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Joining means

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Dedicated to Professor Zsolt Páles on the occasion of his 60th birthday

Abstract. Modifying and generalizing some ideas from [1], we come to the notion of a marginal joint of two arbitrary means given on adjacent intervals. The construction of the joints makes use of the notion of a set-valued joiner. Also, the converse is proved: any mean can be obtained as a marginal joint of its two restrictions, produced with the use of a so-called reconstructing joiner having the smallest values in a sense. We conclude the paper by answering the question when the reconstructing joiner of the mean is a single-valued function.

1. Introduction

Let I be an interval of reals. A function $M : I \times I \to I$ is called a *mean on I* if

$$\min\{x, y\} \le M(x, y) \le \max\{x, y\}$$

for all $x, y \in I$.

Take any interior point ξ of I, and put

$$I_\xi := \{x \in I : \ x \le \xi\}\,, \quad {}_\xi I := \{x \in I : \ \xi \le x\} \tag{1}$$

and

$$I_\xi^\circ := \{x \in I : \ x < \xi\}\,, \quad {}_\xi I^\circ := \{x \in I : \ \xi < x\}\,. \tag{2}$$

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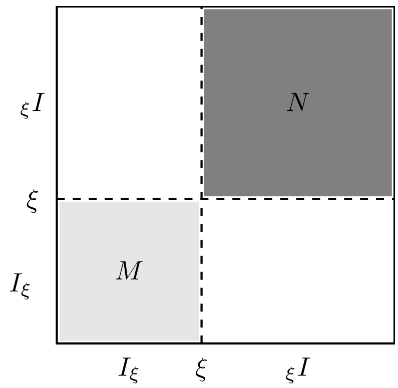


Our main question is as follows.

Problem 1. *Given two arbitrary means M and N on the intervals I_ξ and $_\xi I$, respectively, find a mean, say $M\oplus N$, on the interval I such that*

$$M\oplus N|_{I_\xi \times I_\xi} = M \quad \textit{and} \quad M\oplus N|_{_\xi I \times _\xi I} = N.$$

Any such mean $M\oplus N$ will be called *a joint of M and N*.

Observe that, given any mean K on I, the formula

$$(M\oplus N)(x,y) = \begin{cases} M(x,y), & \text{if } (x,y) \in I_\xi \times I_\xi \\ K(x,y), & \text{if } (x,y) \in I_\xi^\circ \times {_\xi I^\circ} \cup {_\xi I^\circ} \times I_\xi^\circ \\ N(x,y), & \text{if } (x,y) \in {_\xi I} \times {_\xi I} \end{cases}$$

defines a joint of M and N. However, its values taken in the set $I_\xi^\circ \times {_\xi I^\circ} \cup {_\xi I^\circ} \times I_\xi^\circ$ need not be connected with M and N at all. Such trivial joints will not be of interest for us. In the sequel, we will focus on joints carrying information on the means M and N.

In the paper [1], Z. Daróczy and the authors solved Problem 1, assuming that the *marginal functions* $h_1, h_2 : I \to I$, given by

$$h_1(x) = \begin{cases} M(x,\xi), & \text{if } x \in I_\xi, \\ N(x,\xi), & \text{if } x \in {_\xi I}, \end{cases} \tag{3}$$

$$h_2(y) = \begin{cases} M(\xi,y), & \text{if } y \in I_\xi, \\ N(\xi,y), & \text{if } y \in {_\xi I}, \end{cases} \tag{4}$$

are continuous and strictly increasing. Next, we solve Problem 1 for arbitrary means.

2. Joiners

The idea is to modify and generalize some ideas from [1]. The main tool used there to produce joints $M \oplus N$ is the notion of the so-called *joining function.* Now, we replace it by its set-valued analogue, called by us a *joiner*. Using it we construct a set-valued joint with the selections being joints of the means M and N. In what follows, we assume that:

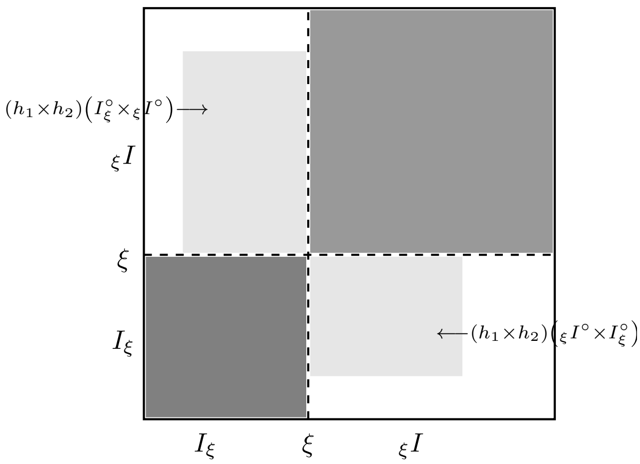
- ξ is an interior point of an interval I;
- the intervals $I_\xi, {}_\xi I$ and I_ξ°; ${}_\xi I^\circ$ are defined by (1) and (2), respectively;
- M and N are means on I_ξ and ${}_\xi I$, respectively;
- $h_1, h_2 : I \to I$ are the marginal functions given by (3) and (4), respectively.

Given any functions $f : I_\xi \to I_\xi$ and $g : {}_\xi I \to {}_\xi I$ satisfying $f(\xi) = g(\xi)$, we use the notation $f \cup g$ for the function mapping I into itself, defined by

$$(f \cup g)(x) := \begin{cases} f(x), & \text{if } x \in I_\xi, \\ g(x), & \text{if } x \in {}_\xi I. \end{cases}$$

Moreover, we define the product $h_1 \times h_2 : I \times I \to I \times I$ by the equality

$$(h_1 \times h_2)(x, y) = (h_1(x), h_2(y)).$$



Definition 1. A multifunction

$$\mathbb{K} : (h_1 \times h_2)\left(I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ\right) \to 2^I \setminus \{\emptyset\}$$

is called a *joiner of the pair* (M, N) if

$$\left(h_1|_{I_\xi} \cup h_2|_{{}_\xi I}\right)^{-1}\left(\mathbb{K}\left(h_1(x), h_2(y)\right)\right) \cap [x, y] \neq \emptyset \quad \text{for } (x, y) \in I_\xi^\circ \times {}_\xi I^\circ, \tag{5}$$

and

$$\left(h_2|_{I_\xi} \cup h_1|_{{}_\xi I}\right)^{-1}\left(\mathbb{K}\left(h_1(x), h_2(y)\right)\right) \cap [y, x] \neq \emptyset \quad \text{for } (x, y) \in {}_\xi I^\circ \times I_\xi^\circ, \tag{6}$$

or, equivalently, if for every $(x, y) \in I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ$ there is a $\kappa(x, y) \in I$ such that

$$\min\{x, y\} \leq \kappa(x, y) \leq \max\{x, y\},$$

and the function $\left(h_1|_{I_\xi} \cup h_2|_{{}_\xi I}\right) \circ \kappa$ is a selection of $K \circ (h_1 \times h_2)$.

Observe that if $\mathbb{K}_1$ is a joiner of the pair (M, N), then so is any $\mathbb{K}_2$ such that

$$\mathbb{K}_1(x, y) \subset \mathbb{K}_2(x, y)$$

for each $(x, y) \in (h_1 \times h_2)\,(I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ)$.

The most trivial example of a joiner is the multifunction $\mathbb{K}$ given by $\mathbb{K}(u, v) = I$: for any point $(x, y) \in I_\xi^\circ \times {}_\xi I^\circ$ we have

$$\begin{aligned}\left(h_1|_{I_\xi} \cup h_2|_{{}_\xi I}\right)^{-1}\left(\mathbb{K}\left(h_1(x), h_2(y)\right)\right) &= \left(h_1|_{I_\xi} \cup h_2|_{{}_\xi I}\right)^{-1}(I) \\ &= h_1^{-1}\left(I_\xi\right) \cup h_2^{-1}\left({}_\xi I\right) = I_\xi \cup {}_\xi I = I,\end{aligned}$$

so condition (5) holds. Similarly, (6) follows directly.

Clearly, such a joiner is not of interest. It is evident that the main point is to consider joiners and set-valued joints with relatively small values.

The following examples of joiners originated in fact in the paper [1].

Example 1. If the marginal functions $h_1, h_2 : I \to I$ are continuous and strictly increasing, and

$$K(x, y) = h_1(x) + h_2(y) - \xi, \qquad (x, y) \in (h_1 \times h_2)\left(I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ\right),$$

then the formula

$$\mathbb{K}(x, y) := \{K(x, y)\} \tag{7}$$

defines a single-valued joiner of the pair (M, N) (cf. [1, Ex. 2.1]).

Example 2. Let $\varphi : I_\xi \to \mathbb{R}$ and $\psi : {}_\xi I \to \mathbb{R}$ be continuous strictly monotonic functions vanishing at ξ, and let $p, q \in (0,1)$. Consider the quasi-arithmetic means $M = A_p^\varphi$ and $N = A_q^\psi$ on the intervals I_ξ and ${}_\xi I$, respectively:

$$M(x,y) = A_p^\varphi(x,y) = \varphi^{-1}\left(p\varphi(x) + (1-p)\varphi(y)\right), \quad x, y \in I_\xi,$$
$$N(x,y) = A_q^\psi(x,y) = \psi^{-1}\left(q\psi(x) + (1-q)\psi(y)\right), \quad x, y \in {}_\xi I.$$

For any $(x,y) \in (h_1 \times h_2)\left(I_\xi^\circ \times {}_\xi I^\circ\right)$, put

$$K(x,y) = \begin{cases} \varphi^{-1}\left(p\left(\varphi(x) + \psi(y)\right)\right), & \text{if } \varphi(x) + \psi(y) < 0, \\ \psi^{-1}\left((1-q)\left(\varphi(x) + \psi(y)\right)\right), & \text{if } \varphi(x) + \psi(y) \geq 0. \end{cases}$$

Similarly, having $(x,y) \in (h_1 \times h_2)\left({}_\xi I^\circ \times I_\xi^\circ\right)$, we put

$$K(x,y) = \begin{cases} \varphi^{-1}\left((1-p)\left(\psi(x) + \varphi(y)\right)\right), & \text{if } \psi(x) + \varphi(y) < 0, \\ \psi^{-1}\left(q\left(\psi(x) + \varphi(y)\right)\right), & \text{if } \psi(x) + \varphi(y) \geq 0. \end{cases}$$

Then $\mathbb{K}$, defined by (7), is a joiner of the pair $\left(A_p^\varphi, A_q^\psi\right)$ (cf. [1, Ex. 2.2]).

More generally, one can easily check (see [1, p. 224]) that if

$$K : (h_1 \times h_2)\left(I_\xi \times {}_\xi I \cup {}_\xi I \times I_\xi\right) \to I$$

is any joining function for the pair (M,N) in the sense of the paper [1], and the marginal functions $h_1, h_2 : I \to I$ are continuous and strictly increasing, then the single-valued multifunction $\mathbb{K}$, given on $(h_1 \times h_2)\left(I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ\right)$ by (7), is a joiner of the pair (M,N).

3. Marginal joints of means

The following result yields a pretty general procedure of joining two means.

Theorem 1. *Let $\mathbb{K}$ be any joiner of the pair (M,N). Then the values of the multifunction $M \oplus_{\mathbb{K}} N$ defined as*

$$\begin{cases} \{M(x,y)\}, & \text{if } (x,y) \in I_\xi \times I_\xi, \\ \left(h_1|_{I_\xi} \cup h_2|_{{}_\xi I}\right)^{-1}\left(\mathbb{K}\left(h_1(x), h_2(y)\right)\right) \cap [x,y], & \text{if } (x,y) \in I_\xi^\circ \times {}_\xi I^\circ, \\ \left(h_2|_{{}_\xi I} \cup h_1|_{I_\xi}\right)^{-1}\left(\mathbb{K}\left(h_1(x), h_2(y)\right)\right) \cap [y,x], & \text{if } (x,y) \in {}_\xi I^\circ \times I_\xi^\circ, \\ \{N(x,y)\}, & \text{if } (x,y) \in {}_\xi I \times {}_\xi I, \end{cases}$$

are non-empty, and every its selection is a mean on the interval I, extending both the means M and N.

Proof. It is enough to follow Definition 1. □

Definition 2. Any mean $M \oplus_{\mathbb{K}} N$ constructed above is called a *marginal* $\mathbb{K}$*-joint of the means* M *and* N.

4. The converse problem

Now, we would like to answer the following question.

Can any mean L *on the interval* I *be reconstructed as a* $\mathbb{K}$*-joint* $M \oplus_{\mathbb{K}} N$ *of the restricted means* $M := L|_{I_\xi \times I_\xi}$ *and* $N := L|_{{}_\xi I \times {}_\xi I}$*, with a suitable joiner* $\mathbb{K}$*?*

However, that question is not well-posed since it can be answered in the following quite trivial way. Namely, if $\mathbb{K}$ is defined by $\mathbb{K}(x,y) = I$, then the joint $M \oplus_{\mathbb{K}} N$ is given by

$$(M \oplus_{\mathbb{K}} N)(x,y) = \begin{cases} \{L(x,y)\}, & \text{if } (x,y) \in I_\xi \times I_\xi \cup {}_\xi I \times {}_\xi I, \\ [x,y], & \text{if } (x,y) \in I_\xi^\circ \times {}_\xi I^\circ, \\ [y,x], & \text{if } (x,y) \in {}_\xi I^\circ \times I_\xi^\circ, \end{cases}$$

and every its selection is a mean on I. Note that L is one of those selections. The reason of that phenomenon is completely clear: the values of the used joiner are too big. So, we will try to answer the following modified question.

Problem 2. *Can any mean* L *on the interval* I *be reconstructed as a* $\mathbb{K}$*-joint* $M \oplus_{\mathbb{K}} N$ *of the restricted means* $M := L|_{I_\xi \times I_\xi}$ *and* $N := L|_{{}_\xi I \times {}_\xi I}$*, with a suitable joiner* $\mathbb{K}$ *with relatively small values?*

Fix a mean L on the interval I, and put

$$M := L|_{I_\xi \times I_\xi} \quad \text{and} \quad N := L|_{{}_\xi I \times {}_\xi I}.$$

Define marginal functions $h_1, h_2 : I \to I$ by the formulas

$$h_1(x) = L(x,\xi) \quad \text{and} \quad h_2(y) = L(\xi, y),$$

respectively. Then

$$h_1(x) = \begin{cases} M(x,\xi), & \text{if } x \in I_\xi, \\ N(x,\xi), & \text{if } x \in {}_\xi I, \end{cases}$$

and

$$h_2(x) = \begin{cases} M(\xi,y), & \text{if } x \in I_\xi, \\ N(\xi,y), & \text{if } x \in {}_\xi I. \end{cases}$$

The formula

$$(u,v)\sim(x,y) :\Longleftrightarrow h_1(u)=h_1(x) \quad \text{and} \quad h_2(v)=h_2(y)$$

defines an equivalence relation in the set $I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ$. Denoting by $(\mathbb{x}_0, \mathbb{y}_0)$ the equivalence class of the point $(x_0, y_0) \in I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ$, we have

$$\begin{aligned}
&(\mathbb{x}_0, \mathbb{y}_0)\\
&\quad = \left\{(x,y) \in I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ : h_1(x) = h_1(x_0) \quad \text{and} \quad h_2(y) = h_2(y_0)\right\}\\
&\quad = \left\{(x,y) \in I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ : (h_1 \times h_2)(x,y) = (h_1(x_0), h_2(y_0))\right\}\\
&\quad = (h_1 \times h_2)^{-1}\left(\{h_1(x_0), h_2(y_0)\}\right) \cap \left(I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ\right).
\end{aligned}$$

This means that the equivalence class $(\mathbb{x}_0, \mathbb{y}_0)$ is the level set of the point $(h_1(x_0),$ $h_2(y_0))$ under the product $h_1 \times h_2$ restricted to $I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ$.

The next result gives a positive answer to the question posed in this section.

Theorem 2. *Let* $\mathbb{K}_0 : (h_1 \times h_2)(I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ) \to 2^I \setminus \{\emptyset\}$ *be given by*

$$\mathbb{K}_0(h_1(x), h_2(y)) = \left(h_1|_{I_\xi} \cup h_2|_{{}_\xi I}\right)\left(L\left((\mathbb{x}_0, \mathbb{y}_0)\right)\right), \quad (x,y) \in I_\xi^\circ \times {}_\xi I^\circ, \tag{8}$$

and

$$\mathbb{K}_0(h_1(x), h_2(y)) = \left(h_2|_{I_\xi} \cup h_1|_{{}_\xi I}\right)\left(L\left((\mathbb{x}_0, \mathbb{y}_0)\right)\right), \quad (x,y) \in {}_\xi I^\circ \times I_\xi^\circ. \tag{9}$$

Then

(i) $\mathbb{K}_0$ *is a joiner of the pair* $\left(L|_{I_\xi \times I_\xi}, L|_{{}_\xi I \times {}_\xi I}\right)$ *and satisfies the condition*

$$L\left((\mathbb{x}_0, \mathbb{y}_0)\right) \subset \left(L|_{I_\xi \times I_\xi} \oplus_{\mathbb{K}_0} L|_{{}_\xi I \times {}_\xi I}\right)(x,y), \quad (x,y) \in I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ,$$

and

(ii) *if* $\mathbb{K}$ *is a joiner of the pair* $\left(L|_{I_\xi \times I_\xi}, L|_{{}_\xi I \times {}_\xi I}\right)$ *and satisfies*

$$L\left((\mathbb{x}_0, \mathbb{y}_0)\right) \subset \left(L|_{I_\xi \times I_\xi} \oplus_{\mathbb{K}_0} L|_{{}_\xi I \times {}_\xi I}\right)(x,y), \quad (x,y) \in I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ, \tag{10}$$

then $\mathbb{K}_0 \subset \mathbb{K}$, *that is,*

(i) *the mean* L *can be reconstructed as a selection for the marginal* $\mathbb{K}_0$*-joint of its restrictions* $L|_{I_\xi \times I_\xi}$ *and* $L|_{{}_\xi I \times {}_\xi I}$,

and

(ii) $\mathbb{K}_0$ *is the smallest (in the sense of inclusion) joiner of the pair* $\left(L|_{I_\xi \times I_\xi},\right.$ $\left.L|_{{}_\xi I \times {}_\xi I}\right)$ *satisfying condition* (10).

Proof. (i) For every $(x,y) \in I_\xi^\circ \times {}_\xi I^\circ$, we have

$$\left(h_1|_{I_\xi} \cup h_2|_{{}_\xi I}\right)^{-1} \left[\left(h_1|_{I_\xi} \cup h_2|_{{}_\xi I}\right) (L((\mathbb{x}_0, \mathbb{y}_0)))\right] \cap [x,y] \\ \supset L((\mathbb{x}_0, \mathbb{y}_0)) \cap [x,y] \ni L(x,y).$$

Similarly, if $(x,y) \in {}_\xi I^\circ \times I_\xi^\circ$, then

$$\left(h_2|_{I_\xi} \cup h_1|_{{}_\xi I}\right)^{-1} \left[\left(h_2|_{I_\xi} \cup h_1|_{{}_\xi I}\right) (L((\mathbb{x}_0, \mathbb{y}_0)))\right] \cap [y,x] \\ \supset L((\mathbb{x}_0, \mathbb{y}_0)) \cap [y,x] \ni L(x,y).$$

Thus $\mathbb{K}_0$ is a joiner of the pair $\left(L_{I_\xi \times I_\xi}, L_{{}_\xi I \times {}_\xi I}\right)$ and

$$\left(L_{I_\xi \times I_\xi} \oplus_{\mathbb{K}_0} L_{{}_\xi I \times {}_\xi I}\right) \supset L\left((\mathbb{x}_0, \mathbb{y}_0)\right),$$

for each $(x,y) \in I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ$.

(ii) Take any joiner $\mathbb{K}$ of the pair $\left(L_{I_\xi \times I_\xi}, L_{{}_\xi I \times {}_\xi I}\right)$ satisfying condition (10). Then, for any $(x,y) \in I_\xi^\circ \times {}_\xi I^\circ$, we have

$$\begin{aligned} \mathbb{K}_0\left(h_1(x), h_2(y)\right) &= \left(h_1|_{I_\xi} \cup h_2|_{{}_\xi I}\right) (L((\mathbb{x}_0, \mathbb{y}_0))) \\ &\subset \left(h_1|_{I_\xi} \cup h_2|_{{}_\xi I}\right) \left[\left(L_{I_\xi \times I_\xi} \oplus_{\mathbb{K}} L_{{}_\xi I \times {}_\xi I}\right)(x,y)\right] \\ &= \left(h_1|_{I_\xi} \cup h_2|_{{}_\xi I}\right) \left[\left(h_1|_{I_\xi} \cup h_2|_{{}_\xi I}\right)^{-1} \left(\mathbb{K}\left(h_1(x), h_2(y)\right)\right) \cap [x,y]\right] \\ &\subset \mathbb{K}\left(h_1(x), h_2(y)\right). \end{aligned}$$

Repeating the calculation for an arbitrary point $(x,y) \in {}_\xi I^\circ \times I_\xi^\circ$, we come to the assertion. □

Definition 3. The multifunction $\mathbb{K}_0$, introduced in Theorem 2, is called *ξ-reconstructing joiner for the mean L.*

5. Reconstructing joiner with singleton values

The following question arises naturally.

Problem 3. *Is it possible that the reconstructing joiner is in fact a single-valued function?*

Below we give a full answer to this question, providing a simple characterization of means with single-valued reconstructing joiners.

We say that a mean L *preserves its ξ-margins* if the equalities $L(u_1,\xi) = L(u_2,\xi)$ and $L(\xi,v_1) = L(\xi,v_2)$ imply that

$$\begin{aligned}&L\left(\min\{L(u_1,v_1),\xi\},\max\{L(u_1,v_1),\xi\}\right)\\&\quad= L\left(\min\{L(u_2,v_2),\xi\},\max\{L(u_2,v_2),\xi\}\right)\end{aligned}$$

for all $(u_1,v_1),(u_2,v_2) \in I_\xi^\circ \times {}_\xi I^\circ$, and

$$\begin{aligned}&L\left(\max\{L(u_1,v_1),\xi\},\min\{L(u_1,v_1),\xi\}\right)\\&\quad= L\left(\max\{L(u_2,v_2),\xi\},\min\{L(u_2,v_2),\xi\}\right)\end{aligned}$$

for all $(u_1,v_1),(u_2,v_2) \in {}_\xi I^\circ \times I_\xi^\circ$.

Remark 1. Observe that for symmetric means the above defining condition becomes much simpler: the equalities $L(u_1,\xi) = L(u_2,\xi)$ and $L(v_1,\xi) = L(v_2,\xi)$ imply that

$$L\left(L(u_1,v_1),\xi\right) = L\left(L(u_2,v_2),\xi\right)$$

for all $(u_1,v_1),(u_2,v_2) \in I_\xi^\circ \times {}_\xi I^\circ$ and $(u_1,v_1),(u_2,v_2) \in {}_\xi I^\circ \times I_\xi^\circ$.

Remark 2. If the marginal functions $L(\cdot,\xi)$ and $L(\xi,\cdot)$ are one-to-one, then the mean L preserves its ξ-margins.

Example 3. Take $I = \mathbb{R}$ and $\xi = 0$, and put

$$L(x,y) = \max\{x,y\}, \quad x,y \in I.$$

Of course, the marginal function $L(\cdot,0)$ is not one-to-one. Nevertheless, L preserves its 0-margins. To see this, take any $(u_1,v_1),(u_2,v_2) \in I_0^\circ \times {}_0 I^\circ$ satisfying $L(u_1,0) = L(u_2,0)$ and $L(v_1,0) = L(v_2,0)$. Then $u_1 < 0 < v_1$ and $u_2 < 0 < v_2$, whence also $L(u_1,v_1) = v_1$ and $L(u_2,v_2) = v_2$. Therefore,

$$L\left(L(u_1,v_1),0\right) = L(v_1,0) = L(v_2,0) = L\left(L(u_2,v_2),0\right).$$

A similar condition holds whenever $(u_1,v_1),(u_2,v_2) \in {}_0 I^\circ \times I_0^\circ$. Thus, by Remark 1, the mean L preserves its 0-margins.

Example 4. Of course, not every mean preserves its margins. To see this, take $\xi = 1$ and define L as the contraharmonic mean on the interval $(0,+\infty)$ by the equality

$$L(x,y) = \frac{x^2+y^2}{x+y}.$$

Suppose that L preserves its 1-margins. As it is symmetric, we may use Remark 1. Take an arbitrary $v \in (1, +\infty)$ and put $u_1 = \frac{1}{5}, u_2 = \frac{2}{3}$, $v_1 = v_2 = v$. Then $(u_1, v_1), (u_2, v_2) \in I_1^\circ \times {}_1I^\circ$,

$$L(u_1, 1) = L\left(\frac{1}{5}, 1\right) = \frac{13}{15} = L\left(\frac{2}{3}, 1\right) = L(u_2, 1),$$

and, of course, $L(v_1, 1) = L(v_2, 1) = L(v, 1)$. Thus, by Remark 1,

$$L(L(u_1, v), 1) = L(L(u_2, v), 1). \tag{11}$$

Since

$$\lim_{v\to\infty} L(u, v) = \lim_{v\to\infty} \frac{u^2 + v^2}{u + v} = +\infty, \quad u \in (0, +\infty),$$

we may choose v in such a way that $L(u_1, v) > 1$ and $L(u_2, v) > 1$. Then, as the restriction of $L(\cdot, 1)$ to $(1, +\infty)$ is one-to-one, we have $L(u_1, v) = L(u_2, v)$. Taking into account that $u_1 = \frac{1}{5}$ and $u_2 = \frac{2}{3}$, we see that

$$\frac{25v^2 + 1}{25v + 5} = \frac{9v^2 + 4}{9v + 6},$$

whence

$$15v^2 - 13v - 2 = 0,$$

that is, $v \in \left\{-\frac{2}{15}, 1\right\}$, a contradiction.

The main result of this section reads as follows.

Theorem 3. *The ξ-reconstructing joiner of the mean L has only singletons among the values if and only if L preserves its ξ-margins.*

Proof. Assume that the reconstructing joiner $\mathbb{K}_0$ given by (8) and (9) is single-valued. Take any points $(u_1, v_1), (u_2, v_2) \in I_\xi^\circ \times {}_\xi I^\circ$ satisfying $L(u_1, \xi) = L(u_2, \xi)$ and $L(\xi, v_1) = L(\xi, v_2)$. Consider the following four possible cases:

(a) $L(u_1, u_2) \le \xi$ and $L(v_1, v_2) \le \xi$;

(b) $L(u_1, u_2) \le \xi < L(v_1, v_2)$;

(c) $L(v_1, v_2) \le \xi < L(u_1, u_2)$;

(d) $\xi < L(u_1, u_2)$ and $\xi < L(v_1, v_2)$.

According to (8), we have

$$\mathbb{K}_0\left(h_1(u_1), h_2(v_1)\right) = h_1\left(L\left((\mathbb{u}_{\mathbb{1}}, \mathbb{v}_{\mathbb{1}})\right) \cap I_\xi\right) \cup h_2\left(L\left((\mathbb{u}_{\mathbb{1}}, \mathbb{v}_{\mathbb{1}})\right) \cap {}_\xi I^\circ\right).$$

Therefore, as the above set is a singleton, we conclude that:

$$h_1\left(L\left(u_1, v_1\right)\right) = h_1\left(L\left(u_2, v_2\right)\right),$$

i.e. $L\left(L\left(u_1, v_1\right), \xi\right) = L\left(L\left(u_2, v_2\right), \xi\right)$ in case (a),

$$h_1\left(L\left(u_1, v_1\right)\right) = h_2\left(L\left(u_2, v_2\right)\right),$$

i.e. $L\left(L\left(u_1, v_1\right), \xi\right) = L\left(\xi, L\left(u_2, v_2\right)\right)$ in case (b),

$$h_2\left(L\left(u_1, v_1\right)\right) = h_1\left(L\left(u_2, v_2\right)\right),$$

i.e. $L\left(\xi, L\left(u_1, v_1\right)\right) = L\left(L\left(u_2, v_2\right), \xi\right)$ in case (c),

$$h_2\left(L\left(u_1, v_1\right)\right) = h_2\left(L\left(u_2, v_2\right)\right),$$

i.e. $L\left(\xi, L\left(u_1, v_1\right)\right) = L\left(\xi, L\left(u_2, v_2\right)\right)$ in case (d). In all cases (a)–(d), the obtained equalities mean that

$$\begin{aligned} &L\left(\min\{L\left(u_1, v_1\right), \xi\}, \max\{L\left(u_1, v_1\right), \xi\}\right) \\ &\quad = L\left(\min\{L\left(u_2, v_2\right), \xi\}, \max\{L\left(u_2, v_2\right), \xi\}\right). \end{aligned}$$

A similar reasoning gives the second desired equality in the case when (u_1, v_1), $(u_2, v_2) \in {}_\xi I^\circ \times I^\circ_\xi$. So L preserves its ξ-margins.

A careful analysis of the above proof shows that also the converse implication holds true. □

Finally, we notice the following immediate consequence of Theorem 3 and Remark 2.

Corollary 1. *If the marginal functions $L(\cdot, \xi)$ and $L(\xi, \cdot)$ are one-to-one, then the ξ-reconstructing joiner of L is single-valued.*

References

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