Publ. Math. Debrecen
91/1-2 (2017), 95–110
DOI: 10.5486/PMD.2017.7668

Integral formulae for codimension-one foliated Randers spaces

By VLADIMIR ROVENSKI (Haifa) and PAWEŁ WALCZAK (Lódź)

Abstract. Integral formulae for foliated Riemannian manifolds provide obstructions for existence of foliations or compact leaves of them with given geometric properties. This paper continues our recent study and presents new integral formulae for codimension-one foliated Randers spaces. Our main goal is a generalization of the Reeb formula (that the total mean curvature of the leaves is zero) and its companion with the total second mean curvature. The paper also extends results by Brito, Langevin and Rosenberg (that total mean curvatures of arbitrary order for a codimension-one foliated Riemannian manifold of constant curvature do not depend on a foliation). All of that is done by a comparison of extrinsic and intrinsic curvatures of the two Riemannian structures which arise in a natural way from a given Randers structure.

Introduction

The two recent decades have brought increasing interest in Finsler spaces (M, F), especially, in extrinsic geometry of their hypersurfaces, see [CS2], [S1], [S2]. Randers metrics $F = \alpha + \beta$, where α is the norm of a Riemannian structure, and β a 1-form of α-norm smaller than 1 on M (which was introduced in [Ra] and appeared in a solution of Zermelo's control problem [BRS]), are of particular interest, see [CS1]. Extrinsic geometry of foliated Riemannian manifolds also became popular since some time (see [RW1] and the bibliography therein). Among other topics of interest, one can find the so-called *integral formulae* (i.e., integral relations for invariants of the shape operator of leaves, e.g., the higher order mean curvatures $\sigma_k (1 \le k \le m)$, and Riemann curvature, see surveys in [RW1], [ARW]). Such formulae provide obstructions for the existence of foliations or compact

Mathematics Subject Classification: Primary: 53C12; Secondary: 53C21.
Key words and phrases: Randers norm, foliation, curvature, integral formula, shape operator.

leaves of them with given geometric properties. The first known integral formula (by G. Reeb, [Re]) for codimension-one foliated closed manifolds tells us that the total mean curvature $H=\sigma_1$ is zero;

$$\int_M \sigma_1 \, \mathrm{d}\, \mathrm{V}_g = 0, \tag{1}$$

thus, either $H \equiv 0$ or $H(x)H(y) < 0$ for some points $x, y \in M$. Its counterpart in the case of the second mean curvature is the formula, according to our knowledge, obtained for the first time in [No],

$$\int_M (2\sigma_2 - \mathrm{Ric}_N)\, \mathrm{d}\, \mathrm{V}_g = 0, \tag{2}$$

where N is a unit normal to the leaves. Formula (2) has been used in [LW1] to prove that codimension-one foliations of a closed Riemannian manifold of either negative Ricci curvature or constant nonzero curvature are far, in a sense defined there, from being totally umbilical, and in [BW], to estimate the energy of a vector field. An infinite series of integral formulae was provided in [RW2]: they include (1) and (2) and generalize the Brito–Langevin–Rosenberg formulae [BLR],

$$\int_M \sigma_k \, \mathrm{d}\, \mathrm{V}_g = \begin{cases} K^{k/2}\binom{m/2}{k/2}\, \mathrm{Vol}_g(M), & m, k \text{ even}, \\ 0, & m \text{ or } k \text{ odd}, \end{cases} \tag{3}$$

which tell us that total mean curvatures (of arbitrary order k) for codimension-one foliations on a closed $(m+1)$-dimensional manifold of constant sectional curvature K depend only on K, k, m and the volume of the manifold, not on a foliation. Using the approach of [RW2] and a unit vector field ν orthogonal in the Finsler sense to the leaves, we studied in [RW3] integral formulae for a codimension-one foliated closed Finsler space (M, F), we defined a new Riemannian structure g on M, and derived its Riemann curvature and the shape operator of the leaves in terms of F.

We produced the integral formulae for (M, F) and for Randers space $(M, \alpha+\beta)$ with $\beta^\sharp$ (i.e., the α-dual of β) tangent to the leaves.

This paper presents new integral formulae for a codimension-one foliated Randers space. Section 1 surveys necessary facts and recent results. Section 3 contains our main results, which generalize (1) and (2), and extend (3); in particular, we generalize some results of [RW3]. All integral formulae of this paper hold when the foliation and the 1-form, the both of them, are defined outside a finite union of closed submanifolds of codimension ≥ 2 under convergence of some integrals (as in Lemma 1 in what follows), leaving details to the readers. The singular case is important since there exist plenty of manifolds which admit no (smooth) codimension-one foliations, while all of them admit such foliations and non-singular 1-forms β outside some "set of singularities".

1. Preliminaries

We work with a closed manifold M equipped with a codimension-one foliation defined on $M\setminus\Sigma$, where Σ is a (possibly empty) union of pairwise disjoint closed submanifolds Σ_i of variable codimensions ≥ 2. Briefly, we say that our foliation admits singularities at points of Σ. For Randers spaces (with metrics $F=\alpha+\beta$), we assume also that β admits singularities, i.e., is defined on $M\setminus\Sigma$. Moreover, the compactness of M can be replaced by the weaker conditions that M has finite F-volume, and 'bounded geometry' in the following sense:

$$\sup\nolimits_M \|R_\nu\|_F<\infty,\quad \sup\nolimits_M \|A\|_F<\infty.$$

Lemma 1 (see [LW2, Lemma 2])**.** *Let* Σ_1, $\operatorname{codim}\Sigma_1\geq 2$, *be a closed submanifold of a Riemannian manifold* (M,a), *and* X *a vector field on* $M\setminus\Sigma_1$ *such that* $\int_M \|X\|^2\,\mathrm{d\,V}_a<\infty$. *Then* $\int_M(\operatorname{div} X)\,\mathrm{d\,V}_a=0$.

Given arbitrary quadratic $m\times m$ real matrices $A_1,\dots,A_k$ and the unit matrix I_m, one can consider the determinant $\det(I_m+t_1A_1+\dots+t_kA_k)$ and express it as a polynomial of real variables $\mathbf{t}=(t_1,\dots,t_k)$. Given $\lambda=(\lambda_1,\dots,\lambda_k)$, a sequence of nonnegative integers with $|\lambda|:=\lambda_1+\dots+\lambda_k\leq m$, we shall denote by $\sigma_\lambda(A_1,\dots,A_k)$ its coefficient at $\mathbf{t}^\lambda=t_1^{\lambda_1}\times\cdots t_k^{\lambda_k}$, see [RW2]:

$$\det(I_m+t_1A_1+\dots+t_kA_k)=\sum\nolimits_{|\lambda|\leq m}\sigma_\lambda(A_1,\dots,A_k)\mathbf{t}^\lambda.$$

The invariants $\sigma_\lambda(A_1,\dots,A_k)$ of real matrices A_i generalize the elementary symmetric functions of a single matrix A. We use them in Section 5.

The *Newton transformations* of an $m\times m$ matrix A (see [RW1]) are defined inductively by $T_0(A)=I_m,\ T_r(A)=\sigma_r(A)I_m-AT_{r-1}(A)$ $(r\geq 1)$. Note that $T_r(\lambda A)=\lambda^rT_r(A)$ for $\lambda\neq 0$ and $\operatorname{Tr}(T_r(A))=(m-r)\sigma_r(A)$. Certainly, $\sigma_r(A)$ coincides with the r-th elementary symmetric polynomial of the eigenvalues of A.

Lemma 2 ([RW3])**.** *Let* C,D,A_i $(i\leq s)$ *be* $m\times m$ *matrices,* $\operatorname{rank}A_i=1$. *Then*

$$\begin{aligned}&\sigma_k(C+D+A_1+\dots+A_s)\\&\quad=\sigma_k(C)+\sum\nolimits_{j>0}\sigma_{k-j,j}(C,D)+\operatorname{Tr}(T_{k-1}(C+D)A_1)\\&\qquad+\dots+\operatorname{Tr}(T_{k-1}(C+D+A_1+\dots+A_{s-1})A_s).\end{aligned}\tag{4}$$

In particular, when $D=0$ *and* $s=1$, $\sigma_k(C+A)=\sigma_k(C)+\operatorname{Tr}(T_{k-1}(C)A)$.

2. The Randers norm

A *Minkowski norm* on a vector space V^{m+1} is a function $F : V^{m+1} \to [0,\infty)$ with the properties of regularity, positive 1-homogeneity and strong convexity, see [S2]. The fundamental tensor

$$g_y(u,v) = \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\big[F^2(y+su+tv)\big]_{|s=t=0} \tag{5}$$

obeys $g_{\lambda y} = g_y$ ($\lambda > 0$), and $\{F(y) \le 1\}$ is a strictly convex set. There exist two F-normal directions $n \in V^{m+1}$ to a hyperplane $W \subset V^{m+1}$, i.e., $g_n(n,w) = 0$ ($w \in W$), which are opposite when F is *reversible*, i.e., $F(-y) = F(y)$ ($y \in V^{m+1}$). Let $N \in V^{m+1}$ be a unit normal to a hyperplane W in V^{m+1} with respect to $\langle\cdot\,,\cdot\rangle$,

$$\langle N,w\rangle = 0 \quad (w \in W), \qquad \alpha(N) = \|N\|_\alpha = \sqrt{\langle N,N\rangle} = 1.$$

Let n be a vector F-normal to W, i.e., $g_n(n,v) = 0$ ($v \in W$), lying in the same half-space as N and such that $\|n\|_\alpha = \alpha(n) = 1$. Set

$$g(u,v) := g_n(u,v), \quad u,v \in V^{m+1}.$$

Then $g(n,n) = F^2(n)$, and $F(n) = 1 + \beta(n)$. By (5), for $y = n$ we have

$$\begin{aligned} g(u,v) &= (1+\beta(n))\langle u,v\rangle + \beta(u)\beta(v) \\ &\quad - \beta(n)\langle n,u\rangle\langle n,v\rangle + \beta(u)\langle n,v\rangle + \beta(v)\langle n,u\rangle. \end{aligned} \tag{6}$$

The 'musical isomorphisms' $\sharp$ and $\flat$ will be used for rank 1 and symmetric rank 2 tensors on Riemannian manifolds. For example, if β is a 1-form on $(V^{m+1},\langle\cdot\,,\cdot\rangle)$ and $v \in V^{m+1}$, then $\langle\beta^\sharp,u\rangle = \beta(u)$ and $v^\flat(u) = \langle v,u\rangle$ for any $u \in V^{m+1}$. The tangent component of a vector, say $\beta^\sharp$, will be denoted by $\beta^{\sharp\top}$, its dual 1-form is $\beta^\top$.

Lemma 3. *Put* $c := (1 - \|\beta^{\sharp\top}\|^2_\alpha)^{\frac12} > 0$ *and* $\hat c = c + \beta(N)$. *Then*

$$n = \hat c N - \beta^\sharp, \quad \text{or, equivalently,} \quad n = cN - \beta^{\sharp\top}, \tag{7}$$

$$g(u,v) = c\hat c(\langle u,v\rangle - \beta(u)\beta(v)), \quad u,v \in W, \tag{8}$$

$$g(n,n) = (c\hat c)^2. \tag{9}$$

The vector $\nu = (c\hat c)^{-1}n$ *is an* F*-unit normal to* W.

Proof. It is similar to the proof of [RW3, Lemma 2.2] for $\beta(N) = 0$. □

Lemma 4. *If $u, U \in W$ and $g(u,v) = \langle U, v\rangle$ for all $v \in W$, then*

$$(c\hat{c})u = U + c^{-2}\beta^{\top}(U)\beta^{\sharp\top}. \tag{10}$$

PROOF. It is similar to the proof of [RW3, Lemma 2.3] for $\beta(N) = 0$. $\square$

Let M^{m+1} be a connected smooth manifold and TM its tangent bundle. A *Finsler structure* F on M is a family of Minkowski norms in tangent spaces T_pM which depend smoothly on a point $p \in M$. Given a transversally oriented codimension-one foliation $\mathcal{F}$ of (M^{m+1}, F), there exists a globally defined smooth vector field n being F-normal to the leaves, which defines a Riemannian metric $g := g_n$ with the Levi–Civita connection ∇. Then $g(n,u) = 0$ ($u \in T\mathcal{F}$) and $g(n,n) = F^2(n)$, see (9), and $\nu = n/F(n)$ is an F-unit normal.

3. Codimension-one foliated Randers spaces

This section generalizes results in [RW3], where the case of $\beta(N) = 0$ has been studied. Let $C^{\sharp}_{\nu}$ be a $(1,1)$-tensor g-dual to the symmetric bilinear form $C_{\nu}(\cdot\,,\cdot\,,\nabla_{\nu}\nu)$. Note that $C^{\sharp}_{n} = \hat{c}^3 C^{\sharp}_{\nu}$. As before, write $\langle\cdot\,,\cdot\rangle$ – a Riemannian metric on M^{m+1}. Let $\mathcal{F}$ be a transversally oriented codimension-one foliation of a Randers space (M^{m+1}, F):

$$F(y) = \sqrt{\langle y, y\rangle} + \beta(y), \quad \|\beta\|_{\alpha} < 1, \quad \beta^{\sharp} \in \Gamma(TM).$$

Let N be a unit α-normal vector field to $\mathcal{F}$, and n an F-normal vector field to $\mathcal{F}$ with the property $\langle n, n\rangle = 1$. Let $\bar{\nabla}$ be the Levi–Civita connection of $\langle\cdot\,,\cdot\rangle$ on M.

The canonical volume forms of F, metrics g and $\langle\cdot\,,\cdot\rangle$ satisfy [CS1]

$$\mathrm{d\,V}_F = \big(1 - \|\beta^{\sharp}\|^2_{\alpha}\big)^{\frac{m+2}{2}}\,\mathrm{d\,V}_a, \quad \mathrm{d}V_g = (c\hat{c})^{m+2}\,\mathrm{d}V_a. \tag{11}$$

Recall that $\nu = (c\hat{c})^{-1}n$. Let $Z = \nabla_{\nu}\nu$ and $\bar{Z} = \bar{\nabla}_N N$ be the curvature vectors of ν- and N-curves for g and $\langle\cdot\,,\cdot\rangle$, respectively.

In the case of $\beta^{\sharp\top} \neq 0$, let $X^{\perp\beta}$ be the projection of $X \in \Gamma(T\mathcal{F})$ on $\beta^{\sharp\perp}$:

$$X^{\perp\beta} = X - \langle X,\ \beta^{\sharp\top}\rangle\|\beta^{\sharp\top}\|^{-2}_{\alpha}\beta^{\sharp\top}. \tag{12}$$

Notation (12) will be used in decompositions of matrices $\tilde{B} = B + \sum_i B_i$, where B_i are rank 1 matrices of the form $U^{\perp\beta} \otimes \beta^{\sharp\top}$, $(U^{\perp\beta})^{\flat} \otimes \beta^{\sharp\top}$ and $f \cdot \beta^{\top} \otimes \beta^{\sharp\top}$ for some $U \in T\mathcal{F}$. The invariants of $\tilde{B}$ and B are close in the sense.

The shape operator $A^g : T\mathcal{F} \to T\mathcal{F}$ of $\mathcal{F}$ is given by

$$A^g(u) = -\nabla_u \nu \quad (u \in T\mathcal{F}). \tag{13}$$

Let L be the leaf through a point $p \in M$, and ρ the local distance function to L in a neighborhood of p. Denote by $\hat\nabla$ the Levi–Civita connection of the (local again) Riemannian metric $\hat g := g_{\nabla\rho}$. Note that $\nabla\rho = \nu$ on L. The shape operator $A : T\mathcal{F} \to T\mathcal{F}$ (self-adjoint for g) is defined by

$$A(u) = -\hat\nabla_u \nu \quad (u \in T\mathcal{F}).$$

The derivative $\bar\nabla u : TM \to TM$ and its conjugate $(\bar\nabla u)^t : TM \to TM$ are $(1,1)$-tensors defined by $(\bar\nabla u)(v) = \bar\nabla_v u$ and $\langle(\bar\nabla u)^t(v), w\rangle = \langle v, (\bar\nabla u)(w)\rangle$ for $v, w \in TM$. The *deformation tensor*, $2\overline{\mathrm{Def}}_u = \bar\nabla u + (\bar\nabla u)^t$, measures the degree to which the flow of a vector field u distorts the metric $\langle\cdot\,,\cdot\rangle$. The same notation $\overline{\mathrm{Def}}_u$ will be used for its $\langle\cdot\,,\cdot\rangle$-dual $(1,1)$-tensor. Set $\overline{\mathrm{Def}}{}_u^\top(v) = (\overline{\mathrm{Def}}_u(v))^\top$.

Proposition 1. *The shape operators of $\mathcal{F}$ satisfy the following:*

$$\begin{aligned} cA^g = \bar A &- \frac12 c^{-1}\hat c^{-2}(\hat c N - \beta^\sharp)(c\hat c) I_m + \hat c^{-1}(\overline{\mathrm{Def}}_{\beta^\sharp})^\top_{|T\mathcal{F}} \\ &+ \frac12\big(U - \bar A(\beta^{\sharp\top})\big)\otimes\beta^\top + \frac12 c^{-2}\Big(\bar A(\beta^{\sharp\top}) - \langle\bar A(\beta^{\sharp\top}), \beta^{\sharp\top}\rangle\beta^{\sharp\top} \\ &+ 2\hat c^{-1}(\overline{\mathrm{Def}}_{\beta^\sharp}\beta^{\sharp\top})^\top + U + \beta(U)\beta^{\sharp\top}\Big)^\flat \otimes \beta^{\sharp\top}, \end{aligned} \tag{14}$$

where $U = \hat c^{-1}(\bar\nabla_{\hat c N - \beta^\sharp}\beta^{\sharp\top})^\top - c\bar Z$. At points $p \in M$ with $\beta^{\sharp\top}(p) \neq 0$, we get

$$\begin{aligned} cA^g = \bar A &- \frac12 c^{-1}\hat c^{-2}(\hat c N - \beta^\sharp)(c\hat c) I_m + \hat c^{-1}(\overline{\mathrm{Def}}_{\beta^\sharp})^\top_{|T\mathcal{F}} \\ &+ \frac12 c^{-2}\Big(2\hat c^{-1}(\overline{\mathrm{Def}}_{\beta^\sharp}\beta^{\sharp\top})^\top + (U + \bar A(\beta^{\sharp\top}))^{\perp\beta}\Big)^\flat \otimes \beta^{\sharp\top} \\ &+ \frac12\Big(U - \bar A(\beta^{\sharp\top})\Big)^{\perp\beta} \otimes \beta^\top + \frac{1}{c^2(1-c^2)}\beta(U)\beta^\top \otimes \beta^{\sharp\top}. \end{aligned} \tag{15}$$

Proof. For the convenience of the readers, we give the proof, which is similar to the proof of [RW3, Proposition 4.1], for $\beta(N) = 0$. By the formula for the Levi–Civita connection and the use of the equalities $g(u,n) = 0 = g(v,n)$ and $g([u,v],n) = 0$, we have

$$2g(\nabla_u n, v) = n(g(u,v)) + g([u,n],v) + g([v,n],u) \quad (u, v \in T\mathcal{F}). \tag{16}$$

One may assume $\bar\nabla^\top_X u = \bar\nabla^\top_X v = 0$ for all $X \in T_pM$ at a given point $p \in M$. Using (6)–(8), we obtain

$$\begin{aligned}
n(g(u,v)) &= n\big(c\hat c(\langle u,v\rangle - \beta(u)\beta(v))\big)\\
&= n(c\hat c)(\langle u,v\rangle - \beta(u)\beta(v)) - c\hat c\big(\beta(u)(\bar\nabla_n(\beta^\top))(v) + (\bar\nabla_n(\beta^\top))(u)\beta(v)\big),\\
g([u,n],v) &= c\hat c\big(\langle [u,n],v\rangle + \langle [u,n],n\rangle\beta(v)\big)\\
&= -c\hat c\langle \hat c\bar A(u) + \bar\nabla_u\beta^\sharp, v\rangle + c\hat c^2\langle \bar A(\beta^{\sharp\top}) + c\bar Z, u\rangle\beta(v),\\
g([v,n],u) &= c\hat c\big(\langle [v,n],u\rangle + \beta(u)\langle [v,n],n\rangle\big)\\
&= -c\hat c\langle \hat c\bar A(v) + \bar\nabla_v\beta^\sharp, u\rangle + c\hat c^2\beta(u)\langle \bar A(\beta^{\sharp\top}) + c\bar Z, v\rangle.
\end{aligned}$$

Substituting the above into (16), we find

$$\begin{aligned}
2g(\nabla_u n, v) &= n(c\hat c)(\langle u,v\rangle - \beta(u)\beta(v)) - 2c\hat c^2\langle \bar A(u), v\rangle - 2c\hat c\langle \overline{\rm Def}_{\beta^\sharp}(u), v\rangle\\
&\quad - c\hat c\big(\beta(u)(\bar\nabla_n(\beta^\top))(v) + (\bar\nabla_n(\beta^\top))(u)\beta(v)\big)\\
&\quad + c\hat c^2\big(\beta(v)\langle \bar A(\beta^{\sharp\top}) + c\bar Z, u\rangle + \beta(u)\langle \bar A(\beta^{\sharp\top}) + c\bar Z, v\rangle\big).
\end{aligned}\tag{17}$$

Assume $g(\nabla_u n, v) = \langle \mathfrak{D}(u), v\rangle$, where $\mathfrak{D} : T\mathcal{F} \to T\mathcal{F}$ is a linear operator. Using Lemma 4 and $g(\nabla_u n, v) = -c\hat c g(A^g(u), v)$, see (13), we obtain from (17)

$$-2(c\hat c)^2 A^g(u) = 2\mathfrak{D}(u) + c^{-2}\langle 2\mathfrak{D}(u), \beta^{\sharp\top}\rangle\beta^{\sharp\top},\tag{18}$$

where

$$\begin{aligned}
2\mathfrak{D}(u) &= n(c\hat c)(u - \beta(u)\beta^{\sharp\top}) - 2c\hat c^2\bar A(u) - 2c\hat c(\overline{\rm Def}_{\beta^\sharp}(u))^\top\\
&\quad - c\hat c\big(\beta(u)(\bar\nabla_n\beta^{\sharp\top})^\top + (\bar\nabla_n(\beta^\top))(u)\beta^{\sharp\top}\big)\\
&\quad + c\hat c^2\big(\langle \bar A(\beta^{\sharp\top}) + c\bar Z, u\rangle\beta^{\sharp\top} + \beta(u)(\bar A(\beta^{\sharp\top}) + c\bar Z)\big).
\end{aligned}\tag{19}$$

From (18) and (19), we obtain (14). For $\beta^{\sharp\top} \ne 0$, we apply (12) with $X = \bar A(\beta^{\sharp\top})$ and $X = U$, and find (15). □

In [RW3], we applied the variational approach to express the Riemann curvature of g in terms of the Riemann curvature and the Cartan torsion of F. In this section, we find a relationship between the Riemann curvature of metrics g and $\langle\cdot\,,\cdot\rangle$ on a Randers space. Observe that

$$C_n(u,v,w) = \frac{1}{m+2}\big(I_n(u)h_n(v,w) + I_n(v)h_n(u,w) + I_n(w)h_n(u,v)\big),$$

where $h_n(u,v) = c\hat c(\langle u,v\rangle - \langle u,n\rangle\langle v,n\rangle)$ is the angular form, see [CS1], and

$$I_n(u) = {\rm Tr}\, C_n(\cdot\,,\cdot,u) = \frac{m+2}{2c\hat c}\langle \beta^\sharp - (c\hat c - 1)u,\ n\rangle.$$

Proposition 2. *We have*

$$Z=(c\hat c)^{-1}\bar Z-c^{-1}\hat c^{-2}\bar\nabla^\top\hat c+c^{-3}\hat c^{-1}\beta(\bar Z-\hat c^{-1}\bar\nabla^\top\hat c)\beta^{\sharp\top} \tag{20}$$

and

$$(c\hat c)C^\sharp_n=\bar C+c^{-2}(\beta^\top\circ\bar C)\otimes\beta^{\sharp\top}, \tag{21}$$

where

$$\begin{aligned}2\bar C=\ &\mathrm{Sym}(\beta^\top\otimes(\bar Z-\hat c^{-1}\bar\nabla^\top c))+(c\hat c)^{-1}\big((\hat c-2c^{-1})\beta^{\sharp\top}(\hat c)+(c-\hat c^{-1})n(\hat c)\big)I_m\\ &+(c\hat c)^{-1}\big((2c^{-1}-3\hat c)\beta^{\sharp\top}(\hat c)+(\hat c^{-1}-3c)n(\hat c)\big)\beta^\top\otimes\beta^{\sharp\top}\\ &+(c\hat c)^{-1}\beta(\bar Z)\big((c\hat c-\hat c^2+2c^{-1}\hat c-1)I_m+(3\hat c^2-3c\hat c-2c^{-1}\hat c+1)\beta^\top\otimes\beta^{\sharp\top}\big).\end{aligned}$$

Proof. It is similar to the proof of [RW3, Proposition 4.3] for $\beta(N)=0$. □

Corollary 1. (i) Let $\bar\nabla\beta=0$ and $\beta(N)=\mathrm{const}$, then $\bar Z=0$ provides $C^\sharp_n=0$. (ii) Let $m>3$, $\beta(N)\ge 0$ and $\|\beta\|_\alpha$ be constant, then $C^\sharp_n=0$ if and only if $\bar Z=0$.

Proof. (i) Since c and $\hat c$ are constant, and by Proposition 2, $\bar C=0$, we get $C^\sharp_n=0$. (ii) It is similar to the proof of [RW3, Corollary 4.4] for $\beta(N)=0$. □

Remark 1. For a codimension-one foliated (M,a) we have, see [RW3]:

$$\langle\bar\nabla_u\bar Z,v\rangle=\langle\bar\nabla_v\bar Z,u\rangle,\quad g(\nabla_u Z,v)=g(\nabla_v Z,u)\qquad(u,v\in T\mathcal F), \tag{22}$$

$$\bar R_N=(\overline{\mathrm{Def}}_{\bar Z})^\top_{|T\mathcal F}+\bar\nabla_N\bar A-\bar A^2-\bar Z^\flat\otimes\bar Z. \tag{23}$$

In [CS1], R_y is expressed (using coordinate presentations) through $\bar R_y$ for $y\in TM$. If $\bar\nabla\beta=0$ (i.e., F is a Berwald structure), then $R_y=\bar R_y$. Alternative formulas with relationship between R_ν and $\bar R_\nu$ follow from (23) and similar formula for g, where A^g and Z are expressed using $\bar A$ and $\bar Z$ given in Propositions 1 and 2.

Given a transversely oriented codimension-one foliation $\mathcal F$ of a closed Finsler manifold (M^{m+1},F), denote by $k_1,k_2,\dots,k_m$ ($k_1\le k_2\le\dots\le k_m$) the eigenvalues of the shape operator A of the leaves of $\mathcal F$. If M is oriented and V_F is the Busemann–Hausdorff volume form on M, then one can consider the integral

$$U^F_{\mathcal F}=\int_M\sum\nolimits_{i<j}(k_i-k_j)^2\,\mathrm{d}\,\mathrm{V}_F,$$

which measures "how far from umbilicity" is $\mathcal F$ (see also [RW1, Example 2.6] for the Riemannian case). Similar measure of non-umbilicity (with different powers of k_i-k_j which made it conformally invariant) for foliated Riemannian manifolds has been considered in [LW1, Section 4.1].

Theorem 1. *Let $\bar\nabla\beta = 0$ on (M, a), and let the Randers metric $F = \alpha + \beta$ with $\overline{\mathrm{Ric}}_N \le -r < 0$. Then*

$$U^F_{\mathcal F} \ge (1 - \|\beta^\sharp\|^2)^{\frac{m+2}{2}} mr \int_M c^{-2}\,\mathrm{d}\,\mathrm{V}_a\,. \tag{24}$$

Proof. One may show that

$$\sum\nolimits_{i<j}(k_i - k_j)^2 = (m-1)\sigma_1^2(A) - 2m\sigma_2(A).$$

Hence, and similarly to (2) formula in [RW3]

$$\int_M \Big(\sigma_2(A) - \frac{1}{2}\,\mathrm{Ric}_\nu\Big)\,\mathrm{d}\,\mathrm{V}_F = 0, \tag{25}$$

we obtain

$$U^F_{\mathcal F} \ge -m\int_M 2\sigma_2(A)\,\mathrm{d}\,\mathrm{V}_F = -m\int_M \mathrm{Ric}_\nu\,\mathrm{d}\,\mathrm{V}_F\,. \tag{26}$$

By $\bar\nabla\beta^\sharp = 0$, we have $\|\beta^\sharp\|_\alpha = \mathrm{const}$ and $\bar R(X,Y)\beta^\sharp = 0$ $(X, Y \in TM)$. Using

$$\overline{\mathrm{Ric}}_n = \overline{\mathrm{Ric}}_{\hat c N - \beta^\sharp} = \hat c^2\,\overline{\mathrm{Ric}}_N + \overline{\mathrm{Ric}}_{\beta^\sharp} - 2\hat c\sum\nolimits_i \bar R(N, b_i, \beta^\sharp, b_i),$$

we obtain $\mathrm{Ric}_\nu = (c\hat c)^{-2}\mathrm{Ric}_n = (c\hat c)^{-2}\overline{\mathrm{Ric}}_n = c^{-2}\overline{\mathrm{Ric}}_N$. From (26), where the volume form is $\mathrm{d}\,\mathrm{V}_F = (1 - \|\beta^\sharp\|^2_\alpha)^{\frac{m+2}{2}}\,\mathrm{d}\,\mathrm{V}_a$, see $(11)_1$, we find

$$U^F_{\mathcal F} \ge -(1 - \|\beta^\sharp\|^2_\alpha)^{\frac{m+2}{2}} m\int_M c^{-2}\,\overline{\mathrm{Ric}}_N\,\mathrm{d}\,\mathrm{V}_a,$$

which reduces to (24) since our assumption $\overline{\mathrm{Ric}}_N \le -r < 0$. □

Let Σ be a union of pairwise disjoint closed submanifolds $\Sigma_i \subset M$ of codimensions ≥ 2. Following [BW] for the Riemannian case, define the energy of a unit vector field X on $M \setminus \Sigma$ by the formula

$$\mathcal E(X) = \frac{m+1}{2}\,\mathrm{Vol}_F(M) + \frac{1}{2}\int_M \|DX\|^2_F\,\mathrm{d}\,\mathrm{Vol}_F\,.$$

Let $X = \nu$ be a unit normal to a codimension-one foliation. By the inequality $\|D\nu\|^2_F \ge \frac{2}{m}\sigma_2(A)$, see [BW] for the Riemannian case, Lemma 1 and (25), we get

Theorem 2. *Let $(M, \alpha + \beta)$ be a codimension-one foliated Randers space with $\bar\nabla\beta = 0$. Then*

$$\mathcal E(\nu) \ge (1 - \|\beta^\sharp\|^2)^{\frac{m+2}{2}}\left(\frac{m+1}{2}\,\mathrm{Vol}_a(M) + \frac{1}{2m}\int_M c^{-2}\,\overline{\mathrm{Ric}}_N\,\mathrm{d}\,\mathrm{V}_a\right). \tag{27}$$

Remark 2. Recall that generally (i.e., when $N(\beta)\neq 0$), $c^2=1-\|\beta^\top\|^2\neq$ const and $1-\|\beta\|^2$ are not the same quantities in (27). If $m\geq 2$, then equality holds in (27) if and only if ν is geodesic and $A=\lambda I_m$. For example, if (M,a) is a round sphere and $c=\mathrm{const}$, then

$$\mathcal{E}(\nu)\geq (1-\|\beta^\sharp\|^2)^{\frac{m+2}{2}}\,\frac{(m+1)c^2+1}{2c^2}\,\mathrm{Vol}_a(S^{m+1}).$$

One can also drop the condition $\bar\nabla\beta=0$ in Theorems 1 and 2, and use $\mathrm{Ric}_n=\overline{\mathrm{Ric}}_n+\Theta(n)$ for a certain (explicitly given in [CS1, p. 54]) function Θ on TM_0.

4. Around the Reeb formula

We will discuss (1) and (2) for Randers spaces. The next formula reduces to (1) when $f=\mathrm{const}$, see [RW1, Lemma 2.5]:

$$\int_M \big(f\sigma_1(\bar A)-N(f)\big)\,\mathrm{d}\,\mathrm{V}_a=0. \tag{28}$$

Theorem 3. *For a codimension-one foliated closed Randers space, we have*

$$\int_M (c\hat c)^m\Big(\frac{m+2}{2}c\hat c N(\hat c)+\Big(\frac{m}{2}\hat c+c\Big)\hat c N(c)-\frac{m+2}{2}c\beta^\sharp(c\hat c)$$
$$-(\hat c-c)c^{-1}\hat c\langle \bar A(\beta^{\sharp\top})+c\bar Z,\beta^\sharp\rangle\Big)\mathrm{d}\,\mathrm{Vol}_a=0. \tag{29}$$

Proof. We calculate

$$\mathrm{Tr}(\overline{\mathrm{Def}}_{\beta^\sharp})^\top_{|T\mathcal{F}}=\overline{\mathrm{div}}\beta^\sharp+\beta(\bar Z)-N(\beta(N)), \tag{30}$$
$$\langle\overline{\mathrm{Def}}_{\beta^\sharp}(\beta^{\sharp\top}),\beta^{\sharp\top}\rangle=-c\beta^{\sharp\top}(c)-\beta(N)\langle\bar A(\beta^{\sharp\top}),\beta^\sharp\rangle. \tag{31}$$

Tracing (14), we then obtain

$$\begin{aligned}
c\sigma_1(A^g)&=\sigma_1(\bar A)-\frac{m}{2}c^{-1}\hat c^{-2}(\hat cN-\beta^\sharp)(c\hat c)+\hat c^{-1}\big(\overline{\mathrm{div}}\beta^\sharp+\beta(\bar Z)-N(\beta(N))\big)\\
&\quad+\frac12\Big(\beta(U)-\langle\bar A(\beta^{\sharp\top}),\beta^\sharp\rangle\Big)+\frac12 c^{-2}\Big(c^2\langle\bar A(\beta^{\sharp\top}),\beta^\sharp\rangle\\
&\quad-2\hat c^{-1}(c\beta^{\sharp\top}(c)+\beta(N)\langle\bar A(\beta^{\sharp\top}),\beta^\sharp\rangle)+(2-c^2)\beta(U)\Big)\\
&=\sigma_1(\bar A)-\frac{m}{2}c^{-1}\hat c^{-2}(\hat cN-\beta^\sharp)(c\hat c)+\hat c^{-1}\overline{\mathrm{div}}\beta^\sharp-\hat c^{-1}N(\hat c)\\
&\quad-(\hat c-c)(c\hat c)^{-1}N(c)-(\hat c-c)c^{-2}\hat c^{-1}\langle\bar A(\beta^{\sharp\top})+c\bar Z,\beta^\sharp\rangle.
\end{aligned}\tag{32}$$

By (32), (1) for $\langle\cdot\,,\cdot\rangle$ and g, and using $\mathrm{d}\,\mathrm{V}_g=(c\hat c)^{m+2}\,\mathrm{d}\,\mathrm{V}_a$, see $(11)_2$, the Divergence Theorem and $f\,\overline{\mathrm{div}}\beta^\sharp=\overline{\mathrm{div}}(f\beta^\sharp)-\beta^\sharp(f)$ with $f=(c\hat c)^{m+1}$, we get

$$\int_M (c\hat c)^{m+2}c^{-1}\Big(\sigma_1(\bar A)-\frac{m}{2}(c\hat c)^{-1}N(c\hat c)-\hat c^{-1}N(\hat c)-\frac{m+2}{2}c^{-1}\hat c^{-2}\beta^\sharp(c\hat c)$$
$$-(\hat c-c)(c\hat c)^{-1}N(c)-(\hat c-c)c^{-2}\hat c^{-1}\langle\bar A(\beta^{\sharp\top})+c\bar Z,\beta^\sharp\rangle\Big)\,\mathrm{d}\,\mathrm{Vol}_a=0, \tag{33}$$

which is the Reeb formula when $\beta=0$. Applying (28), we obtain (29). □

Remark 3. If c and $\beta(N)\neq 0$ are constant, then (29) reduces to

$$\int_M\langle\bar A(\beta^{\sharp\top})+c\bar Z,\ \beta^\sharp\rangle\,\mathrm{d}\,\mathrm{V}_a=0. \tag{34}$$

For $\beta=0$, we have $c=1=\hat c$; hence, (33) reduces to the Reeb formula. The following application of (34) seems to be interesting. Let $\beta(\bar Z)=0$, and a unit vector field $X\in\Gamma(T\mathcal F)$ be an eigenvector of $\bar A$ corresponding to an eigenvalue $\lambda: M\to\mathbb R$. By Theorem 3, $\beta^\sharp=\varepsilon' X+\varepsilon N$, where $\varepsilon=\mathrm{const}\in(-1,1)$ and $\varepsilon'=\mathrm{const}\in(0,\sqrt{1-\varepsilon^2})$, obeys (34). Note that $c^2=1-\varepsilon^2-(\varepsilon')^2$, $\beta(N)=\varepsilon$ and $c\hat c=1+\varepsilon$. Thus, assuming $\varepsilon\neq 0$, we get $\int_M\lambda\,\mathrm{d}\,\mathrm{V}_a=0$. Consequently, either $\lambda\equiv 0$ on M or $\lambda(x)\cdot\lambda(y)<0$ for some points x and y of M. This implies the Reeb formula $\int_M\sigma_1(\bar A)\,\mathrm{d}\,\mathrm{V}_a=\sum_i\int_M\lambda_i\,\mathrm{d}\,\mathrm{V}_a=0$ when $\bar Z=0$.

The next theorem generalizes (2), using the approach of foliated Randers spaces: given a Riemannian manifold (M,a) with a vector field $\beta^\sharp$ of small norm, we associate a Randers space $(M,\alpha+\beta)$. Recall that $F=\alpha+\beta$ is Berwald if and only if $\bar\nabla\beta^\sharp=0$. In this case, the Finsler metric and the source metric $\langle\cdot\,,\cdot\rangle$ have equal Riemann curvatures: $R_y=\bar R_y$ for $y\in TM_0$, see Remark 1.

Theorem 4. *Let (M,a) admit a non-trivial parallel vector field $\beta^\sharp$ (say, $\|\beta^\sharp\|_\alpha<1$), which is nowhere orthogonal to a codimension-one foliation $\mathcal F$. Then*

$$\int_M c^{-2}\left(\sigma_2(\bar A+cC^\sharp_\nu)+\left(\frac{c-2\hat c}{c\hat c}\langle\bar A(\beta^{\sharp\top}),\beta^\sharp\rangle-\frac{\hat c-c}{c^2\hat c}\beta(\bar Z)\right)\sigma_1(\bar A+cC^\sharp_\nu)\right.$$
$$+\frac{\hat c-c}{c\hat c(1-c^2)}\langle(\bar A+cC^\sharp_\nu)(\beta^{\sharp\top}),\beta^{\sharp\top}\rangle\langle\bar A(\beta^{\sharp\top}),\beta^{\sharp\top}\rangle-\frac{c(c-2\hat c)^2(1-c^2)}{4c\hat c^2}\|\bar Z^{\perp\beta}\|^2_\alpha$$
$$+\frac{c-2\hat c}{c\hat c(1-c^2)}\beta(\bar Z)\langle(\bar A+cC^\sharp_\nu)(\beta^{\sharp\top}),\beta^{\sharp\top}\rangle-\frac{1-(c-2\hat c)^2}{4\hat c^2}\|\bar A(\beta^{\sharp\top})^{\perp\beta}\|^2_\alpha$$
$$-\frac{(c-2\hat c)(1-c^2+2c\hat c)}{2\hat c^2}\langle\bar A(\beta^{\sharp\top})^{\perp\beta},\bar Z\rangle-\frac{1+c^2-2c\hat c}{2\hat c}\langle\bar A(\beta^{\sharp\top})^{\perp\beta},C^\sharp_\nu(\beta^{\sharp\top})\rangle$$
$$\left.-\frac{(c-2\hat c)(1+c^2)}{2\hat c}\langle C^\sharp_\nu(\beta^{\sharp\top})^{\perp\beta},\bar Z^{\perp\beta}\rangle-\frac12\overline{\mathrm{Ric}}_N\right)\mathrm{d}\,\mathrm{V}_a=0. \tag{35}$$

Furthermore, if $\beta(N)=\mathrm{const}$, N *being a unit normal to* $\mathcal{F}$, *then* (35) *reads*

$$\begin{aligned}\int_M \Big(& c\,\mathrm{Tr}(C^\sharp_\nu)\sigma_1(\bar A) - c\,\mathrm{Tr}(\bar A C^\sharp_\nu) - \frac{1-(c-2\hat c)^2}{4\hat c^2}\|\bar A(\beta^{\sharp\top})\|^2_\alpha \\ &- \frac{(c-2\hat c)(1-c^2+2c\hat c)}{2\hat c^2}\langle \bar A(\beta^{\sharp\top}), \bar Z\rangle - \frac{1+c^2-2c\hat c}{2\hat c}\langle \bar A(\beta^{\sharp\top}), C^\sharp_\nu(\beta^{\sharp\top})\rangle \\ &- \frac{c(c-2\hat c)^2(1-c^2)}{4c\hat c^2}\|\bar Z^{\perp\beta}\|^2_\alpha - \frac{(c-2\hat c)(1+c^2)}{2\hat c}\langle C^\sharp_\nu(\beta^{\sharp\top}), \bar Z\rangle\Big)\,\mathrm{d\,V}_a = 0. \quad (36)\end{aligned}$$

PROOF. Note that $c<1$ when $\beta^{\sharp\top}\neq 0$ on a Randers space $(M,\alpha+\beta)$. For $\bar\nabla\beta^\sharp=0$, we get $(\bar\nabla_n\beta^{\sharp\top})^\top = -\beta(N)(\bar A(\beta^{\sharp\top})+c\bar Z)$. We have, see [RW1],

$$A - A^g = C^\sharp_\nu, \quad (37)$$

and $cA^g=\bar A + A_1 + A_2 + A_3$, see (15), where $A_1 = U_1^\flat\otimes\beta^{\sharp\top}$, $A_2 = U_2\otimes\beta^\top$, $A_3 = a_3\beta^\top\otimes\beta^{\sharp\top}$ are rank 1 matrices, and

$$\begin{aligned} a_3 &= \frac{c-2\hat c}{c\hat c(1-c^2)}\beta(\bar Z) - \frac{\hat c - c}{c^2\hat c(1-c^2)}\langle \bar A(\beta^{\sharp\top}),\beta^\sharp\rangle, \\ U_1 &= \frac{1}{2c\hat c}\big(\bar A(\beta^{\sharp\top}) + (c-2\hat c)\bar Z\big)^{\perp\beta}, \qquad U_2 = \frac{c-2\hat c}{2\hat c}\big(\bar A(\beta^{\sharp\top}) + c\bar Z\big)^{\perp\beta}. \quad (38)\end{aligned}$$

Thus, $cA = \bar A + cC^\sharp_\nu + A_1 + A_2 + A_3$. We have $\sigma_1(A_1)=\sigma_1(A_2)=\sigma_2(A_i)=0$. Recall the following identity for square matrices:

$$\sigma_2\Big(\sum\nolimits_i A_i\Big) = \sum\nolimits_i \sigma_2(A_i) + \sum\nolimits_{i<j}\big((\mathrm{Tr}\,A_i)(\mathrm{Tr}\,A_j) - \mathrm{Tr}(A_iA_j)\big).$$

By the above, we obtain

$$c^2\sigma_2(A) = \sigma_2(\bar A + cC^\sharp_\nu) + \sigma_1(A_3)\sigma_1(\bar A + cC^\sharp_\nu) - Q,$$

where $\sigma_1(A_3) = a_3(1-c^2)$ and

$$\begin{aligned} Q :&= \mathrm{Tr}(A_1A_2 + A_1(\bar A + cC^\sharp_\nu) + A_2(\bar A + cC^\sharp_\nu) + A_3(\bar A + cC^\sharp_\nu)) \\ &= -\frac{\hat c - c}{c\hat c(1-c^2)}\langle(\bar A + cC^\sharp_\nu)(\beta^{\sharp\top}),\beta^{\sharp\top}\rangle\langle\bar A(\beta^{\sharp\top}),\beta^{\sharp\top}\rangle \\ &\quad - \frac{c-2\hat c}{c\hat c(1-c^2)}\beta(\bar Z)\langle(\bar A + cC^\sharp_\nu)(\beta^{\sharp\top}),\beta^{\sharp\top}\rangle + \frac{c(c-2\hat c)^2(1-c^2)}{4c\hat c^2}\|\bar Z^{\perp\beta}\|^2_\alpha \\ &\quad + \frac{1-(c-2\hat c)^2}{4\hat c^2}\|\bar A(\beta^{\sharp\top})^{\perp\beta}\|^2_\alpha + \frac{(c-2\hat c)(1-c^2+2c\hat c)}{2\hat c^2}\langle\bar A(\beta^{\sharp\top})^{\perp\beta},\bar Z\rangle\end{aligned}$$

$$+\frac{1+c^2-2c\hat{c}}{2\hat{c}}\langle\bar{A}(\beta^{\sharp\top})^{\perp\beta}, C^\sharp_\nu(\beta^{\sharp\top})\rangle+\frac{(c-2\hat{c})(1+c^2)}{2\hat{c}}\langle\bar{Z}^{\perp\beta}, C^\sharp_\nu(\beta^{\sharp\top})\rangle.$$

The condition $\bar\nabla\beta^\sharp=0$ implies $\|\beta^\sharp\|_\alpha=\mathrm{const}$ and $\bar{R}(X,Y)\beta^\sharp=0 (X,Y\in TM)$. Using equality

$$\overline{\mathrm{Ric}}_n=\overline{\mathrm{Ric}}_{\hat{c}N-\beta^\sharp}=\hat{c}^2\overline{\mathrm{Ric}}_N+\overline{\mathrm{Ric}}_{\beta^\sharp}-2\hat{c}\sum\nolimits_i\bar{R}(N,b_i,\beta^\sharp,b_i),$$

we obtain $\mathrm{Ric}_\nu=(c\hat{c})^{-2}\mathrm{Ric}_n=(c\hat{c})^{-2}\overline{\mathrm{Ric}}_n=c^{-2}\overline{\mathrm{Ric}}_N$. From (2) for F, where the volume form is $\mathrm{d\,V}_F=(1-\|\beta^\sharp\|^2_\alpha)^{\frac{m+2}{2}}\,\mathrm{d\,V}_a$, see $(11)_1$, we get (35). Since $\lim\limits_{\beta\to0}\bar{A}(\beta^{\sharp\top})^{\perp\beta}=0$, (35) reduces to (2) when $\beta\to0$. If $\beta(N)=\mathrm{const}$, then $\beta(\bar{Z})=0$ and $\langle\bar{A}(\beta^{\sharp\top}),\beta^{\sharp\top}\rangle=0$:

$$\begin{aligned}0&=\langle\bar\nabla_N\beta^\sharp,N\rangle=\langle\bar\nabla_N(\beta^{\sharp\top}+\beta(N)N),N\rangle=-\langle\beta^\sharp,\bar{Z}\rangle,\\0&=\langle\bar\nabla_{\beta^{\sharp\top}}\beta^\sharp,N\rangle=\langle\bar\nabla_{\beta^{\sharp\top}}(\beta^{\sharp\top}+\beta(N)N),N\rangle=-\langle\bar{A}(\beta^{\sharp\top}),\beta^\sharp\rangle.\end{aligned}\qquad(39)$$

Hence, by (4) for $\sigma_2(\bar{A}+cC^\sharp_\nu)$ and by (2), we reduce (35) to (36). □

Remark that a parallel vector field $\beta^\sharp$ forms a constant angle with (the leaves of) $\mathcal{F}$ if and only if $\beta(N)=\mathrm{const}$ (e.g., $\beta(N)=0$) and $\|\beta^{\sharp\top}\|_\alpha=\mathrm{const}$.

Corollary 2. *Assume that a Riemannian manifold (M,a) admits a non-trivial parallel vector field $\beta^\sharp$, which forms a constant angle with the leaves of a Riemannian ($\bar{Z}=0$) foliation $\mathcal{F}$, and $2\beta(N)+c\neq1$. Then $\bar{A}(\beta^{\sharp\top})=0$ on M. If, in addition, $\mathcal{F}$ is totally umbilical ($\bar{A}=\bar{H}\cdot I_m$), then $\mathcal{F}$ is totally geodesic.*

Proof. Let $\|\beta^\sharp\|_\alpha<1$. By conditions and Corollary 1 (i), $\bar{Z}=0$ yields $C^\sharp_\nu=0$ on a Randers space $(M,\alpha+\beta)$. Since c and $\hat{c}$ are constant, (36) reads

$$\frac{1-(2\hat{c}-c)^2}{4\hat{c}^2}\int_M\|\bar{A}(\beta^{\sharp\top})\|^2_\alpha\,\mathrm{d\,V}_a=0.$$

By conditions, $1-(2\hat{c}-c)^2$ is nonzero. This yields $\bar{A}(\beta^{\sharp\top})=0$ on M. If $\mathcal{F}$ is a totally umbilical foliation, then $0=\langle\bar{A}(\beta^{\sharp\top}),\beta^{\sharp\top}\rangle=\bar{H}\|\beta^{\sharp\top}\|^2_\alpha$, hence $\bar{H}=0$. □

Remark 4. Using the formula in [CS1, Lemma 4.2.2] for Ric–$\overline{\mathrm{Ric}}$ (see also Remark 2), one may generalize (35) (completing it with more terms) for foliated Randers spaces without additional condition $\bar\nabla\beta^\sharp=0$.

5. Around Brito–Langevin–Rosenberg formula

Results of this section are valid for a codimension-one foliation and 1-form with singularities (according to [RW2, Theorem 2 and Corollary 4] and Lemma 1).

Observe that if a rank 1 matrix $A := U\otimes\beta$ has trace zero, i.e., $\beta(U)=0$, then

$$A^2 = U(\beta^\sharp)^t\cdot U(\beta^\sharp)^t = U\beta(U)(\beta^\sharp)^t = \beta(U)A = 0.$$

Define the quantity

$$\delta := -\frac{1}{2}c^{-1}\hat{c}^{-2}(\hat{c}N-\beta^\sharp)(c\hat{c}).$$

Let our Randers space be Berwald, and $\beta^\sharp$ be nowhere orthogonal to $\mathcal{F}$:

$$\bar\nabla\beta^\sharp = 0,\qquad \beta^{\sharp\top}\neq 0, \tag{40}$$

for $\beta\perp\mathcal{F}$, see Remark 5. If, in addition, $\langle\cdot\,,\cdot\rangle$ has constant curvature $\bar K$, then $\bar K=0$, because only flat space forms admit parallel vector fields.

Theorem 5. *A codimension-one foliated closed Randers–Berwald space, with* (40) *and constant sectional curvature $\bar K$ of $\langle\cdot\,,\cdot\rangle$, obeys for $1\le k\le m$,*

$$\begin{aligned}\int_M\Big(&\delta(m-k+1)\sigma_{k-1}(\bar A)+\sum\nolimits_{j>0}\sigma_{k-j,j}(\bar A+\delta I_m, cC^\sharp_\nu)\\ &+\langle T_{k-1}(\bar A+\delta I_m{+}cC^\sharp_\nu)(\beta^{\sharp\top}),U_1\rangle+\beta\big(T_{k-1}(\bar A+\delta I_m{+}cC^\sharp_\nu+U_1^\flat\otimes\beta^{\sharp\top})(U_2)\big)\\ &+a_3\beta\big(T_{k-1}(\bar A+\delta I_m+cC^\sharp_\nu+U_1^\flat\otimes\beta^{\sharp\top}+U_2\otimes\beta^\top)(\beta^{\sharp\top})\big)\Big)\,\mathrm{d\,V}_a=0,\end{aligned}\tag{41}$$

where U_1,U_2 and a_3 are given in (38). *Moreover, if $\beta(N)=\mathrm{const}$ and $\bar Z=0$, then*

$$\int_M\frac{1+c^2-2c\hat c}{2c\hat c}\big\langle T_{k-1}(\bar A)(\beta^{\sharp\top}),\bar A(\beta^{\sharp\top})^{\perp\beta}\big\rangle\,\mathrm{d\,V}_a=0.$$

Proof. As was shown, $\bar K=0$, and $R_y=\bar R_y=0$ for $y\in TM_0$. By assumptions, $c<1$ and $\|\beta\|_\alpha=\mathrm{const}$. By (15) and (37),

$$cA = cA^g + cC^\sharp_\nu = \bar A+\delta I_m + cC^\sharp_\nu + A_1+A_2+A_3,$$

where A_i are three rank ≤ 1 matrices, $A_1 = U_1^\flat\otimes\beta^{\sharp\top}$, $A_2 = U_2\otimes\beta^\top$ and $A_3 = a_3\beta^\top\otimes\beta^{\sharp\top}$. By (4) with $C=\bar A+\delta I_m$ and $D=cC^\sharp_\nu$, we have

$$\begin{aligned}c^k\sigma_k(A) = {}&\sigma_k(\bar A+\delta I_m)+\sum\nolimits_{j>0}\sigma_{k-j,j}(\bar A+\delta I_m, cC^\sharp_\nu)\\ &+U_1^\flat(T_{k-1}(\bar A{+}\delta I_m{+}cC^\sharp_\nu)(\beta^{\sharp\top}))+\beta(T_{k-1}(\bar A+\delta I_m{+}cC^\sharp_\nu{+}A_1)(U_2))\\ &+a_3\beta(T_{k-1}(\bar A+\delta I_m+cC^\sharp_\nu+A_1+A_2)(\beta^{\sharp\top})).\end{aligned}\tag{42}$$

Recall that $\mathrm{d\,V}_F = (1-\|\beta^\sharp\|^2_\alpha)^{\frac{m+2}{2}}\,\mathrm{d\,V}_a$, see $(11)_1$. Comparing the analogue of (3) for F, see [RW3], when $K=0$ with $\int_M \sigma_k(\bar A_p)\,\mathrm{d\,V}_a = 0$, and using $\sigma_k(\bar A + \delta I_m) = \sigma_k(\bar A) + \delta(m-k+1)\sigma_{k-1}(\bar A)$, see (4) and (42), we find (41). If $\beta(N) =$ const and $\bar Z = 0$, then c and $\hat c$ are constant; hence, $\delta = 0$. Then $\langle \bar A(\beta^{\sharp\top}), \beta^{\sharp\top}\rangle = 0$, see (39), and $a_3 = 0$. Thus, the second claim follows from Corollary 1 (i) and (41). □

Example 1. For $k = 1$ and $\bar Z = 0$, (41) yields the Reeb type formula $\int_M \frac{\hat c - c}{c^2\hat c}\langle \bar A(\beta^{\sharp\top}), \beta^\sharp\rangle\,\mathrm{d\,V}_a = 0$, see also (34); thus, if $\hat c \ne c$, then $\bar A(\beta^{\sharp\top}) = 0$ on M.

Corollary 3. *Assume that* $(M^{m+1}, \alpha+\beta)$ *is a closed Randers–Berwald space of constant sectional curvature* $\bar K = 0$ *of* $\langle\cdot\,,\cdot\rangle$, *endowed with a codimension-one totally geodesic* (*for our metric* a) *foliation, and that conditions* (40) *hold. Then*

$$\int_M \Big(c^k\sigma_k(C^\sharp_\nu) + \frac{c-2\hat c}{2c\hat c}\langle T_{k-1}(C^\sharp_\nu + \delta I_m)(\beta^{\sharp\top}), \bar Z^{\perp\beta}\rangle$$
$$+ \frac{c(c-2\hat c)}{2\hat c}\beta\left(T_{k-1}\Big(cC^\sharp_\nu + \delta I_m + \frac{c-2\hat c}{2c\hat c}(\bar Z^{\perp\beta})^\flat \otimes \beta^{\sharp\top}\Big)(\bar Z^{\perp\beta})\right)$$
$$+ \frac{c-2\hat c}{c\hat c(1-c^2)}\beta(\bar Z)\beta\Big(T_{k-1}\Big(cC^\sharp_\nu + \delta I_m + \frac{c-2\hat c}{2c\hat c}(\bar Z^{\perp\beta})^\flat \otimes \beta^{\sharp\top}$$
$$+\frac{c(c-2\hat c)}{2\hat c}\bar Z^{\perp\beta}\otimes\beta^{\top}\Big)(\beta^{\sharp\top})\Big)\Big)\,\mathrm{d\,V}_a = 0, \quad 1\le k\le m. \tag{43}$$

Remark 5. (i) For $\beta(N) = 0$, (43) reduces to formula (4.23) in [RW3]. Similar integral formulae exist for totally umbilical foliations (for $\beta(N)\equiv 0$, see [RW3]). Non-flat closed Riemannian manifolds of constant curvature do not admit such foliations. (ii) Let $\beta^\sharp = fN$, for a smooth function $f: M\to(-1,1)$. Then $c=1$ and $\beta(N) = f$. Theorems 3 and 5 yield trivial identities in this case.

References

[ARW] K. Andrzejewski, V. Rovenski and P. Walczak, Integral formulas in foliation theory, In: Geometry and Its Applications, Springer Proceedings in Mathematics and Statistics, Vol. **72**, *Springer*, *Cham*, 2014, 73–82.

[BRS] D. Bao, C. Robles and Z. Shen, Zermelo navigation on Riemannian manifolds, *J. Differential Geom.* **66** (2004), 377–435.

[BLR] F. Brito, R. Langevin and H. Rosenberg, Intégrales de courbure sur des variétés feuilletées, *J. Differential Geom.* **16** (1981), 19–55.

[BW] F. Brito and P. Walczak, On the energy of unit vector fields with isolated singularities, *Ann. Polon. Math.* **73** (2000), 269–274.

[CS1] X. Cheng and Z. Shen, Finsler Geometry. An Approach via Randers Spaces, *Springer, Heidelberg*, 2012.

[CS2] S. S. Chern and Z. Shen, Riemann–Finsler Geometry, *World Scientific Publishing Co., Hackensack, NJ*, 2005.

[LW1] R. Langevin and P. Walczak, Conformal geometry of foliations, *Geom. Dedicata* **132** (2008), 135–178.

[LW2] M. Lużyńczyk and P. Walczak, New integral formulae for two complementary orthogonal distributions on Riemannian manifolds, *Ann. Global Anal. Geom.* **48** (2015), 195–209.

[No] T. Nora, Seconde forme fondamentale d'une application et d'un feuilletage, *Thése, Université de Limoges* (1983), 115 pp.

[Ra] G. Randers, On an asymmetrical metric in the fourspace of general relativity, *Phys. Rev. (2)* **59** (1941), 195–199.

[Re] G. Reeb, Sur la courbure moyenne des variétés intégrales d'une équation de Pfaff $\omega = 0$, *C. R. Acad. Sci. Paris* **231** (1950), 101–102.

[RW1] V. Rovenski and P. Walczak, Topics in Extrinsic Geometry of Codimension-One Foliations, *Springer, New York*, 2011.

[RW2] V. Rovenski and P. Walczak, Integral formulae on foliated symmetric spaces, *Math. Ann.* **352** (2012), 223–237.

[RW3] V. Rovenski and P. Walczak, Integral formulae for codimension-one foliated Finsler manifolds, *Balkan J. Geom. Appl.* **21** (2016), 76–102.

[S1] Z. Shen, On Finsler geometry of submanifolds, *Math. Ann.* **311** (1998), 549–576.

[S2] Z. Shen, Lectures on Finsler Geometry, *World Scientific Publishing Co., Singapore*, 2001.

VLADIMIR ROVENSKI
DEPARTMENT OF MATHEMATICS
FACULTY OF NATURAL SCIENCES
UNIVERSITY OF HAIFA
MOUNT CARMEL
31905 HAIFA
ISRAEL

E-mail: vrovenski@univ.haifa.ac.il
URL: http://math.haifa.ac.il/ROVENSKI/rovenski.html

PAWEŁ WALCZAK
DEPARTMENT OF GEOMETRY
FACULTY OF MATHEMATICS
AND COMPUTER SCIENCE
UNIVERSITY OF ŁÓDZ
UL. 22 BANACH STR., 90–238
ŁÓDŹ
POLAND

E-mail: pawelwal@math.uni.lodz.pl
URL: http://math.uni.lodz.pl/~pawelwal/

(Received April 6, 2016; revised September 11, 2016)