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Linear divisibility sequences and Salem numbers

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Abstract. We study linear divisibility sequences of order 4, finding a characterization by means of their characteristic polynomials and providing their factorization as a product of linear divisibility sequences of order 2. Moreover, we show new interesting connections between linear divisibility sequences and Salem numbers. Specifically, we generate linear divisibility sequences of order 4 by means of Salem numbers modulo 1.

1. Introduction

The study of linear divisibility sequences is a fruitful and fascinating branch of number theory, whose relevance is apparent from many open questions and some unexpected connections with other topics such as cryptography, elliptic curves and algebraic integers, as pointed out in [22], [13] and [21]. We may say that the most cited example of a sequence $(a_n)_{n=0}^{+\infty}$ having the divisibility property

$$m|n \Rightarrow a_m|a_n$$

is the classic Fibonacci sequence, which is the most known linear divisibility sequence of order 2. During the years, the work of many mathematicians has offered a deep insight and many generalizations to the subject, for instance, extending the concept to matrix divisibility sequences as in [9]. The well-known Lucas–Lehmer theory offers a detailed analysis of linear recurrence sequence of order 2, while strong divisibility sequences and polynomial divisibility sequences are deeply studied in [12] and [15]. Linear recurrence sequences of higher order turned out to be

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an intriguing challenge, examined since the paper of Hall [11] concerning sequences of order 3, to the general results presented by Bézivin, Pethő and Van der Poorten in [4] on the characterization of divisibility sequences, and lately revisited in [2]. The aim of this paper is to shine new light to linear divisibility sequences of order 4, which recently have been deeply examined in [18], [23] and [24]. In particular, Williams and Guy in their papers [23], [24] introduced and studied a class of linear divisibility sequences of order 4 that extends the Lucas–Lehmer theory for divisibility sequences of order 2. In Section 2, we consider these sequences, proving that all the (non-degenerate) divisibility sequences of order 4 have the same characteristic polynomials as the sequences of Williams and Guy do. Moreover, we provide all the factorizations of divisibility sequences of order 4 into the product of divisibility sequences of order 2. Since the construction of divisibility sequences by means of powers of algebraic integers is a research field that has been recently developed (see, e.g., [21]), we present in Section 3 a way to generate linear divisibility sequences of order 4 by means of powers of Salem numbers. This result is particularly interesting, since connections between Salem numbers and divisibility sequences have been studied only in some particular cases (see, e.g., [17]).

2. Standard linear divisibility sequences

Definition 1. Given a ring $\mathcal{R}$, a sequence $a = (a_n)_{n=0}^{+\infty}$, with $a_i \in \mathcal{R}$, is a *divisibility sequence* if

$$m|n \Rightarrow a_m|a_n.$$

Conventionally, we will consider $a_0 = 0$.

In the following, we will deal with *linear divisibility sequences* (LDSs), i.e., divisibility sequences that satisfy a linear recurrence. Classic LDSs are the Lucas sequences, i.e., linear recurrence sequences whose characteristic polynomial is $x^2 - hx + k$ and with initial conditions $0, 1$.

In [23] and [24], the authors introduced and studied some linear divisibility sequences of order 4. We recall these sequences in the following definition.

Definition 2. Let us consider linear recurrence sequences of order 4 over $\mathbb{Z}$ with characteristic polynomial

$$x^4 - px^3 + (q + 2r)x^2 - prx + r^2,$$

and with initial conditions

$$0, 1, p, p^2 - q - 3r.$$

We say that these sequences are *standard* LDSs of order 4, and we call the previous polynomial as a *standard polynomial*.

In the next theorem, we will prove that the product of two LDSs of order 2 is a standard LDS of order 4. First of all, we recall the definition of the Kronecker product of two matrices and an important lemma proved in [8].

Definition 3. Given any matrices A and B, with dimensions $m \times n$ and $p \times q$, respectively, the Kronecker product $A \otimes B$ is a matrix $mp \times nq$ defined as follows:

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.$$

Lemma 1. *Let $a = (a_n)_{n=0}^{+\infty}$ and $b = (b_n)_{n=0}^{+\infty}$ be linear recurrence sequences with characteristic polynomials $f(x)$ and $g(x)$, respectively. The sequence $ab = (a_n b_n)_{n=0}^{+\infty}$ is a linear recurrence sequence that recurs with $f(x) \otimes g(x)$, the characteristic polynomial of the matrix $F \otimes G$ (Kronecker product of matrices), where F and G are the companion matrices of $f(x)$ and $g(x)$, respectively.*

Remark 1. The previous lemma also corresponds to the following statement. Let $a = (a_n)_{n=0}^{+\infty}$ and $b = (b_n)_{n=0}^{+\infty}$ be linear recurrence sequences whose characteristic polynomials have roots $\alpha_1, ..., \alpha_s$ and $\beta_1, ..., \beta_t$, respectively. Then, the sequence $c = (c_n)_{n=0}^{+\infty} = (a_n b_n)_{n=0}^{+\infty}$ is also a linear recurrence sequence whose characteristic polynomial has roots $\gamma_1, ..., \gamma_{st}$, where

$$(\gamma_1, ..., \gamma_{st}) = (\alpha_1, ..., \alpha_s) \otimes (\beta_1, ..., \beta_t).$$

Theorem 1. *Let $a = (a_n)_{n=0}^{+\infty}$ and $b = (b_n)_{n=0}^{+\infty}$ be LDSs of order 2 with characteristic polynomials $x^2 - h_1 x + k_1$, $x^2 - h_2 x + k_2$, respectively, and initial conditions $0, 1$. The sequence $ab = (a_n b_n)_{n=0}^{+\infty}$ is a standard LDS of order 4 with initial conditions $0, 1, h_1 h_2, \left(h_1^2 - k_1\right)\left(h_2^2 - k_2\right)$.*

Proof. Since a and b are LDSs, it immediately follows that ab is a divisibility sequence, and, by Lemma 1, we know that it is a linear recurrence sequence of order 4 whose characteristic polynomial is

$$x^4 - h_1 h_2 x^3 + (k_1 h_1^2 - k_2 h_1^2 + 2k_1 k_2)x^2 + h_1 k_1 h_2 k_2 x + k_1^2 k_2^2.$$

By Definition 2, ab is a standard LDS for $p = h_1 h_2$, $q = h_1^2 k_2 + k_1\left(h_2^2 - 4k_2\right)$, $r = k_1 k_2$. The initial conditions can be directly calculated. $\square$

Moreover, we prove that all the LDSs of order 4 have characteristic polynomial equal to the characteristic polynomial of standard LDSs.

Theorem 2. *Let $a = (a_n)_{n=0}^{+\infty}$ be a non-degenerate LDS of order 4 with $a_0 = 0$ and $a_1 = 1$. Then its characteristic polynomial is*

$$x^4 - px^3 + (q + 2r)x^2 - prx + r^2, \tag{1}$$

for some p, q, r.

Proof. Since we deal with the non-degenerate case, i.e., the ratios between the roots of the characteristic polynomial are not roots of unity, the characteristic polynomial of a has four distinct roots α, β, γ, δ. From well-known results on the characterization of divisibility sequences (see [4] and [2]), the sequence a is a divisor of the sequence $b = (b_n)_{n=0}^{+\infty}$, where

$$b_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \cdot \frac{\alpha^n - \gamma^n}{\alpha - \gamma} \cdot \frac{\alpha^n - \delta^n}{\alpha - \delta} \cdot \frac{\beta^n - \gamma^n}{\beta - \gamma} \cdot \frac{\beta^n - \delta^n}{\beta - \delta} \cdot \frac{\gamma^n - \delta^n}{\gamma - \delta}.$$

In other words, there exists a sequence $c = (c_n)_{n=0}^{+\infty}$ such that $b_n = a_n c_n$, for any index n.

By Lemma 1 and Remark 1, the sequence b can be written as the product of six Lucas sequences with characteristic polynomials having roots (α, β), (α, γ), (α, δ), (β, γ), (β, δ), (γ, δ), respectively. Thus, without loss of generality, we may suppose that the roots of the characteristic polynomial of b are the entries of the following vector with 64 components:

$$B = (\alpha, \beta) \otimes (\alpha, \gamma) \otimes (\alpha, \delta) \otimes (\beta, \gamma) \otimes (\beta, \delta) \otimes (\gamma, \delta),$$

where all the roots appear with the due multiplicity. We can write the vector B as

$$B = \left(\alpha C', \gamma C', \beta C', \frac{\beta\gamma}{\alpha} C'\right), \tag{2}$$

where

$$C' = (\alpha^2, \alpha\delta) \otimes (\beta, \gamma) \otimes (\beta, \delta) \otimes (\gamma, \delta)$$

is a vector with 16 components. Moreover, $B = A \otimes C$, where C is a vector whose components are the roots of the characteristic polynomial of c (appearing with the due multiplicity), and

$$A = (\omega_1, \omega_2, \omega_3, \omega_4),$$

a certain permutation of $(\alpha, \beta, \gamma, \delta)$. Thus, we have

$$B = (\omega_1 C, \omega_2 C, \omega_3 C, \omega_4 C), \tag{3}$$

where C clearly has 16 components. Subtracting (3) from (2), we obtain

$$\left(w_1 C - \alpha C', w_2 C - \gamma C', w_3 C - \beta C', w_4 C - \frac{\beta\gamma}{\alpha} C'\right) = O_{64},$$

where O_{64} is the zero vector with 64 components. Thus

$$w_1 C - \alpha C' = w_2 C - \gamma C' = w_3 C - \beta C' = w_4 C - \frac{\beta\gamma}{\alpha} C' = O_{16},$$

where O_{16} is the zero vector with 16 components. From $w_1 C - \alpha C' = O_{16}$, it follows that $C = \dfrac{\alpha}{w_1} C'$ and

$$\left(\frac{w_2\alpha}{w_1} - \gamma\right) C' = \left(\frac{w_3\alpha}{w_1} - \beta\right) C' = \left(\frac{w_4\alpha}{w_1} - \frac{\beta\gamma}{\alpha}\right) C' = O_{16}.$$

Since $C' \neq O_{16}$, clearly,

$$\frac{w_2\alpha}{w_1} - \gamma = \frac{w_3\alpha}{w_1} - \beta = \frac{w_4\alpha}{w_1} - \frac{\beta\gamma}{\alpha} = 0,$$

and from these equalities it is straightforward to obtain that $w_1 w_4 = w_2 w_3$, i.e., the characteristic polynomial of a must be of the form (1). $\square$

Now, we show how any standard LDS can be factorized as a product of two LDS of order 2 over $\mathbb{C}$.

Definition 4. Given the sequences $(u_n)_{n=0}^{+\infty}, (v_n)_{n=0}^{+\infty}, (s_n)_{n=0}^{+\infty}, (t_n)_{n=0}^{+\infty}$ over a ring $\mathcal{R}$, we say that the sequences $(u_n v_n)_{n=0}^{+\infty}$ and $(s_n t_n)_{n=0}^{+\infty}$ are *equivalent* if

$$u_n = \lambda^{n-1} s_n, \quad v_n = \lambda^{1-n} t_n,$$

where $\lambda \in \mathcal{R}$ is a unit.

Theorem 3. *Let $a = (a_n)_{n=0}^{+\infty}$ be a standard LDS over $\mathbb{Z}$. Then $a_n = b_n c_n$, for all $n \geq 0$, where $b = (b_n)_{n=0}^{+\infty}$ and $c = (c_n)_{n=0}^{+\infty}$ are LDSs of order 2 over $\mathbb{C}$ with initial conditions $0, 1$ and characteristic polynomials*

$$\begin{cases} x^2 - \dfrac{\sqrt{q + 4r + 2p\sqrt{r}} \pm \sqrt{q + 4r - 2p\sqrt{r}}}{2\sqrt{r}} x + 1, \\ x^2 - \dfrac{\sqrt{q + 4r + 2p\sqrt{r}} \mp \sqrt{q + 4r - 2p\sqrt{r}}}{2} x + r, \end{cases}$$

when $p \neq 0$*. Moreover, when* $p = 0$ *and* $q + 4r \neq 0$, $q \neq 0$ *(to avoid degenerate cases) we have the two possible families of characteristic polynomials for* b *and* c *given by*

$$\begin{cases} x^2 + 1 \\ x^2 - \sqrt{q+4r}\,x + r \end{cases} \quad \textit{and} \quad \begin{cases} x^2 + 1 \\ x^2 - \sqrt{q}\,x - r. \end{cases}$$

These are all the families of non-equivalent factorizations of a *over* $\mathbb{C}$*.*

Proof. We want to factorize a standard polynomial into the Kronecker product of two polynomials of degree 2, i.e., we want to find h_1, h_2, k_1, k_2 such that

$$(x^2 - h_1 x + k_1) \otimes (x^2 - h_2 x + k_2) = x^4 - px^3 + (q + 2r)x^2 - px + r^2.$$

Let us observe that the characteristic polynomial of a must have distinct non-zero roots in order to guarantee that a is an LDS of order 4. Let γ_1, γ_2 and σ_1, σ_2 be the roots of $x^2 - h_1 x + k_1$ and $x^2 - h_2 x + k_2$, respectively. We have

$$\begin{cases} (\gamma_1 + \gamma_2)(\sigma_1 + \sigma_2) = p \\ (\gamma_1^2 + \gamma_2^2)\sigma_1\sigma_2 + \gamma_1\gamma_2(\sigma_1 + \sigma_2)^2 = q + 2r \\ \gamma_1\gamma_2\sigma_1\sigma_2(\gamma_1 + \gamma_2)(\sigma_1 + \sigma_2) = pr \\ (\gamma_1\gamma_2\sigma_1\sigma_2)^2 = r^2. \end{cases} \tag{4}$$

When $p \neq 0$, these conditions are equivalent to the system

$$\begin{cases} k_1 k_2 = r, \\ h_1 h_2 = p, \\ h_1^2 k_2 + h_2^2 k_1 = q + 4r, \end{cases} \tag{5}$$

which is a particular case of

$$\begin{cases} k_1 k_2 = A, \\ h_1 h_2 = B, \\ h_1^2 k_2 + h_2^2 k_1 = C, \end{cases}$$

where $A \neq 0$, since we suppose that the standard polynomial has non-zero roots. Thus

$$A\left(\frac{h_1^2}{k_1}\right)^2 - C\left(\frac{h_1^2}{k_1}\right) + B^2 = 0,$$

from which we have

$$h_1 = \pm\sqrt{k_1}\frac{\sqrt{C+2B\sqrt{A}} \pm \sqrt{C-2B\sqrt{A}}}{2\sqrt{A}},$$

and

$$h_2 = \pm\frac{\sqrt{C+2B\sqrt{A}} \mp \sqrt{C-2B\sqrt{A}}}{2\sqrt{k_1}}.$$

Therefore, the solutions of the system (5) are

$$\begin{cases} h_1 = \pm\sqrt{k_1}\dfrac{\sqrt{q+4r+2p\sqrt{r}} \pm \sqrt{q+4r-2p\sqrt{r}}}{2\sqrt{r}} \\ h_2 = \pm\dfrac{\sqrt{q+4r+2p\sqrt{r}} \mp \sqrt{q+4r-2p\sqrt{r}}}{2\sqrt{k_1}} \\ k_2 = \dfrac{r}{k_1}. \end{cases}$$

Set

$$\lambda = \pm\sqrt{k_1}, \quad s = \frac{\sqrt{q+4r+2p\sqrt{r}} \pm \sqrt{q+4r-2p\sqrt{r}}}{2\sqrt{r}},$$

$$\bar{s} = \frac{\sqrt{q+4r+2p\sqrt{r}} \mp \sqrt{q+4r-2p\sqrt{r}}}{2}.$$

Considering the solutions of the system (5), we find $x^2 - h_1x + k_1 = x^2 - s\lambda x + \lambda^2$ and $x^2 - h_2x + k_2 = x^2 - \frac{\bar{s}}{\lambda}x + \frac{r}{\lambda^2}$, whose roots are

$$\gamma_{1,2} = \lambda\left(\frac{s \pm \sqrt{s^2-4}}{2}\right), \quad \sigma_{1,2} = \frac{1}{\lambda}\left(\frac{\bar{s} \pm \sqrt{\bar{s}^2-4}}{2}\right).$$

In this case, we have $u_n = \lambda^{n-1}b_n$ and $v_n = \lambda^{1-n}c_n$, where b and c are Lucas sequences with characteristic polynomials $x^2 - sx + 1$ and $x^2 - \bar{s}x + r$, respectively. When $p = 0$ in conditions (4), we may suppose $\gamma_1 + \gamma_2 = h_1 = 0$ and find the two systems

$$\begin{cases} h_1 = 0 \\ k_1k_2 = r \\ h_2^2k_1 = q+4r \end{cases} \quad \textit{and} \quad \begin{cases} h_1 = 0 \\ k_1k_2 = -r \\ h_2^2k_1 = q, \end{cases}$$

with respective solutions

$$\begin{cases} h_1 = 0 \\ h_2 = \pm\sqrt{\frac{q+4r}{k_1}} \\ k_2 = \frac{r}{k_1} \end{cases} \quad and \quad \begin{cases} h_1 = 0 \\ h_2 = \pm\sqrt{\frac{q}{k_1}} \\ k_2 = -\frac{r}{k_1}, \end{cases}$$

which give, with analogous considerations as in the case $p \neq 0$, with $\lambda = \pm\sqrt{k_1}$, the two families of characteristic polynomials for b and c related to this case. □

Remark 2. It would be interesting to find when the previous factorizations determine sequences in $\mathbb{Z}$ or $\mathbb{Z}[i]$.

In the next section, we see a new connection between LDSs of order 4 and Salem numbers.

3. Construction of linear divisibility sequences by means of Salem numbers of order 4

The Salem numbers have been introduced in 1944 by Raphael Salem [20], and they are closely related to the Pisot numbers [19]. There are several results regarding Pisot numbers and recurrence sequences [5], [6], [7]. In the following, we relate Salem numbers and LDS.

There are many equivalent definitions of Salem numbers, here we report the following one.

Definition 5. A *Salem number* is a real algebraic integer $\tau > 1$ of degree $d \geq 4$ such that all the conjugate elements lie on the unit circle, unless τ and τ^{-1}.

In the following, we work with Salem numbers of degree 4, which can be characterized as follows (see [3, p. 81]).

Proposition 1. *The Salem numbers of degree 4 are all the real roots, $\tau > 1$, of the following polynomials with integer coefficients*

$$x^4 + tx^3 + cx^2 - tx + 1, \tag{6}$$

where

$$2(t-1) < c < -2(t+1).$$

It is immediate to note that the previous polynomials are standard polynomials for $p = -t$, $q = -2 + c$, $r = 1$. In [18], the author studied properties of the sequences having characteristic polynomials (6), also highlighting divisibility properties and factorization. Moreover, in [17], the divisibility property of the sequences was proved in the cases $c = -1, -3$.

Definition 6. Salem standard polynomials are the polynomials

$$x^4 - px^3 + (q+2)x^2 - px + 1$$

with

$$2(-p-1) < 2 + q < -2(-p+1).$$

The study of the distribution modulo 1 of the powers of a given real number greater than 1 is a rich and classic research field (see, e.g, [14]). In the following, we use the same notation of [3, p. 61)].

Definition 7. Given a real number α, let $E(\alpha)$ be the nearest integer to α, i.e., $\alpha = E(\alpha) + \epsilon(\alpha)$, where $\epsilon(\alpha) \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ is called α *modulo 1.*

In the original work of Salem [20], he proved that if α is a Pisot number, then α^n modulo 1 tends to zero, and if α is a Salem number, then α^n modulo 1 is dense in the unit interval. Further results on the distribution modulo 1 of the Salem numbers can be found, e.g., in [1] and [26]. Moreover, integer and fractional parts of Pisot and Salem numbers have been studied, e.g., in [10] and [27].

Let $\mathcal{R} \subseteq \mathbb{C}$ be a ring and $\alpha \in \mathcal{R}$, then the sequence $(\alpha^n)_{n=0}^{+\infty}$ is clearly an LDS. Given a couple of irrational numbers λ and α, it is interesting to study when the sequence $(E(\lambda\alpha^n))_{n=0}^{+\infty}$ is an LDS.

Example 1. If we consider $\frac{1}{\sqrt{5}}$ and the golden mean ϕ, it is well-known that

$$E\left(\frac{1}{\sqrt{5}}\phi^n\right) = F_n,$$

where F_n is the n-th Fibonacci number, consequently, we get an LDS.

Let $g(x)$ be a Salem standard polynomial, having real roots $\alpha > 1$, α^{-1}, and complex roots γ, γ^{-1} with norm 1. Let $(u_n)_{n=0}^{+\infty}$ be a standard LDS with characteristic polynomial $g(x)$. By the Binet formula, there exist $\lambda, \lambda_1, \lambda_2, \lambda_3$ such that

$$u_n = \lambda\alpha^n + \lambda_1\alpha^{-n} + \lambda_2\gamma^n + \lambda_3\gamma^{-n}.$$

Since

$$|u_n - \lambda\alpha^n| \leq |\lambda_1\alpha^{-n}| + |\lambda_2| + |\lambda_3|,$$

for all $\epsilon > 0$, with n sufficiently large, we have

$$|u_n - \lambda\alpha^n| \leq \epsilon + |\lambda_2| + |\lambda_3|.$$

Thus, if $|\lambda_2| + |\lambda_3| < \frac{1}{2}$, there exists n_0 such that

$$u_n = E(\lambda\alpha^n), \quad \forall n > n_0,$$

and if $|\lambda_1\alpha^{-1}| + |\lambda_2| + |\lambda_3| < \frac{1}{2}$, then

$$u_n = E(\lambda\alpha^n), \quad \forall n \geq 1.$$

An interesting case is given by the Salem standard polynomial

$$x^4 - tx^3 + tx^2 - tx + 1$$

for $t \geq 6$. In this case, we have the Salem numbers

$$\alpha = \frac{1}{4}\left(t + \sqrt{(t-4)t+8} + \sqrt{2}\sqrt{t(t + \sqrt{(t-4)t+8} - 2) - 4}\right)$$

and

$$\lambda = \frac{1}{\sqrt{(t-4)t+8}}.$$

Thus, we can determine infinitely many LDSs generated by powers of a Salem number, specifically the sequences

$$(\theta_n(t))_{n=1}^{+\infty} = E(\lambda\alpha^n), \quad \forall t \geq 6 \in \mathbb{Z}.$$

For example, when $t = 6$ and $t = 7$, respectively, we have the LDSs

$$1, 6, 29, 144, 725, 3654, 18409, \ldots,$$

and

$$1, 7, 41, 245, 8897, 53621, \ldots.$$

These sequences appear to be new, since they are not listed in OEIS [16]. Moreover, as a consequence, we have the following property:

$$d|n \Rightarrow E(\lambda\alpha^d)|E(\lambda\alpha^n).$$

Finally, in the following proposition we characterize all the Salem standard polynomials that yield LDSs of this kind.

Proposition 2. *With the above notation, if* $|\lambda_1\alpha^{-1}|+|\lambda_2|+|\lambda_3|<\frac{1}{2}$, *then the integer coefficients* p, q *of* $g(x)$ *must satisfy the following inequalities:*

$$\begin{cases} 2\le p\le 8, & -4-2p<q<\dfrac{p^4+8p^3-160p-400}{4p^2+32p+64}, \\ p>8, & -4-2p<q<-4+2p. \end{cases}$$

Proof. The real root $\alpha>1$ of $g(x)$ can be written as

$$\alpha=\frac{\left(p+\sqrt{p^2-4q}+\sqrt{(p+\sqrt{p^2-4q})^2-16}\right)}{4}.$$

Moreover, by the Binet formula

$$\lambda=\lambda_1=\frac{\alpha\gamma}{(\alpha-\gamma)(\alpha\gamma-1)},\quad \lambda_2=\lambda_3=-\frac{\alpha\gamma}{(\alpha-\gamma)(\alpha\gamma-1)}.$$

Thus, from $|\lambda_1\alpha^{-1}|+|\lambda_2|+|\lambda_3|<\frac{1}{2}$ we get

$$|(\alpha-\gamma)(\alpha\gamma-1)|>2\alpha+2.$$

Posing $\gamma=a+ib$, with some calculations we find

$$\alpha^4-4a\alpha^3+2(2a^2-7)\alpha^2-4(a+4)\alpha-3>0,$$

from which we have

$$\alpha>2+a+\sqrt{(a+2)^2+1},$$

since $-1<a<1$ and $\alpha>1$. Using the explicit expression of α and the equality $a=\frac{p-\sqrt{p^2-4q}}{4}$, we finally obtain

$$\frac{1}{4}\left(p+\sqrt{-16+(-p-\sqrt{p^2-4q})^2}+\sqrt{p^2-4q}\right)$$
$$>2+\frac{p}{4}+\sqrt{1+\frac{1}{16}(8+p-\sqrt{p^2-4q})^2}-\frac{1}{4}\sqrt{p^2-4q},$$

whose solutions are

$$\begin{cases} 2\le p\le 8, & -4-2p<q<\dfrac{p^4+8p^3-160p-400}{4p^2+32p+64}, \\ p>8, & -4-2p<q<-4+2p. \end{cases}$$ □

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