abelian and $A_G = 1$;*

(ii) *$B \trianglelefteq G$ and $A_G = 1$;*

(iii) *$B \trianglelefteq G$, $A \cap B = 1$ and $C_A(B) = 1$ (so that G is a faithful split extension of B by A).*

Proof. To show that (i) implies (ii), we note that if B is not normal in G, then $B^G = \Omega_1(A)B$ by Lemma 2.5. But B^G is abelian, so B centralises $\Omega_1(A)$. Hence $1 \neq \Omega_1(A) \leqslant A \cap Z(G) \leqslant A_G$, and a contradiction arises.

To show that (ii) implies (iii), we note that $A \cap B \leqslant A \cap Z(G) \leqslant A_G = 1$. We similarly have $C_A(B) \leqslant A \cap Z(G) = 1$. Therefore, G is a faithful split extension of B by A.

Finally, if G is a faithful split extension of B by A, then $B \trianglelefteq G$, so $B^G = B$ and B^G is elementary abelian. In addition, $A_G \cap B \leqslant A \cap B = 1$, so $[A_G, B] \leqslant A_G \cap B = 1$. Hence $A_G \leqslant C_A(B) = 1$, so (i) follows from (iii). □

For odd p, the final case we need to consider is where the elementary abelian factor has an (elementary) abelian normal closure and the cyclic factor has a non-trivial core. In fact, only the latter condition is required, as seen in the following result.

Theorem 2.15. *Let p be an odd prime, and let $G = AB$ be a non-cyclic, finite p-group for subgroups A and B such that A is cyclic of order p^k and B is elementary abelian. If $A_G \neq 1$, then*

(i) *$\Omega_1(A)B$ is an elementary abelian, normal subgroup of G;*

(ii) *if N is an elementary abelian, normal subgroup of G, then $N \leqslant \Omega_1(A)B$ (so $\Omega_1(A)B$ is the unique maximal elementary abelian, normal subgroup in G, and, in particular, $\Omega_1(A)B$ is characteristic in G);*

(iii) *B^G is elementary abelian;*

(iv) *G has exponent p^k;*

(v) *G has a normal subgroup, W, such that*

  (a) *W is elementary abelian;*

  (b) *$W = \Omega_1(A) \times \widehat{B}$, for a suitable subgroup $\widehat{B} \leqslant B$;*

  (c) *G/W is abelian of type (p, p^{k-1});*

(vi) *G' is elementary abelian and $r(G') \leq r(B)$;*

(vii) *for $\widetilde{B} = \Omega_1(A)B$, we have $G = A\widetilde{B}$, where $\widetilde{B}$ is an elementary abelian, normal subgroup of G and $A \cap \widetilde{B} = \Omega_1(A)$.*

Proof. Since $A_G \neq 1$, we have $A_G \cap Z(G) \neq 1$, so $\Omega_1(A) \leqslant Z(G)$. Hence, by Lemma 2.5, $\Omega_1(A)B$ is an elementary abelian, normal subgroup of G, in accordance with (i).

For (ii), if N is an elementary abelian, normal subgroup of G such that $N \nleqslant \Omega_1(A)B$, then, by Lemma 2.3 (iii), we have $\Omega_2(A) \leqslant \Omega_1(A)BN$. Hence $\exp(\Omega_1(A)BN) \geq p^2$. But $\Omega_1(A)BN$ is the product of two normal elementary abelian subgroups, so $\exp(\Omega_1(A)BN) = p$ by Lemma 2.4, and a contradiction ensues.

We observe that (iii) follows from Theorem 2.6 (i). For (iv), we note that $G/\Omega_1(A)B \cong A/\Omega_1(A) \cong C_{p^{k-1}}$ and $\exp(\Omega_1(A)B) = p$. Since A is cyclic of order p^k, we conclude that $\exp(G) = p^k$.

Since G is non-cyclic, $\Omega_1(A)$ is a proper subgroup of $\Omega_1(A)B$. Therefore, since G is a finite p-group and $\Omega_1(A) \leqslant Z(G)$, there exists a normal subgroup $W \trianglelefteq G$ with $\Omega_1(A) \leqslant W \leqslant \Omega_1(A)B$, and such that $|\Omega_1(A)B : W| = p$. It follows that $W = \Omega_1(A)(W \cap B)$. If $\Omega_1(A) \cap (W \cap B) = 1$, then we let $\widehat{B} = W \cap B$ and have $W = \Omega_1(A) \times \widehat{B}$. If $\Omega_1(A) \cap (W \cap B) \neq 1$, then $\Omega_1(A) \leqslant W \cap B$. We let $\widehat{B}$ be a complement for $\Omega_1(A)$ in $W \cap B$ and see that $W = \Omega_1(A) \times \widehat{B}$. Now, $\Omega_1(A)B/W \cong C_p$ and $\Omega_1(A)B/W \trianglelefteq G/W$, so $\Omega_1(A)B/W \leqslant Z(G/W)$. In addition, $AW/W \cong A/(A \cap W) = A/\Omega_1(A) \cong C_{p^{k-1}}$. We further have $\Omega_1(A)B \cap AW = (\Omega_1(A)B \cap A)W = \Omega_1(A)W = W$, so $\Omega_1(A)B/W \cap AW/W = 1_{G/W}$. Hence $G/W = \Omega_1(A)B/W \times AW/W \cong C_p \times C_{p^{k-1}}$, so G/W is abelian of type (p, p^{k-1}) and (v) is established.

For (vi), we see that $G^{'} \leqslant W$, so $G^{'}$ is elementary abelian and $r(G^{'}) \leq r(W)$. If $\Omega_1(A) \cap B = 1$, then $|\Omega_1(A)B : B| = p = |\Omega_1(A)B : W|$, so $r(W) = r(B)$. If $\Omega_1(A) \leqslant B$, then $|\Omega_1(A)B : W| = |B : W| = p$, so $r(W) = r(B) - 1$. Hence $r(G^{'}) \leq r(B)$.

Finally, we note that (vii) follows directly from (i). □

Now suppose, as in the preceding theorem, that $G = AB$, where A is cyclic of order p^k, B is elementary abelian and $A_G \neq 1$. Let $A = \langle x \rangle$, and let $U = \Omega_1(A)B$. Then $A \cap U = \Omega_1(A) = \langle x^{p^{k-1}} \rangle$. In addition, let $u_1 = x^{p^{k-1}}$, and let $\{u_2, \ldots, u_t\}$ be a minimal generating set for a complement for $\langle u_1 \rangle$ in U. In particular, we have $r(U) = t$. Let $\langle \widetilde{x} \rangle$ be isomorphic to C_{p^k}. Since conjugation by x induces an automorphism of order at most p^{k-1} on U, we may let $\langle \widetilde{x} \rangle$ act on U by letting $u_i^{\widetilde{x}} = u_i^x$, and see that, under this action, U is centralised by $\langle \widetilde{x}^{p^{k-1}} \rangle$. We let $\widetilde{G}$ be the semi-direct product of U by $\langle \widetilde{x} \rangle$, so that

$$\widetilde{G} = \left\langle \begin{array}{c|l} u_1, \ldots, u_t & u_1^p = \cdots = u_t^p = \widetilde{x}^{p^k} = 1;\ [u_i, u_j] = 1,\ i, j = 1, \ldots, t \\ \widetilde{x} & u_i^{\widetilde{x}} = u_i^x,\ i = 1, \ldots, t \end{array} \right\rangle.$$

We further observe that $|\widetilde{G}| = p^{t+k} = p|G|$, and that $\langle u_1, \widetilde{x}^{p^{k-1}} \rangle \leqslant Z(\widetilde{G})$. Since the appropriate relations are satisfied, we have $G \cong \widetilde{G}/\langle u_1^{-1} \widetilde{x}^{p^{k-1}} \rangle$. Bearing Theorem 2.14 in mind, we then see that, for the odd prime p, finite p-groups that factorise as the product of a cyclic subgroup and an elementary abelian subgroup whose normal closure is abelian can either be realised as a faithful split extension of an elementary abelian group by a cyclic group, or as a split extension of an elementary abelian group by a cyclic group, modulo a subgroup of order p in its centre.

For the finite p-group $G = AB$, where A is cyclic, we observe that $\exp(G) \geq \exp(A) = |A|$. Our next result gives an upper bound for $\exp(G)$ in such cases.

Lemma 2.16. *Let $G = AB$ be a finite p-group for subgroups A and B such that A is cyclic. Then* $\exp(G) \leq \exp(A)\exp(B) = |A|\exp(B)$.

Proof. We let $|A| = p^k$ and suppose that $\exp(G) > \exp(A)\exp(B)$, say $\exp(G) = p^{k+s}\exp(B)$, where $s \geq 1$. Since G is a finite p-group, G possesses an element y whose order is equal to $\exp(G)$, so $o(y) = p^{k+s}\exp(B)$. Now $\langle y\rangle \cap B$ is cyclic, so $|\langle y\rangle \cap B|$ is a divisor of $\exp(B)$. Hence

$$|G| \geq |\langle y\rangle B| = \frac{|\langle y\rangle||B|}{|\langle y\rangle \cap B|} \geq \frac{|\langle y\rangle||B|}{\exp(B)} = \frac{p^{k+s}\exp(B)|B|}{\exp(B)} = p^{k+s}|B|.$$

But $|G| \leq |A||B| = p^k|B|$, and a contradiction arises since $s \geq 1$. $\square$

Combining Lemma 2.16 with Theorems 2.6 (x) and (xi), 2.14 and 2.15 (iv) and (vi), we have the following bounds on $\exp(G)$ and $|G^{'}|$ for a finite p-group G that is the product of a cyclic subgroup and an elementary abelian subgroup, where p is an odd prime.

Corollary 2.17. *Let p be an odd prime, and let $G = AB$ be a finite p-group for subgroups A and B such that A is cyclic of order p^k and B is elementary abelian. Then:*

(i) $p^k \leq \exp(G) \leq p^{k+1}$;

(ii) $G^{'}$ *is elementary abelian of rank at most* $r(B)$.

We finally present an example to show that there exist groups for which the hypotheses of Theorems 2.14 and 2.15 are both satisfied. Thus, in contrast to the case where the normal closure of the elementary abelian factor is non-abelian (see Corollary 2.7), quite dissimilar factorisations can occur in the case where a finite p-group (for p odd) is the product of a cyclic subgroup and an elementary abelian subgroup whose normal closure is abelian. The example also shows that the upper bounds on $\exp(G)$ and $r(G^{'})$ can be attained.

Example 2.18. Let p be an odd prime, and let $k \geq 2$. We let the wreath product $G = C_p wr C_{p^{k-1}}$ be presented as follows:

$$G = \left\langle \begin{matrix} w_1, \ldots, w_{p^{k-1}} \\ \theta \end{matrix} \;\middle|\; \begin{matrix} w_1^p = \cdots = w_{p^{k-1}}^p = \theta^{p^{k-1}} = 1;\ [w_i, w_j] = 1,\ i, j = 1, \ldots, p^{k-1} \\ w_i^\theta = w_{i+1},\ i = 1, \ldots, p^{k-1} - 1;\ w_{p^{k-1}}^\theta = w_1 \end{matrix} \right\rangle.$$

Then $G = AB$, where $A = \langle \theta \rangle \cong C_{p^{k-1}}$ and $B = \langle w_1, \ldots, w_{p^{k-1}} \rangle$ is elementary abelian of rank p^{k-1}. In this case, the hypotheses of Theorem 2.14 are satisfied. We let $x = \theta w_1$. We can confirm that $o(x) = p^k$. We let $\widetilde{A} = \langle x \rangle \cong C_{p^k}$ and $\widetilde{B} = \langle w_2, \ldots, w_{p^{k-1}} \rangle$. Then $\widetilde{B}$ is elementary abelian of rank $p^{k-1} - 1$, and we see that G also factorises as $G = \widetilde{A}\widetilde{B}$, where the hypotheses of Theorem 2.15 are satisfied.

Since $\exp(C_p wr C_{P^{k-1}}) = p^k$, we see that the upper bound on $\exp(G)$ is attained by the factorisation $G = AB$. In addition, $G^{'} = \langle w_1 w_2^{-1}, \ldots, w_{p^{k-1}-1} w_{p^{k-1}}^{-1} \rangle$, so $r(G^{'}) = p^{k-1} - 1$. Hence the upper bound on $r(G^{'})$ is attained by the factorisation $G = \widetilde{A}\widetilde{B}$.

3. The case $p = 2$

Turning to the case where $p = 2$, we first examine some cases where the normal closure of the elementary abelian factor is also elementary abelian. The following two results will be used in the proofs of Lemmas 3.3 and 3.7, respectively.

Lemma 3.1. *Let $G = B_1 B_2$ be a finite 2-group for subgroups B_1 and B_2 such that B_i is elementary abelian and $|G : B_i| = 2$ for $i = 1, 2$. Suppose that there exists an element $g \in G$ such that $o(g) = 2$ and such that $g \notin B_1 \cup B_2$. Then G is elementary abelian.*

Proof. We let $Z = B_1 \cap B_2$. Since the B_i are abelian, we have $Z \leqslant Z(B_1 B_2) = Z(G)$. In addition, we have $|G : Z| = 4$ and see that $|B_i : Z| = 2$ for $i = 1, 2$. Letting $b_i \in B_i$ be such that $B_i = \langle b_i \rangle Z$ for $i = 1, 2$, we then have $G/Z = B_1/Z \times B_2/Z = \langle b_1 \rangle Z/Z \times \langle b_2 \rangle Z/Z \cong C_2 \times C_2$. Since $g \notin B_1 \cup B_2$, there exists $z \in Z$ such that $g = b_1 b_2 z$. Now $g^2 = z^2 = 1$, so $(b_1 b_2 z)^2 = (b_1 b_2)^2 z^2 = (b_1 b_2)^2 = 1$. But $b_i \in B_i$, so $o(b_i) = 2$ for $i = 1, 2$. Hence $[b_1, b_2] = b_1 b_2 b_1 b_2 = (b_1 b_2)^2 = 1$. Thus b_1 and b_2 commute, so $G = \langle b_1, b_2 \rangle Z$ is abelian. Since G is generated by elements of order 2, it follows that G is elementary abelian. $\square$

Lemma 3.2. *Let G be a finite 2-group, and let $W \leqslant G$ and $x \in G$ be such that*

(i) *W is elementary abelian of rank 3;*

(ii) *$[\langle x \rangle, W] \leqslant W$;*

(iii) *$[\langle x^2 \rangle, W] = 1$.*

Then $|C_W(x)| \geq 4$.

Proof. Let $W = \langle w_1, w_2, w_3\rangle$. Since G is a 2-group and x normalises W, we have $C_W(x) \neq 1$. Hence we may assume that $w_1 \in C_W(x)$. If $|C_W(x)| < 4$, then $C_W(x) = \langle w_1\rangle \cong C_2$. Since x^2 centralises W, we have $(w_2w_2^x)^x = w_2^x w_2 = w_2w_2^x$, so $w_2w_2^x \in C_W(x) = \langle w_1\rangle$. If $w_2w_2^x = 1$, then $w_2^x = w_2^{-1} = w_2$, so $w_2 \in C_W(x) = \langle w_1\rangle$, which is a contradiction. Hence $w_2w_2^x = w_1$, and, similarly, $w_3w_3^x = w_1$. Then $w_2w_2^xw_3w_3^x = w_1^2 = 1$, so $w_2w_3(w_2w_3)^x = 1$. Hence $w_2w_3 \in C_W(x)$. Once more a contradiction arises, so we conclude that $|C_W(x)| \geq 4$. $\square$

Lemma 3.3. *Let $G = AB$ be a finite 2-group for subgroups A and B such that A is cyclic and B is elementary abelian. Then B^G is abelian (and thus elementary abelian) if and only if $B^G \leqslant \Omega_1(A)B$.*

Proof. By Lemma 2.3, $B^G = \Omega_s(A)B$ for a suitable s. If B^G is abelian, then, by Lemma 2.1, we have $\exp(\Omega_s(A)) \leq 2$, so $s \leq 1$. Hence $B^G \leqslant \Omega_1(A)B$. Conversely, suppose that $B^G \leqslant \Omega_1(A)B$. If $B^G = B \trianglelefteq G$, then B^G is elementary abelian. If B is not normal in G, then B is a proper subgroup of B^G, so $B^G = \Omega_1(A)B$ and $|B^G : B| = |\Omega_1(A)B : B| = 2$. Letting $A = \langle x\rangle \cong C_{2^k}$, we see that $B^x \leqslant B^G = \Omega_1(A)B$, but that $B^x \neq B$ (as otherwise $N_G(B) = AB = G$). Then $|\Omega_1(A)B : B^x| = 2$, and by comparison of orders, we have $\Omega_1(A)B = BB^x$. Now $x^{2^{k-1}} \in \Omega_1(A)B \setminus B$, so $x^{2^{k-1}} \notin B \cup B^x$. Since $o(x^{2^{k-1}}) = 2$, we may apply Lemma 3.1 to conclude that B^G is elementary abelian. $\square$

Our next result gives a condition under which Lemma 3.3 can be applied.

Lemma 3.4. *Let $G = AB$ be a finite p-group for subgroups A and B such that A is cyclic and B is elementary abelian. If $B \leqslant C_G(\Omega_2(A))$, then $\Omega_1(G) = \Omega_1(A)B$.*

Proof. We let $A \cong C_{p^k}$, and note that the result is trivial for $k = 1$. For $k \geq 2$, we use induction on k. For $k = 2$, B centralises A, so G is abelian. Let $g \in G$ be such that $g^p = 1$. Then $g = ab$, where $a \in A$ and $b \in B$. Since B is elementary abelian, we have $a^p = a^pb^p = g^p = 1$. Hence $\Omega_1(G) \leqslant \Omega_1(A)B$. Since the reverse inclusion is evident, it follows that $\Omega_1(G) = \Omega_1(A)B$.

We now assume that the result holds for some $k \geq 2$ and consider the case $G = AB$, where $A \cong C_{p^{k+1}}$, B is elementary abelian and $B \leqslant C_G(\Omega_2(A))$. By Lemma 2.3, $\Omega_k(A)B$ is the unique maximal subgroup of G that contains B. Now $\Omega_k(A) \cong C_{p^k}$ and $\Omega_2(\Omega_k(A)) = \Omega_2(A)$ centralises B. By induction, $\Omega_1(\Omega_k(A)B) = \Omega_1(\Omega_k(A))B = \Omega_1(A)B$. In particular, $\Omega_1(A)B$ is characteristic in $\Omega_k(A)B$ and is thus normal in G. Then

$$G/\Omega_1(A)B = A\Omega_1(A)B/\Omega_1(A)B \cong A/(A \cap \Omega_1(A)B) = A/\Omega_1(A) \cong C_{p^k}.$$

Now $k \geq 2$, so $\Omega_1(A)B$ is a proper subgroup of $\Omega_k(A)B$. Thus $\Omega_k(A)B/\Omega_1(A)B$ contains the unique subgroup of order p in $G/\Omega_1(A)B$. By Lemma 2.2, we have $\Omega_1(G) \leqslant \Omega_k(A)B$. Hence $\Omega_1(G) \leqslant \Omega_1(\Omega_k(A)B) = \Omega_1(A)B$, and we conclude that $\Omega_1(G) = \Omega_1(A)B$. □

Corollary 3.5. *Let $G = AB$ be a finite 2-group for subgroups A and B such that A is cyclic and B is elementary abelian. If $B \leqslant C_G(\Omega_2(A))$, then B^G is elementary abelian.*

Proof. By Lemma 3.4, we have $\Omega_1(A)B = \Omega_1(G)$. Hence $\Omega_1(A)B$ is characteristic in G, so $B^G \leqslant \Omega_1(A)B$. By Lemma 3.3 it follows that B^G is elementary abelian. □

We apply Lemma 3.3 to provide a partial analogue to Theorem 2.15 in the case where $p = 2$.

Theorem 3.6. *Let $G = AB$ be a non-cyclic, finite 2-group for subgroups A and B such that A is cyclic of order 2^k and B is elementary abelian. If B^G is abelian and $A_G \neq 1$, then*

(i) *$\Omega_1(A)B$ is an elementary abelian, normal subgroup of G;*

(ii) *B^G is elementary abelian;*

(iii) *G has exponent 2^k;*

(iv) *G has a normal subgroup, W, such that*

  (a) *W is elementary abelian;*

  (b) *$W = \Omega_1(A) \times \widehat{B}$, for a suitable subgroup $\widehat{B} \leqslant B$;*

  (c) *G/W is abelian of type $(2, 2^{k-1})$;*

(v) *G' is elementary abelian and $r(G') \leq r(B)$;*

(vi) *for $\widetilde{B} = \Omega_1(A)B$, we have $G = A\widetilde{B}$, where $\widetilde{B}$ is an elementary abelian, normal subgroup of G and $A \cap \widetilde{B} = \Omega_1(A)$.*

Proof. Since $A_G \neq 1$, we have $\Omega_1(A) \leqslant Z(G)$, so $\Omega_1(A)B$ is elementary abelian. Moreover, $B^G \leqslant \Omega_1(A)B$, by Lemma 3.3. Since $G/B^G \cong A/(A \cap B^G)$, which is cyclic, we then have $\Omega_1(A)B \trianglelefteq G$, so (i) follows. We further observe that (ii) follows immediately from Lemma 2.1. Substituting $p = 2$, the remainder of the proof now follows that of Theorem 2.15 (iv)–(vii), respectively. □

We come to the main result used to describe the structure of a finite 2-group that is the product of an elementary abelian subgroup and a cyclic subgroup in the case where the normal closure of the elementary abelian factor is non-abelian.

Lemma 3.7. *Let $G = AB$ be a finite 2-group for subgroups A and B such that A is cyclic and B is elementary abelian. If $B^G \nleqslant \Omega_1(A)B$, then $|G : C_G(\Omega_2(A))| = 2$.*

Proof. Let $|A| = 2^k$. We note first that $k \geq 3$, as otherwise $B^G \leqslant \Omega_1(A)B$ by Lemma 2.3. We let $A = \langle x \rangle$, and let $A_i = \Omega_i(A)$ $(= \langle x^{2^{k-i}} \rangle)$, $i = 1, \dots, k$. We further let $C = C_G(A_2)$ $(= C_G(\Omega_2(A)))$, and let $B_1 = C_B(A_2)$. Thus $B_1 = B \cap C$ and $C = AB_1$. If $C = G$, then $B \leqslant C_G(A_2)$, so, by Lemma 3.4, $B^G \leqslant \Omega_1(A)B$, which is excluded. Hence $|G : C| \geq 2$.

Now $B^G \nleqslant A_1B$, so A_1B is not normal in G. By Lemma 2.3, A_1B is a proper normal subgroup of A_2B, so, again by Lemma 2.3, we have $N_G(A_1B) = A_sB$, where $2 \leq s < k$. Since $s < k$, we have $|A_{s+1}B : A_sB| = 2$, so $x^{2^{k-s-1}}$ normalises A_sB, but does not normalise A_1B. We let $x_1 = x^{2^{k-s-1}}$. Then A_1B and $(A_1B)^{x_1} = A_1B^{x_1}$ are distinct normal subgroups of A_sB.

Since $A_sB/A_1B \cong A_s/(A_s \cap A_1B) \cong A_s/A_1 = C_{2^{s-1}}$, we see that A_2B/A_1B is the unique subgroup of order 2 in A_sB/A_1B. In addition, A_1B and $A_1B^{x_1}$ are generated by elements of order 2. Hence, by Lemma 2.2, we have $A_1BA_1B^{x_1} = A_1BB^{x_1} \leqslant \Omega_1(A_sB) \leqslant A_2B$. Since A_1B and $A_1B^{x_1}$ are then distinct subgroups of index 2 in A_2B, we have $A_1BB^{x_1} = A_2B$. But $A_1 = \Phi(A_2) \leqslant \Phi(A_2B)$, so $A_2B = \langle B, B^{x_1} \rangle$. Since B is abelian, we have $B \cap B^{x_1} \leqslant Z(\langle B, B^{x_1} \rangle) = Z(A_2B)$. In particular, $B \cap B^{x_1} \leqslant C_G(A_2) = C$, so $B \cap B^{x_1} \leqslant B_1$.

Now $A_1 \cong C_2$, so either $A_1 \leqslant B$ or $A_1 \cap B = 1$. If $A_1 \leqslant B$, then $A_1 \leqslant B^{x_1}$. Applying Lemma 2.3, we further have $|A_2B : B| = |A_2B : B^{x_1}| = 2$. Thus $A_2B = BB^{x_1}$ and $|B : B \cap B^{x_1}| = 2$. Since $|B : B_1| \leq |B : B \cap B^{x_1}|$, we have $|G| = \dfrac{|B||C|}{|B \cap C|} = \dfrac{|B||C|}{|B_1|} \leq 2|C|$. It then follows that $|G : C| = 2$. We may thus assume that $A_1 \cap B = 1$ $(= A \cap B)$. Then $|A_2B : B| = |A_2B : B^{x_1}| = 4$, so $|B : B \cap B^{x_1}| \leq 4$. Hence $|C| = |A||B_1| \geq |A||B \cap B^{x_1}| \geq \dfrac{|A||B|}{4} = \dfrac{|G|}{4}$, so $|G : C| \leq 4$. It follows that either $|G : C| = 2$ or $|G : C| = 4$. Thus, we may now further assume that $|G : C| = 4$. Then $|B : B_1| = 4$. Since B is elementary abelian, there exist y_1 and $y_2 \in B$ such that $B = B_1\langle y_1, y_2 \rangle$. Hence $G = C\langle y_1, y_2 \rangle$. Since G is a 2-group, we may assume, without loss of generality, that

$$C = AB_1 \trianglelefteq C\langle y_1 \rangle = AB_1\langle y_1 \rangle \trianglelefteq G = AB_1\langle y_1, y_2 \rangle.$$

We show that A_1B_1 is normal in G. By Lemma 3.4, we have $A_1B_1 = \Omega_1(C)$. Hence A_1B_1 is characteristic in C, so $A_1B_1 \trianglelefteq C\langle y_1 \rangle$. Then $C/A_1B_1 = AB_1/A_1B_1$ is a cyclic normal subgroup of order 2^{k-1} in $C\langle y_1 \rangle/A_1B_1$. But A_2B_1/A_1B_1 is characteristic in C/A_1B_1, so $A_2B_1 \trianglelefteq C\langle y_1 \rangle$. Since A_2B_1 is an abelian 2-group, we have $\Phi(A_2B_1) = \Phi(A_2)\Phi(B_1) = A_1$. Hence $A_1 \trianglelefteq C\langle y_1\rangle$. Since $A_1 \cong C_2$, we then have $A_1 \leqslant Z(C\langle y_1\rangle)$. But $Z(C\langle y_1\rangle) \leqslant C$, as otherwise the contradiction $C\langle y_1\rangle = CZ(C\langle y_1\rangle) \leqslant C_G(A_2) = C$ would result. In addition, if $Z(C\langle y_1\rangle) \nleqslant A_1B_1$, then, by minimality, $A_2B_1/A_1B_1 \leqslant Z(C\langle y_1\rangle)A_1B_1/A_1B_1$, so $A_2 \leqslant Z(C\langle y_1\rangle)A_1B_1$. But y_1 centralises both $Z(C\langle y_1\rangle)$ and A_1B_1, so y_1 centralises A_2, and once more a contradiction ensues. Therefore, $Z(C\langle y_1\rangle) \leqslant A_1B_1$. Now $A_1 \leqslant Z(C\langle y_1\rangle)$, so $A_1B_1 = B_1Z(C\langle y_1\rangle)$. But $Z(C\langle y_1\rangle)$ is characteristic in $C\langle y_1\rangle$, and hence normal in G. In addition, B is abelian, so y_2 centralises B_1. Therefore, y_2 normalises $A_1B_1 = B_1Z(C\langle y_1\rangle)$. But we already have $A_1B_1 \trianglelefteq C\langle y_1\rangle$. Since $G = C\langle y_1, y_2\rangle$, we conclude that $A_1B_1 \trianglelefteq G$.

Next, we show that $A_1 \leqslant Z(G)$ and that $B^G \leqslant C_G(A_1B_1)$. We note that $A_{k-1}B_1/A_1B_1 = \Phi(AB_1/A_1B_1) \leqslant \Phi(C\langle y_1\rangle/A_1B_1)$. In addition, $A_{k-1}B_1/A_1B_1$ is characteristic in AB_1/A_1B_1, so $A_{k-1}B_1 \trianglelefteq C\langle y_1\rangle$. We further have $|C\langle y_1\rangle/A_1B_1 : A_{k-1}B_1/A_1B_1| = |AB_1\langle y_1\rangle : A_{k-1}B_1| = 4$. But $y_1A_{k-1}B_1$ and (by Lemma 2.3) $xA_{k-1}B_1$ are distinct elements of order 2 in $C\langle y_1\rangle/A_{k-1}B_1$, so $C\langle y_1\rangle/A_{k-1}B_1$ is elementary abelian. Hence $A_{k-1}B_1/A_1B_1 = \Phi(C\langle y_1\rangle/A_1B_1)$, so $A_{k-1}B_1/A_1B_1 \trianglelefteq G/A_1B_1$. In addition, A_2B_1/A_1B_1 is characteristic in $A_{k-1}B_1/A_1B_1$, so $A_2B_1 \trianglelefteq G$. Now, from the above, $A_1 = \Phi(A_2B_1)$, so $A_1 \trianglelefteq G$. Since $A_1 \cong C_2$, we conclude that $A_1 \leqslant Z(G)$. Then A_1B_1 is a normal subgroup of G that is centralised by B, so it follows that $B^G \leqslant C_G(A_1B_1)$.

By Lemma 2.3, we have $B^G = A_sB$, where $s \geq 0$. Since $B^G \nleqslant A_1B$, we again apply Lemma 2.3 to see that $A_2B \leqslant B^G$. We suppose first that $B^G = A_2B$. From the preceding paragraph, we have $A_2B_1 \trianglelefteq G$. But $A_2B_1/A_1B_1 \cong C_2$, so $A_2B_1/A_1B_1 \leqslant Z(G/A_1B_1)$. Now $A_1B/A_1B_1 = \langle y_1, y_2\rangle A_1B_1/A_1B_1 \cong C_2 \times C_2$ and $A_1B/A_1B_1 \cap A_2B_1/A_1B_1 = A_1B_1/A_1B_1 = 1_{G/A_1B_1}$. Therefore, B^G/A_1B_1 is the direct product of A_1B/A_1B_1 and A_2B_1/A_1B_1, so B^G/A_1B_1 is elementary abelian of rank 3. Since $A_{k-1}B_1/A_1B_1$ is normal in G/A_1B_1, we have $[\langle y_1, y_2\rangle, A_{k-1}]A_1B_1/A_1B_1 \leqslant B^G/A_1B_1 \cap A_{k-1}B_1/A_1B_1 = A_2B_1/A_1B_1$. Hence, by minimality, we see that either $[\langle y_1, y_2\rangle, A_{k-1}]A_1B_1 = A_2B_1$ or $[\langle y_1, y_2\rangle, A_{k-1}] \leqslant A_1B_1$.

If $[\langle y_1, y_2\rangle, A_{k-1}]A_1B_1 = A_2B_1$, then $A_2 \leqslant [\langle y_1, y_2\rangle, A_{k-1}]A_1B_1 \leqslant B^{A_{k-1}B}A_1$. It follows that $A_1 = \Phi(A_2) \leqslant \Phi(B^{A_{k-1}B}A_1)$. Then $B^{A_{k-1}B}A_1 = B^{A_{k-1}B}$, so we have $A_2 \leqslant B^{A_{k-1}B}$. Since $\langle y_1, y_2\rangle \cong C_2 \times C_2$ and $[\langle y_1, y_2\rangle, A_{k-1}]A_1B_1/A_1B_1 = A_2B_1/A_1B_1 \cong C_2$, there exists a non-trivial element $1 \neq y \in \langle y_1, y_2\rangle$ such that conjugation by y induces the identity automorphism on $A_{k-1}B_1/A_1B_1$ (which is cyclic of order 2^{k-2}). Then $[A_{k-1}, \langle y\rangle] \leqslant A_1B_1$, so $\langle y\rangle A_1B_1 \trianglelefteq A_{k-1}B$. Now $\langle y\rangle B_1 \leqslant B$ and $A_1 \leqslant Z(G)$, so B centralises $\langle y\rangle A_1B_1$. Thus $A_2 \leqslant B^{A_{k-1}B} \leqslant C_{A_{k-1}B}(\langle y\rangle A_1B_1)$. But then y centralises A_2, which is a contradiction.

We may thus assume that $[\langle y_1, y_2\rangle, A_{k-1}] \leqslant A_1B_1$. Recalling that $A = \langle x\rangle$, so that $A_{k-1} = \langle x^2\rangle$, we then have $[\langle y_1, y_2\rangle, \langle x^2\rangle] \leqslant A_1B_1$. It follows that $[\langle x^2\rangle, B^G] \leqslant A_1B_1$. Since B^G/A_1B_1 is elementary abelian of rank 3, we may apply Lemma 3.2 to see that $|C_{B^G/A_1B_1}(xA_1B_1)| \geq 4$. Hence, by comparison of orders, $A_1B/A_1B_1 \cap C_{B^G/A_1B_1}(xA_1B_1) \neq 1_{G/A_1B_1}$. Since $B = \langle y_1, y_2\rangle B_1$, it follows that there exists $1 \neq y \in \langle y_1, y_2\rangle$ such that $[y, x] \in A_1B_1$. Then $\langle y\rangle A_1B_1 \trianglelefteq G$. Since $A_1 \leqslant Z(G)$, we see that B centralises both $\langle y\rangle B_1$ and A_1. Hence $B^G \leqslant C_G(\langle y\rangle A_1B_1)$. But $A_2 \leqslant B^G$ so y centralises A_2, and once more a contradiction ensues.

We now have $B^G \nleqslant A_2B$. If $k = 3$, then, by Lemma 2.3, we have $B^G \leqslant A_2B$, which is excluded. Thus $k > 3$. Again by Lemma 2.3, we have $A_3B \leqslant B^G \leqslant C_G(A_1B_1)$. From the above we have $A_{k-1}B_1/A_1B_1 = \Phi(C\langle y_1\rangle/A_1B_1) \trianglelefteq G/A_1B_1$. Since A_3B_1/A_1B_1 is characteristic in $A_{k-1}B_1/A_1B_1$, we have $A_3B_1 \trianglelefteq G$. But $A_3 \leqslant B^G \leqslant C_G(A_1B_1)$, so A_3B_1 is abelian. Then $A_2 = \Phi(A_3) = \Phi(A_3B_1)$, so $A_2 \trianglelefteq G$. But $A_2 \cong C_4$ and $\mathrm{Aut}(C_4) \cong C_2$. Since $\langle y_1, y_2\rangle \cong C_2 \times C_2$, we have a final contradiction $1 \neq C_{\langle y_1, y_2\rangle}(A_2)$. We thus conclude that $|G : C_G(\Omega_2(A))| = 2$. □

We apply Lemma 3.7 to establish some properties of products of elementary abelian and cyclic 2-groups in the case where the normal closure of the elementary abelian factor is non-abelian.

Theorem 3.8. *Let $G = AB$ be a finite 2-group for subgroups A and B such that A is cyclic of order 2^k and B is elementary abelian. Let $B_1 = C_B(\Omega_2(A))$. If B^G is not abelian, then*

(i) $k \geq 3$;

(ii) *$\Omega_1(A)B_1$ is an elementary abelian, normal subgroup of G;*

(iii) *$\Omega_1(A) \leqslant Z(G)$ (so, in particular, $A_G \neq 1$);*

(iv) $B^G \leqslant C_G(\Omega_1(A)B_1)$;

(v) $\Omega_1(Z(G/\Omega_1(A)B_1)) = \Omega_2(A)B_1/\Omega_1(A)B_1$;

(vi) *if N is an elementary abelian, normal subgroup of G, then $N \leqslant \Omega_1(A)B_1$ (so $\Omega_1(A)B_1$ is the unique maximal elementary abelian, normal subgroup in G, and, in particular, $\Omega_1(A)B_1$ is characteristic in G);*

(vii) *G has exponent 2^k;*

(viii) *if $k = 3$, then $G/\Omega_1(A)B_1 \cong \langle x, y \mid x^4 = y^2 = 1,\ x^y = x^{-1}\rangle$ (the dihedral group of order 8);*

(ix) *if $k \geq 4$, then $G/\Omega_1(A)B_1$ is isomorphic to one of the following groups*

(a) *$\langle x, y \mid x^{2^{k-1}} = y^2 = 1,\ x^y = x^{-1}\rangle$ (the dihedral group of order 2^k);*

(b) $\langle x, y \mid x^{2^{k-1}} = y^2 = 1,\ x^y = x^{-1+2^{k-2}}\rangle$ *(the quasi-dihedral, or semi-dihedral, group of order 2^k);*

(c) $\langle x, y \mid x^{2^{k-1}} = y^2 = 1,\ x^y = x^{1+2^{k-2}}\rangle$*;*

(x) *if $G/\Omega_1(A)B_1$ is dihedral or quasi-dihedral of order 2^k, then*

(a) *G' is abelian of rank at most $r(B)$;*

(b) $\exp(G') = 2^{k-1}$*;*

(c) $\Phi(G') = \Omega_{k-2}(A) \cong C_{2^{k-2}}$*;*

(d) *there exists an element $x_1 \in G$ and a subgroup $\widehat{B} \leqslant B_1$ such that $o(x_1) = 2^{k-1}$ and $G' = \widehat{B} \times \langle x_1\rangle$;*

(xi) *if $G/\Omega_1(A)B_1$ is isomorphic to the group $\langle x, y \mid x^{2^{k-1}} = y^2 = 1,\ x^y = x^{1+2^{k-2}}\rangle$, then*

(a) *G' is abelian of rank at most $r(B)$;*

(b) $\exp(G') = 4$*;*

(c) $\Phi(G') = \Omega_1(A) \cong C_2$*;*

(d) *there exists an element $x_1 \in G$ and a subgroup $\widehat{B} \leqslant B_1$ such that $o(x_1) = 4$ and $G' = \widehat{B} \times \langle x_1\rangle$.*

Proof. As in the proof of Lemma 3.7, we let $A_i = \Omega_i(A)$ (for $i = 1, \ldots, k$), and let $C = C_G(A_2)$ $(= C_G(\Omega_2(A)))$, so that $C = AB_1$. If $k < 3$, then $B^G \leqslant A_1B \trianglelefteq G$ by Lemma 2.3. Hence B^G is abelian by Lemma 3.3, which is excluded, so (i) follows.

For (ii) we see, by Lemma 3.7, that $|G : C| = 2$, so $C \trianglelefteq G$. By Lemma 3.4, $A_1B_1 = \Omega_1(C)$, so A_1B_1 is characteristic in C, and hence normal in G. Now A_1 has order 2 and is centralised by B_1. Since B_1 is elementary abelian, we see that A_1B_1 is an elementary abelian, normal subgroup of G.

For (iii), we note that $A_2 \cong C_4$ since $k \geq 3$. In addition, $C/A_1B_1 = AB_1/A_1B_1$ is a cyclic normal subgroup of G/A_1B_1. Hence A_2B_1/A_1B_1 is characteristic in C/A_1B_1, and is thus normal in G/A_1B_1. It follows that A_2B_1 is normal in G. But A_2B_1 is abelian, so $\Phi(A_2B_1) = \Phi(A_2)\Phi(B_1) = A_1$. Thus A_1 is a normal subgroup of order 2 in G, so $A_1 \leqslant Z(G)$.

Now $B_1 \leqslant B$, which is abelian, and B centralises A_1 by (iii). Therefore, B centralises the normal subgroup A_1B_1, so (iv) follows by the normality of $C_G(A_1B_1)$.

For (v), we observe that G/A_1B_1 is non-abelian, as otherwise $A_1B/A_1B_1 \trianglelefteq G/A_1B_1$ and $B^G \leqslant A_1B$. It follows that $Z(G/A_1B_1) \leqslant AB_1/A_1B_1$, as otherwise $G/A_1B_1 = Z(G/A_1B_1)(AB_1/A_1B_1)$, which is abelian. By comparison of orders, we then see that $\Omega_1(Z(G/A_1B_1)) = \Omega_1(AB_1/A_1B_1) = A_2B_1/A_1B_1$.

Now, if N is an elementary abelian, normal subgroup of G, we see that if $N \nleqslant A_1B_1$, then there exists a subgroup $N_1 \leqslant N$ with $N_1A_1B_1/A_1B_1 \cong C_2$ and such that $N_1A_1B_1/A_1B_1 \trianglelefteq G/A_1B_1$. Then $N_1A_1B_1/A_1B_1 \leqslant Z(G/A_1B_1)$, so, by comparison of orders, we have $N_1A_1B_1/A_1B_1 = \Omega_1(Z(G/A_1B_1)) = A_2B_1/A_1B_1$. Hence $N_1A_1B_1 = A_2B_1$. Since A_2B_1 is abelian, and N_1 and A_1B_1 are elementary abelian, we see that $\exp(A_2B_1) = 2$. But $A_2 \cong C_4$, and a contradiction ensues. Thus $N \leqslant A_1B_1$, in accordance with (vi).

From the above, G/A_1B_1 is non-abelian and $|G/A_1B_1| = 2|AB_1/A_1B_1| = 2^k$, so $\exp(G/A_1B_1) \leq 2^{k-1}$. Since A_1B_1 is elementary abelian and $A \cong C_{2^k}$, it follows that $\exp(G) = 2^k$, so (vii) holds.

For (viii) and (ix), we note that, since A_1B_1 is elementary abelian by (ii), we have $A \cap A_1B_1 = A_1$. It follows that $C/A_1B_1 = AB_1/A_1B_1 \cong A/(A \cap A_1B_1) = A/A_1 \cong C_{2^{k-1}}$. Since B^G is non-abelian, we have $B^G \nleqslant A_1B_1$ by Lemma 3.3, so, in particular, $B \nleqslant A_1B_1$. In addition, $|G : C| = 2$, so $|B : B \cap A_1B_1| \leq |B : B_1| \leq 2$. Therefore, $|B : B \cap A_1B_1| = 2$. Hence, letting $y \in B \setminus B \cap A_1B_1$, we have $A_1B/A_1B_1 \cong B/(B \cap A_1B_1) = \langle y\rangle B_1/B_1 \cong C_2$. Since $A_1B/A_1B_1 \cap C/A_1B_1 = (A_1B \cap AB_1)/A_1B_1 = 1_{G/A_1B_1}$, it follows that G/A_1B_1 is a group of order 2^k that is the semi-direct product of $C/A_1B_1 \cong C_{2^{k-1}}$, by $A_1B/A_1B_1 \cong C_2$. Moreover, G/A_1B_1 is non-abelian. Thus, if $k = 3$, then G/A_1B_1 is isomorphic to the dihedral group of order 8, whereas if $k \geq 4$, we see, by, say, [3, Theorem 1.2], that G/A_1B_1 is isomorphic to either the dihedral group of order 2^k, the quasi-dihedral group of order 2^k or the group $\langle x, y \mid x^{2^{k-1}} = y^2 = 1,\ x^y = x^{1+2^{k-2}}\rangle$.

To see that (x) holds, we let $y \in B \setminus (B \cap A_1B_1)$, and let $A = \langle x\rangle$. If G/A_1B_1 is either dihedral or quasi-dihedral, then $G' A_1B_1/A_1B_1 = \langle [x, y]\rangle A_1B_1/A_1B_1 = A_{k-1}B_1/A_1B_1$. By (iv), we have $y \in B^G \leqslant C_G(A_1B_1) \trianglelefteq G$. Since A_1B_1 is abelian, we see, by normality, that $G' A_1B_1 = \langle [x, y]\rangle A_1B_1 = A_{k-1}B_1 \leqslant C_G(A_1B_1)$. But A_{k-1} is cyclic and B_1 is abelian, so $G' A_1B_1$ is abelian. In particular, G' is abelian and $r(G') \leq r(B_1)+1 \leq r(B)-1+1 = r(B)$. We further see that $\exp(A_{k-1}B_1) = o(x^2) = 2^{k-1}$. Therefore, since $\exp(A_1B_1) = 2$, we have $\exp(G') = 2^{k-1}$. In addition, we have $\Phi(A_{k-1}B_1) = \Phi(A_{k-1}) = A_{k-2} = \Phi(G' A_1B_1) = \Phi(G')$, so $\Phi(G') = A_{k-2} = \langle x^4\rangle$. Now, since G' has exponent 2^{k-1}, we let x_1 be an element of order 2^{k-1} in G'. Since B_1 is elementary abelian, we have $\Omega_1(A_{k-1}B_1) = A_1B_1$. Hence, by comparison of orders, we see that $A_{k-1}B_1 = A_1B_1\langle x_1\rangle$. But $\Phi(A_1B_1\langle x_1\rangle) = \Phi(\langle x_1\rangle) = \langle x_1^2\rangle$, so $\langle x_1^2\rangle = \langle x^4\rangle$. Thus, since $k \geq 3$, it follows that $A_1 = \langle x^{2^{k-1}}\rangle = \langle x_1^{2^{k-2}}\rangle$, so $A_{k-1}B_1 = A_1B_1\langle x_1\rangle = B_1\langle x_1\rangle$. Since $x_1 \in G'$, we then have $G' = (G' \cap B_1)\langle x_1\rangle$. If $B_1 \cap \langle x_1\rangle = 1$, then we let $\widehat{B} = G' \cap B_1$ and have $G' = \widehat{B} \times \langle x_1\rangle$. If $B_1 \cap \langle x_1\rangle \neq 1$, then $B_1 \cap \langle x_1\rangle = G' \cap B_1 \cap \langle x_1\rangle = \langle x_1^{2^{k-2}}\rangle \cong C_2$.

Letting $\widehat{B}$ be a complement for $\langle x_1^{2^{k-2}} \rangle$ in $G' \cap B_1$, we then see, by comparison of orders, that $G' = \widehat{B} \times \langle x_1 \rangle$. Thus (x) has been established.

For (xi), we again let $y \in B \setminus (B \cap A_1 B_1)$. Since G/A_1B_1 is isomorphic to the group $\langle x, y \mid x^{2^{k-1}} = y^2 = 1,\ x^y = x^{1+2^{k-2}} \rangle$, we see that $G' A_1B_1/A_1B_1 = \langle [x,y] \rangle A_1B_1/A_1B_1 = A_2B_1/A_1B_1$. We can then repeat the proof of part (x), with minor adjustments, to show that (xi) holds. □

We apply Theorem 3.8 to provide a partial analogue to Corollary 2.7 in the case where $p = 2$.

Corollary 3.9. *Let $G = AB$ be a finite 2-group for subgroups A and B such that A is cyclic and B is elementary abelian. If B^G is not abelian and if G is also the product of the cyclic subgroup $\widetilde{A}$ and the elementary abelian subgroup $\widetilde{B}$, then*

(i) *$\widetilde{B}^G$ is non-abelian;*

(ii) *$\Omega_1(\widetilde{A}) C_{\widetilde{B}}(\Omega_2(\widetilde{A})) = \Omega_1(A) C_B(\Omega_2(A))$;*

(iii) *$\widetilde{A} \cong A$;*

(iv) *$\Omega_1(\widetilde{A}) = \Omega_1(A)$.*

Proof. Let $W = \Omega_1(A) C_B(\Omega_2(A))$. Then, by Theorem 3.8 (vi), W is the unique maximal elementary abelian, normal subgroup in G. If $\widetilde{B}^G$ is abelian (and hence elementary abelian), then $\widetilde{B}^G \leqslant W$. It follows that G/W is isomorphic to a factor group of $G/\widetilde{B}^G$, which in turn is isomorphic to a factor group of $\widetilde{A}$. But $\widetilde{A}$ is cyclic and G/W is non-abelian by Theorem 3.8 (viii) and (ix), so a contradiction arises. Hence $\widetilde{B}^G$ is non-abelian, in accordance with (i), and (ii) then follows by Theorem 3.8 (vi). We further have $\exp(G) = |\widetilde{A}| = |A|$ by Theorem 3.8 (vii), so $\widetilde{A} \cong A$. Finally, by Theorem 3.8 (x)(c) and (xi)(c), we have $\Omega_1(A) = \Omega_1(\Phi(G')) = \Omega_1(\widetilde{A})$. □

We present some examples of factorised 2-groups. They will be used to provide a characterisation of the groups that satisfy the hypotheses of Theorem 3.8, similar to that given by Theorem 2.13 for p odd.

Example 3.10. Let $k \geq 3$, and let $G = \langle x, y \mid x^{2^k} = y^2 = 1,\ x^y = x^{-1} \rangle$. Thus G is isomorphic to the dihedral group of order 2^{k+1}. We have $G = AB$, where $A = \langle x \rangle \cong C_{2^k}$ and $B = \langle y \rangle \cong C_2$. We note that $B^G = \langle x^2, y \rangle$, which is isomorphic to the dihedral group of order 2^k. Hence B^G is non-abelian. We observe that G also admits the factorisation $G = \widetilde{A}\widetilde{B}$, where $\widetilde{A} = A$ and $\widetilde{B} = \langle x^{2^{k-1}}, xy \rangle \cong C_2 \times C_2$. In this case, $\widetilde{B} \ncong B$. In addition, $\widetilde{B}^G = \langle x^2, xy \rangle \neq B^G$.

Example 3.11. Letting $k \geq 3$, we present the quasi-dihedral (or semi-dihedral) group of order 2^{k+1} as follows: $G = \langle x, y \mid x^{2^k} = y^2 = 1,\ x^y = x^{-1+2^{k-1}} \rangle$. Then $G = AB$, where $A = \langle x \rangle \cong C_{2^k}$ and $B = \langle y \rangle \cong C_2$. As in Example 3.10, we have $B^G = \langle x^2, y \rangle$, which is isomorphic to the dihedral group of order 2^k, and is thus non-abelian.

Example 3.12. Let $G = \langle w, x, y \mid w^2 = x^{2^k} = y^2 = 1, w^x = w, w^y = w, x^y = x^{-1}w \rangle$, where $k \geq 3$. G is the split extension of $\langle w, x \rangle \cong C_2 \times C_{2^k}$ by $\langle y \rangle \cong C_2$. We have $G = AB$, where $A = \langle x \rangle \cong C_{2^k}$ and $B = \langle w, y \rangle \cong C_2 \times C_2$. In this case, $B^G = \langle w, x^2, y \rangle = \langle w \rangle \times \langle x^2, y \rangle$. Thus B^G is isomorphic to the direct product of a cyclic group of order 2 and a dihedral group of order 2^k, so B^G is non-abelian.

Example 3.13. Let $k \geq 4$, and let $G = \langle w, x, y \mid w^2 = x^{2^k} = y^2 = 1, w^x = wx^{2^{k-1}}, w^y = w, x^y = x^{-1+2^{k-2}}w \rangle$. We can confirm that $\langle w, x \rangle = \langle x \rangle \langle w \rangle$ is a non-abelian group of order 2^{k+1} with $Z(\langle w, x \rangle) = \langle x^2 \rangle$. We can further confirm that $(x^y)^2 = x^{-1}x^{2^{k-2}}wx^{-1}x^{2^{k-2}}w = x^{-1}wx^{-1}wx^{2^{k-1}} = w^x w^{x^2} x^{-2} x^{2^{k-1}} = wx^{2^{k-1}}wx^{-2}x^{2^{k-1}} = w^2x^{-2}x^{2^k} = x^{-2}$. Hence $o(x^y) = o(x) = 2^k$, and we see that $(x^y)^{2^{k-1}} = (x^{-2})^{2^{k-2}} = x^{-2^{k-1}} = x^{2^{k-1}}$. It now follows that $(w^y)^{x^y} = w^{x^{-1}x^{2^{k-2}}w} = w^x = wx^{2^{k-1}} = w^y(x^y)^{2^{k-1}}$. Since the relevant relations are satisfied, we see that y defines an automorphism of $\langle w, x \rangle$. It further follows that $x^{y^2} = (x^{-1}x^{2^{k-2}}w)^y = wxx^{-2^{k-2}}x^{-2^{k-2}}w = xx^{-1}wxwx^{-2^{k-1}} = xw^xwx^{2^{k-1}} = xwx^{2^{k-1}}wx^{2^{k-1}} = xw^2x^{2^k} = x$. This confirms that conjugation by y induces an automorphism of order 2 on $\langle w, x \rangle$. Hence G is the split extension of $\langle w, x \rangle$ by $\langle y \rangle$, where $\langle y \rangle \cong C_2$. We have $G = AB$, where $A = \langle x \rangle \cong C_{2^k}$ and $B = \langle w, y \rangle \cong C_2 \times C_2$. In this case, $B^G = \langle w, x^2, y \rangle = \langle w \rangle \times \langle x^2, y \rangle$, which is isomorphic to the direct product of a cyclic group of order 2 and a dihedral group of order 2^k. Thus B^G is once more non-abelian.

We describe our final example in more detail and show that it satisfies the hypotheses of Theorem 3.8.

Example 3.14. We let $k \geq 4$, and let $W = \langle w_1, \ldots, w_{2^{k-2}} \rangle$ be an elementary abelian 2-group of rank 2^{k-2}. Let $\langle x \rangle \cong C_{2^k}$, and let x act on W as follows:

$$w_i^x = w_{i+1}, \quad i = 1, \ldots, 2^{k-2} - 1, \quad w_{2^{k-2}}^x = w_1.$$

We note that $C_{\langle x \rangle}(W) = \langle x^{2^{k-2}} \rangle \cong C_4$. We let E_k be the semi-direct product of W by $\langle x \rangle$, so that

$$E_k = \left\langle \begin{matrix} w_1, \ldots, w_{2^{k-2}} \\ x \end{matrix} \;\middle|\; \begin{matrix} w_1^2 = \cdots = w_{2^{k-2}}^2 = x^{2^k} = 1;\ [w_i, w_j] = 1,\ i, j = 1, \ldots, 2^{k-2} \\ w_i^x = w_{i+1},\ i = 1, \ldots, 2^{k-2} - 1;\ w_{2^{k-2}}^x = w_1 \end{matrix} \right\rangle.$$

We let $z = w_1 \cdots w_{2^{k-2}}$ and observe that $Z(E_k) = \langle z, x^{2^{k-2}} \rangle \cong C_2 \times C_4$. Thus $zx^{2^{k-1}}$ is an element of order 2 in $Z(E_k)$. We let F_k be the factor group $F_k = E_k/\langle zx^{2^{k-1}} \rangle$. Then F_k can be presented as follows:

$$F_k = \left\langle \begin{matrix} w_1, \ldots, w_{2^{k-2}} \\ x \end{matrix} \;\middle|\; \begin{matrix} w_1^2 = \cdots = w_{2^{k-2}}^2 = x^{2^k} = 1;\ [w_i, w_j] = 1,\ i,j = 1, \ldots, 2^{k-2} \\ w_i^x = w_{i+1},\ i = 1, \ldots, 2^{k-2} - 1;\ w_{2^{k-2}}^x = w_1;\ x^{2^{k-1}} = w_1 \cdots w_{2^{k-2}} \end{matrix} \right\rangle.$$

We again let $z = w_1 \cdots w_{2^{k-2}}$ and identify W with the corresponding subgroup of F_k. We observe that $r(W) = 2^{k-2}$ and that $W \cap Z(F_k) = \langle z \rangle = \langle x^{2^{k-1}} \rangle$. Since F_k is the product of the abelian subgroups $\langle x \rangle$ and W, we have

$$Z(F_k) = (Z(F_k) \cap \langle x \rangle)(Z(F_k) \cap W) = \langle x^{2^{k-2}} \rangle \langle z \rangle = \langle x^{2^{k-2}} \rangle \cong C_4.$$

We define a mapping, y, by

$$w_i^y = w_i, \qquad i = 1, \ldots, 2^{k-2}, \qquad x^y = x^{1+2^{k-2}} w_1.$$

We show that y extends to an automorphism, of order 2, of F_k. Since F_k is generated by $w_1^y, \ldots, w_{2^{k-2}}^y$ and x^y, we need only confirm that the appropriate relations are satisfied. We observe first that $(w_1^y)^2 = \cdots = (w_{2^{k-2}}^y)^2 = 1$ and that $[w_i^y, w_j^y] = 1$, for $i, j = 1, \ldots 2^{k-2}$. Since W is abelian and $x^{2^{k-2}} \in Z(F_k)$, we further see that $(w_i^y)^{x^y} = w_{i+1} = w_{i+1}^y$, for $i = 1, \ldots, 2^{k-2} - 1$, and that $(w_{2^{k-2}}^y)^{x^y} = w_1 = w_1^y$. For the remaining relations, we note that $(x^y)^2 = x^{1+2^{k-2}} w_1 x^{1+2^{k-2}} w_1 = x w_1 x w_1 (x^{2^{k-2}})^2 = x^2 w_1^x w_1 x^{2^{k-1}} = x^2 w_2 w_1 x^{2^{k-1}}$. Then $(x^y)^4 = (x^y)^2 (x^y)^2 = x^2 w_2 w_1 x^{2^{k-1}} x^2 w_2 w_1 x^{2^{k-1}} = x^4 (w_2 w_1)^{x^2} w_2 w_1 (x^{2^{k-1}})^2 = x^4 w_4 w_3 w_2 w_1 x^{2^k}$. Hence $(x^y)^4 = x^4 w_1 w_2 w_3 w_4$. More generally, for $k \geq 5$ and $3 \leq s \leq k-2$, we see inductively that

$$\begin{aligned} (x^y)^{2^s} &= (x^y)^{2^{s-1}} (x^y)^{2^{s-1}} = x^{2^{s-1}} w_1 \cdots w_{2^{s-1}} x^{2^{s-1}} w_1 \cdots w_{2^{s-1}} \\ &= x^{2^s} (w_1 \cdots w_{2^{s-1}})^{x^{x^{s-1}}} w_1 \cdots w_{2^{s-1}} \\ &= x^{2^s} w_{2^{s-1}+1} \cdots w_{2^{s-1}+2^{s-1}} w_1 \cdots w_{2^{s-1}}. \end{aligned}$$

Hence $(x^y)^{2^s} = x^{2^s} w_1 \cdots w_{2^s}$, for $3 \leq s \leq k-2$. In particular, we see that $(x^y)^{2^{k-2}} = x^{2^{k-2}} w_1 \cdots w_{2^{k-2}} = x^{2^{k-2}} x^{2^{k-1}}$. Thus $(x^y)^{2^{k-2}} = (x^{2^{k-2}})^3 = (x^{2^{k-2}})^{-1}$, so $(x^y)^{2^k} = ((x^y)^{2^{k-2}})^4 = ((x^{2^{k-2}})^{-1})^4 = 1$. In addition, we have $(x^y)^{2^{k-1}} = (x^y)^{2^{k-2}} (x^y)^{2^{k-2}} = (x^{2^{k-2}})^{-1} (x^{2^{k-2}})^{-1} = x^{-2^{k-1}} = x^{2^{k-1}}$. But $x^{2^{k-1}} = w_1 \cdots w_{2^{k-2}} = w_1^y \cdots w_{2^{k-2}}^y$, so the relation $(x^y)^{2^{k-1}} = w_1^y \cdots w_{2^{k-2}}^y$ is satisfied. This confirms that y defines an automorphism of F_k.

To show that y has order 2, we note that $x^{y^2} = (xx^{2^{k-2}}w_1)^y = x^y(x^y)^{2^{k-2}}w_1^y$. Now $x^{2^{k-2}}$ centralises W, and, from the above, $(x^y)^{2^{k-2}} = (x^{2^{k-2}})^{-1}$. Hence $x^{y^2} = x^{1+2^{k-2}}w_1(x^{2^{k-2}})^{-1}w_1 = xx^{2^{k-2}}(x^{2^{k-2}})^{-1}w_1^2 = x$. Since $w_i^y = w_i,\ i = 1,\ldots,2^{k-2}$, we see that y has order 2 in $\mathrm{Aut}(F_k)$.

We now let J_k be the semi-direct product of F_k by $\langle y\rangle$, so that

$$J_k = \left\langle \begin{array}{c|l} w_1,\ldots,w_{2^{k-2}} & w_1^2 = \cdots = w_{2^{k-2}}^2 = x^{2^k} = y^2 = 1;\ [w_i,w_j] = 1,\ i,j = 1,\ldots,2^{k-2} \\ x & w_i^x = w_{i+1},\ i = 1,\ldots,2^{k-2}-1;\ w_{2^{k-2}}^x = w_1;\ x^{2^{k-1}} = w_1\cdots w_{2^{k-2}} \\ y & w_i^y = w_i,\ i = 1,\ldots,2^{k-2};\ x^y = x^{1+2^{k-2}}w_1 \end{array} \right\rangle.$$

We have $[x,y] = x^{2^{k-2}}w_1 \notin W$, so J_k/W is non-abelian. Hence $Z(J_k) \leqslant F_k$, as otherwise $J_k/W = (F_k/W)(Z(J_k)W/W)$, which is abelian. Thus $Z(J_k) \leqslant \langle x^{2^{k-2}}\rangle$. But, from the above, $(x^{2^{k-2}})^y = (x^{2^{k-2}})^{-1} \neq x^{2^{k-2}}$. It follows that:

$$Z(J_k) = \langle x^{2^{k-1}}\rangle \cong C_2.$$

We see that $J_k = AB$, where $A = \langle x\rangle$ is cyclic of order 2^k, and $B = W\langle y\rangle$ is elementary abelian of rank $1 + 2^{k-2}$. Since $[x,y] = x^{2^{k-2}}w_1$, we have $(B)^{J_k} \nleqslant \Omega_1(A)B$. Hence $(B)^{J_k}$ is non-abelian by Lemma 3.3. Since y does not centralise $x^{2^{k-2}}$, we have $y \notin C_{J_k}(\Omega_2(A))$. It follows that $C_{J_k}(\Omega_2(A)) = A\langle w_1,\ldots,w_{2^{k-2}}\rangle = F_k$, so $|J_k : C_{J_k}(\Omega_2(A))| = 2$, in accordance with Lemma 3.7. We observe that $\Omega_1(A) = \langle x^{2^{k-1}}\rangle \leqslant W = B \cap F_k = C_B(\Omega_2(A))$. We have $AW/W \cong A/\Omega_1(A) \cong C_{2^{k-1}}$, and $\langle y\rangle W/W \cong C_2$. Since $[x,y] = x^{2^{k-2}}w_1$, we further have $\langle [x,y]\rangle W/W = \Omega_2(A)W/W \cong C_2$. Now $|J_k/W| = 2^k$, where $k \geq 4$. It follows, by, say, [3, Theorem 1.2], that J_k/W is isomorphic to the non-abelian group of order 2^k given by $\langle x,y \mid x^{2^{k-1}} = y^2 = 1, x^y = x^{1+2^{k-2}}\rangle$.

Our next result uses Examples 3.10–3.14 to provide an alternative description of the groups that satisfy the hypotheses of Theorem 3.8.

Theorem 3.15. *The following are equivalent for the finite 2-group G:*

(i) *$G = AB$ for subgroups A and B such that A is cyclic, B is elementary abelian and B^G is non-abelian;*

(ii) *G is of the form $G = AW\langle\widetilde{y}\rangle$, where*
   - (a) *$A \cong C_{2^k}$, for $k \geq 3$;*
   - (b) *W is an elementary abelian, normal subgroup of G;*
   - (c) *$\langle\widetilde{y}\rangle \cong C_2$;*
   - (d) *$[W,\langle\widetilde{y}\rangle] = 1$;*
   - (e) *$\Omega_1(A) \leqslant W$;*

(f) $\langle A, \widetilde{y}\rangle$ *is isomorphic to one of the following:*

(i) $\langle x, y \mid x^{2^k} = y^2 = 1, x^y = x^{-1}\rangle$ *(the dihedral group of order* 2^{k+1}*);*

(ii) $\langle x, y \mid x^{2^k} = y^2 = 1, x^y = x^{-1+2^{k-1}}\rangle$ *(the quasi-dihedral, or semi-dihedral, group of order* 2^{k+1}*);*

(iii) $\langle w, x, y \mid w^2 = x^{2^k} = y^2 = 1, w^x = w, w^y = w, x^y = x^{-1}w\rangle$*;*

(iv) $\langle w, x, y \mid w^2 = x^{2^k} = y^2 = 1, w^x = wx^{2^{k-1}}, w^y = w, x^y = x^{-1+2^{k-2}}w\rangle$ *(with* $k \geq 4$*);*

(v) J_k*, as in Example 3.14 (with* $k \geq 4$*).*

Proof. To show that (i) implies (ii), we let $A = \langle\widetilde{x}\rangle$. By Theorem 3.8 (i), $A \cong C_{2^k}$, where $k \geq 3$. As in Theorem 3.8, we let $B_1 = C_B(\Omega_2(A))$. Then $|B : B_1| = 2$ by Lemmas 3.3 and 3.7. Moreover, by Theorem 3.8 (vi), $\Omega_1(A)B_1$ is the unique maximal elementary abelian, normal subgroup of G. We let $W = \Omega_1(A)B_1$. Since $|B : B_1| = 2$, we have $B \cap W = B_1$, as otherwise $B \leqslant W$ and B^G is elementary abelian. Letting $\widetilde{y} \in B \setminus B_1$, we have $B = B_1\langle\widetilde{y}\rangle$, so $G = AB = A\Omega_1(A)B_1\langle\widetilde{y}\rangle = AW\langle\widetilde{y}\rangle$. Since $\Omega_1(A) \leqslant Z(G)$, by Theorem 3.8 (iii), and B is abelian, we have $[W, \langle\widetilde{y}\rangle] = 1$. Thus G satisfies (ii)(a)–(e).

For (ii) (f), we note that G/W is the product of the normal subgroup $AW/W \cong A/\Omega_1(A) \cong C_{2^{k-1}}$, and $\langle\widetilde{y}\rangle W/W \cong C_2$. By Theorem 3.8 (viii) and (ix), G/W is isomorphic either to the dihedral group of order 2^k, the quasi-dihedral group of order 2^k, or the group $\langle x, y \mid x^{2^{k-1}} = y^2 = 1, x^y = x^{1+2^{k-2}}\rangle$. In the latter two cases, we may assume that $k \geq 4$ by Theorem 3.8 (viii). Since G/W is non-abelian in each case, we note further that $\widetilde{y} \notin AW$. We deal with these three cases in turn.

If G/W is dihedral of order 2^k, we may assume that $\widetilde{x}^{\widetilde{y}} = \widetilde{x}^{-1}\widetilde{w}$, where $\widetilde{w} \in W$. Since $\widetilde{y}$ centralises W and $o(\widetilde{y}) = 2$, we have

$$\widetilde{x} = \widetilde{x}^{\widetilde{y}^2} = (\widetilde{x}^{-1}\widetilde{w})^{\widetilde{y}} = (\widetilde{x}^{\widetilde{y}})^{-1}\widetilde{w}^{\widetilde{y}} = \widetilde{w}^{-1}\widetilde{x}\widetilde{w}.$$

Thus $\widetilde{w}$ commutes with $\widetilde{x}$, so $\widetilde{w} \in Z(\langle A, \widetilde{y}\rangle)$ and $\langle A, \widetilde{y}\rangle = A\langle\widetilde{w}\rangle\langle\widetilde{y}\rangle$. If $\widetilde{w} = 1$, then $\widetilde{x}^{\widetilde{y}} = \widetilde{x}^{-1}$, so $\langle A, \widetilde{y}\rangle$ is dihedral of order 2^{k+1}. If $1 \neq \widetilde{w} \in A$, then $\widetilde{w} = \widetilde{x}^{2^{k-1}}$. Hence $\widetilde{x}^{\widetilde{y}} = \widetilde{x}^{-1+2^{k-1}}$, so $\langle A, \widetilde{y}\rangle$ is isomorphic to the quasi-dihedral group of order 2^{k+1}. Finally, if $\widetilde{w} \notin A$, then $\langle A, \widetilde{w}\rangle = A \times \langle\widetilde{w}\rangle \cong C_{2^k} \times C_2$. Since $\langle A, \widetilde{y}\rangle$ is the split extension of $\langle A, \widetilde{w}\rangle$ by $\langle\widetilde{y}\rangle$, we see that $\langle A, \widetilde{y}\rangle$ is isomorphic to the group $\langle w, x, y \mid w^2 = x^{2^k} = y^2 = 1, w^x = w, w^y = w, x^y = x^{-1}w\rangle$.

If G/W is quasi-dihedral of order 2^k, we may assume that $k \geq 4$, and that $\widetilde{x}^{\widetilde{y}} = \widetilde{x}^{-1+2^{k-2}}\widetilde{w}$, where $\widetilde{w} \in W$. By normality, we have $\widetilde{x}^{-2+2^{k-2}} = [\widetilde{x}, \widetilde{y}]\widetilde{w} \in$

$C_G(W)$. Since $k \geq 4$, it follows that $\widetilde{x}^2 \in C_G(W)$, so $\langle \widetilde{x}^2 \rangle W$ is abelian. Now $\widetilde{y}$ normalises $\langle \widetilde{x}^2 \rangle W$, so $\widetilde{y}$ also normalises $\Phi(\langle \widetilde{x}^2 \rangle W) = \Phi(\langle \widetilde{x}^2 \rangle) = \langle \widetilde{x}^4 \rangle$. Hence $\widetilde{y}$ normalises $\Omega_2(\langle \widetilde{x}^4 \rangle) = \langle \widetilde{x}^{2^{k-2}} \rangle = \Omega_2(A)$. Since $\widetilde{y} \notin B_1 = C_B(\Omega_2(A))$, we have $(\widetilde{x}^{2^{k-2}})^{\widetilde{y}} = (\widetilde{x}^{2^{k-2}})^{-1} = \widetilde{x}^{-2^{k-2}}$. Now $\Omega_1(A) \leqslant Z(G)$, so $\widetilde{x} = \widetilde{x}^{\widetilde{y}^2} = (\widetilde{x}^{-1+2^{k-2}}\widetilde{w})^{\widetilde{y}} = (\widetilde{x}^{\widetilde{y}})^{-1}(\widetilde{x}^{2^{k-2}})^{\widetilde{y}}\widetilde{w} = \widetilde{w}^{-1}\widetilde{x}\widetilde{x}^{-2^{k-2}}\widetilde{x}^{-2^{k-2}}\widetilde{w} = \widetilde{w}^{-1}\widetilde{x}\widetilde{x}^{-2^{k-1}}\widetilde{w} = \widetilde{w}^{-1}\widetilde{x}\widetilde{w}\widetilde{x}^{-2^{k-1}}$. Hence $\widetilde{x} = \widetilde{w}^{-1}\widetilde{x}\widetilde{w}\widetilde{x}^{2^{k-1}}$, so $\widetilde{x}^{-1}\widetilde{w}\widetilde{x} = \widetilde{w}\widetilde{x}^{2^{k-1}}$. Equivalently, we have $\widetilde{w}^{\widetilde{x}} = \widetilde{w}\widetilde{x}^{2^{k-1}}$. Since $\widetilde{x}^{2^{k-1}} \neq 1$, we see, in particular, that $\widetilde{w} \neq 1$ and $\widetilde{w} \notin \langle \widetilde{x} \rangle = A$. In addition, $\langle A, \widetilde{w} \rangle = A\langle \widetilde{w} \rangle$. Thus $\langle A, \widetilde{y} \rangle$ is the split extension of $A\langle \widetilde{w} \rangle$ by $\langle \widetilde{y} \rangle$. It follows that $\langle A, \widetilde{y} \rangle$ is isomorphic to the group $\langle w, x, y \mid w^2 = x^{2^k} = y^2 = 1, w^x = wx^{2^{k-1}}, w^y = w, x^y = x^{-1+2^{k-2}}w \rangle$.

For our third case, we have $G/W \cong \langle x, y \mid x^{2^{k-1}} = y^2 = 1, x^y = x^{1+2^{k-2}} \rangle$. Here we may assume that $k \geq 4$, and that $\widetilde{x}^{\widetilde{y}} = \widetilde{x}^{!+2^{k-2}}\widetilde{w}$, where $\widetilde{w} \in W$. Since $W \leqslant C_G(\Omega_2(A))$, we see that $\langle \widetilde{w}^{\widetilde{x}^i} \mid i = 1, \ldots, 2^k \rangle = \langle \widetilde{w}, \widetilde{w}^{\widetilde{x}}, \ldots, \widetilde{w}^{\widetilde{x}^{2^{k-2}-1}} \rangle$. Letting $W_1 = \langle \widetilde{w}, \widetilde{w}^{\widetilde{x}}, \ldots, \widetilde{w}^{\widetilde{x}^{2^{k-2}-1}} \rangle$, we have $W_1 \leqslant W$, so W_1 is elementary abelian and is centralised by $\widetilde{y}$. In addition, W_1 is normalised by $\widetilde{x}$, so $\langle A, \widetilde{y} \rangle = AW_1\langle \widetilde{y} \rangle$. Since $\widetilde{x}^{2^{k-2}}$ and $\widetilde{y}$ commute with $\widetilde{w}$, we have

$$\widetilde{x} = \widetilde{x}^{\widetilde{y}^2} = (\widetilde{x}^{1+2^{k-2}}\widetilde{w})^{\widetilde{y}} = \widetilde{x}^{\widetilde{y}}(\widetilde{x}^{2^{k-2}})^{\widetilde{y}}\widetilde{w} = \widetilde{x}\widetilde{x}^{2^{k-2}}\widetilde{w}(\widetilde{x}^{2^{k-2}})^{\widetilde{y}}\widetilde{w} = \widetilde{x}\widetilde{x}^{2^{k-2}}(\widetilde{x}^{2^{k-2}})^{\widetilde{y}}\widetilde{w}^2.$$

Hence $\widetilde{x} = \widetilde{x}\widetilde{x}^{2^{k-2}}(\widetilde{x}^{2^{k-2}})^{\widetilde{y}}$, so $(\widetilde{x}^{2^{k-2}})^{\widetilde{y}} = \widetilde{x}^{-2^{k-2}}$. Since $k \geq 4$, we have $\widetilde{x}^{2^{2k-4}} = 1$, so we can also evaluate $(\widetilde{x}^{2^{k-2}})^{\widetilde{y}}$ as

$$(\widetilde{x}^{2^{k-2}})^{\widetilde{y}} = (\widetilde{x}^{\widetilde{y}})^{2^{k-2}} = (\widetilde{x}^{1+2^{k-2}}\widetilde{w})^{2^{k-2}} = (\widetilde{x}\widetilde{w})^{2^{k-2}}\widetilde{x}^{2^{2k-4}} = (\widetilde{x}\widetilde{w})^{2^{k-2}}.$$

But

$$(\widetilde{x}\widetilde{w})^{2^{k-2}} = \widetilde{w}^{\widetilde{x}^{-1}} \cdots \widetilde{w}^{\widetilde{x}^{-2^{k-2}}}\widetilde{x}^{2^{k-2}} = \widetilde{w}^{\widetilde{x}^{2^{k-2}-1}} \cdots \widetilde{w}^{\widetilde{x}}\widetilde{w}\widetilde{x}^{2^{k-2}},$$

so $\widetilde{w}\widetilde{w}^{\widetilde{x}} \cdots \widetilde{w}^{\widetilde{x}^{2^{k-2}-1}}\widetilde{x}^{2^{k-2}} = \widetilde{x}^{-2^{k-2}}$, and it follows that $\widetilde{w}\widetilde{w}^{\widetilde{x}} \cdots \widetilde{w}^{\widetilde{x}^{2^{k-2}-1}} = \widetilde{x}^{-2^{k-1}} = \widetilde{x}^{2^{k-1}}$. Therefore, since the relevant relations are satisfied, we see that the mapping ϕ defined by

$$\phi(w_1) = \widetilde{w}, \phi(w_2) = \widetilde{w}^{\widetilde{x}}, \ldots, \phi(w_{2^{k-2}}) = \widetilde{w}^{\widetilde{x}^{2^{k-2}-1}}, \phi(x) = \widetilde{x}, \quad \text{and} \quad \phi(y) = \widetilde{y},$$

extends to an epimorphism from J_k (as in Example 3.14) to $\langle A, \widetilde{x} \rangle$. Since $Z(J_k) = \langle x^{2^{k-1}} \rangle \cong C_2$ and $\phi(x^{2^{k-1}}) = \widetilde{x}^{2^{k-1}} \neq 1$, we see that $\ker(\phi) = 1$. Thus ϕ defines an isomorphism, and so we conclude that (i) implies (ii).

Conversely, if G is of the form given by (ii), then G is the product of the cyclic subgroup, A, and the elementary abelian subgroup $B = W\langle \widetilde{y} \rangle$. If B^G is abelian, then, by (ii)(e) and Lemma 3.3, we have $\Omega_1(A) \leqslant W\langle \widetilde{y} \rangle = B \trianglelefteq G$. In particular, $G/B \cong A/\Omega_1(A)$, so G/B is cyclic. Letting $N = \langle A, \widetilde{y} \rangle \cap B$, it follows that N is an elementary abelian, normal subgroup of $\langle A, \widetilde{y}\rangle$ such that $\langle A, \widetilde{y}\rangle/N$ is cyclic. But, as noted in Examples 3.10–3.14, $\langle A, \widetilde{y}\rangle$ satisfies the hypotheses of Theorem 3.8, so by Theorem 3.8 (vi), (viii) and (ix), $\langle A, \widetilde{y}\rangle$ has a unique maximal elementary abelian, normal subgroup, whose quotient is non-abelian. Thus a contradiction arises, so we conclude that B^G is non-abelian and (i) follows. □

Noting that the cyclic factor will have a non-trivial core whenever the elementary abelian factor has a non-abelian normal closure, the following partial analogue to Theorem 2.14 covers the remaining case for $p = 2$, in which namely the cyclic factor has a trivial core and the normal closure of the elementary abelian factor is abelian.

Theorem 3.16. *Let $G = AB$ be a finite 2-group for subgroups A and B such that A is cyclic and B is elementary abelian. Then the following are equivalent:*

(i) $A_G = 1$;

(ii) $B \trianglelefteq G$, $A \cap B = 1$ *and* $C_A(B) = 1$ *(so that G is a faithful split extension of B by A).*

Proof. If $A_G = 1$, then $A \cap Z(G) = 1$, so, by Theorem 3.8 (iii), B^G is abelian. If B is not normal in G, then $B^G = \Omega_1(A)B$ by Lemma 3.3, so $\Omega_1(A)$ is centralised by B. Then $1 \neq \Omega_1(A) \leqslant A \cap Z(G) \leqslant A_G$, and a contradiction arises. Hence $B \trianglelefteq G$. We further see that $A \cap B \leqslant A \cap Z(G) = 1$ and $C_A(B) \leqslant A \cap Z(G) = 1$. Thus G is a faithful split extension of B by A. Since (i) clearly follows from (ii), the proof is complete. □

We conclude this section with a result analogous to Corollary 2.17 that is a consequence of Lemma 2.16 and Theorems 3.6 (iii) and (v), 3.8 (vii), (x)(a) and (xi)(a), and 3.16.

Corollary 3.17. *Let $G = AB$ be a finite 2-group for subgroups A and B such that A is cyclic of order 2^k and B is elementary abelian. Then*

(i) $2^k \leq \exp(G) \leq 2^{k+1}$;

(ii) G' *is abelian of rank at most* $r(B)$.

4. Groups of exponent p^{k+1} and concluding remarks

We recall from Lemma 2.16 that if the finite p-group $G = AB$ is the product of a cyclic subgroup A, of order p^k, and an elementary abelian subgroup B, then $p^k \leq \exp(G) \leq p^{k+1}$. As a consequence of Theorems 2.6 (x), 2.15 (iv), 3.6 (iii) and 3.8 (vii), we see that if $\exp(G) = p^{k+1}$, then either Theorem 2.14 (iii) or Theorem 3.16 applies, so that G is a faithful split extension of B by A. We provide more information on the structure of G in this particular case.

Theorem 4.1. *Let $G = AB$ be a finite p-group for subgroups A and B such that A is cyclic of order p^k and B is elementary abelian. Then the following are equivalent:*

(i) *G has a subgroup isomorphic to the wreath product $C_p wr C_{p^k}$;*

(ii) *G has exponent p^{k+1}.*

Proof. It is well-known that the wreath product $C_p wr C_{p^k}$ has exponent p^{k+1}. Hence (ii) follows from (i) by Lemma 2.16. Conversely, if G has exponent p^{k+1}, then we let $g \in G$ be such that $o(g) = p^{k+1}$. We let $A = \langle x \rangle$ and observe that, since $G = AB$, we have $g = x^t w$, for a suitable positive integer t and a suitable element $w \in B$. By Lemma 2.16, we have $\exp(\langle x^p \rangle B) \leq p^{1+k-1} = p^k$, so, without loss of generality, we may further assume that $g = xw$. Since G has exponent p^{k+1}, we see, by Theorem 2.6 (x) or Theorem 3.8 (vii), that B^G is abelian, and hence elementary abelian. Thus $\langle w \rangle^{\langle w,x \rangle} = \langle w, w^x, \dots, w^{x^{p^k-1}} \rangle$ is an elementary abelian, normal subgroup of $\langle w, x \rangle$ such that $\langle w, x \rangle = \langle w, w^x, \dots, w^{x^{p^k-1}} \rangle \langle x \rangle$. We let $H \cong C_p wr C_{p^k}$. Since the requisite relations are satisfied by $w, w^x, \dots,$ $w^{x^{p^k-1}}$ and x, we see that $\langle w, x \rangle$ is isomorphic to a factor group of H. Thus $\langle w, x \rangle \cong H/N$, for a suitable subgroup $N \trianglelefteq H$. But $Z(H) \cong C_p$ and $H/Z(H)$ has exponent p^k. Hence, if N is non-trivial, then $Z(H) \leqslant N$ and $\langle w, x \rangle \cong H/N$ has exponent p^k, which is a contradiction. We conclude that $N = 1$, so $\langle w, x \rangle \cong H \cong C_p wr C_{p^k}$, as desired. □

Our final result provides bounds for the rank of a maximal normal elementary abelian subgroup in a finite p-group that factorises as the product of a cyclic subgroup and an elementary abelian subgroup.

Theorem 4.2. *Let $G = AB$ be a finite p-group for subgroups A and B such that A is cyclic and B is elementary abelian. Let N be an elementary abelian, normal subgroup of maximal order in G. Then $r(B) - 1 \leq r(N) \leq r(B) + 1$.*

Proof. We have $|AN| = \dfrac{|A||N|}{|A \cap N|} \leqslant |G| = \dfrac{|A||B|}{|A \cap B|}$, so $|N| \leq \dfrac{|A \cap N||B|}{|A \cap B|} \leq$ $|A \cap N||B|$. But A is cyclic and N is elementary abelian, so $|A \cap N| \leq p$. Thus $|N| \leq p|B|$, so $r(N) \leq r(B) + 1$. On the other hand, letting $A \cong C_{p^k}$, we see, by Theorems 2.6 (viii), 2.14 (iii), 2.15 (v)(c), 3.6 (iv)(c), 3.8 (viii) and (ix), and 3.16 (ii), that G has a elementary abelian, normal subgroup of index p^k, so

$|G : N| \le p^k$. Hence $|N| \ge \dfrac{|G|}{p^k} = \dfrac{|A||B|}{|A \cap B|p^k} = \dfrac{|B|}{|A \cap B|}$. As above, we have $|A \cap B| \le p$, so $|N| \ge \dfrac{|B|}{p}$. It follows that $r(N) \ge r(B) - 1$. □

The results in this paper indicate that it may be possible to provide a detailed account of the structure of finite groups that are the product of an abelian subgroup and a cyclic subgroup. However, the variety of examples presented, particularly in the case $p = 2$, suggests that a comprehensive understanding of the structure of groups that are the product of two abelian subgroups in general may be difficult to achieve.

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