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On a generalization of a functional equation associated with the distance between the probability distributions

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Abstract. In this paper, the functional equation

$$f(pr, qs) + f(ps, qr) = g(p, q)\, f(r, s) + g(r, s)\, f(p, q) \qquad (p, q, r, s \in\,]0,\, 1]\,)$$

where f and g are complex-valued functions defined on the open-closed unit interval $]0,\, 1]$, is solved without any regularity assumptions. This functional equation is a generalization of a functional equation which was instrumental in the characterization of the symmetric divergence of degree α in [*J. Math. Anal. Appl.*, **139** (1989), 280-292].

1. Introduction

Let $\Gamma_n^0 = \{P = (p_1, p_2, ..., p_n) \mid 0 < p_k < 1,\ \sum_{k=1}^n p_k = 1\}$ denote the set of all n-ary discrete probability distributions, that is, Γ_n^0 is the class of discrete distributions on a finite set Ω of cardinality n. For P and Q in Γ_n^0, Kullback and Leibler [9] (see also [8]) defined directed divergence as

$$D_n(P||Q) = \sum_{k=1}^{n} p_k \, \log \frac{p_k}{q_k}. \tag{1.1}$$

This measure is nonnegative and attains minimum when $P = Q$. Thus, it serves as a distance measure between the distributions P and Q. It is frequently used in statistics, pattern recognition, coding theory, signal processing and information theory. However, this directed divergence is neither symmetric nor does it satisfy the triangle inequality and thus its

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application as a metric is limited. So, in [4] the notion of symmetric divergence between any two probability distributions P and Q in Γ_n^0, was introduced as

$$J_n(P, Q) = D_n(P||Q) + D_n(Q||P) \tag{1.2}$$

to restore the symmetry. In explicit form J_n is given by

$$J_n(P, Q) = \sum_{k=1}^{n}(p_k - q_k) \log \frac{p_k}{q_k}. \tag{1.3}$$

The measure (1.3) is called the J-divergence in honor of JEFFREYS who first used this measure in connection with some estimation problems in [4]. A well known generalization of the J-divergence (see [3]) is the symmetric divergence of degree α and is given by

$$J_{n,\alpha}(P, Q) = \frac{\sum_{k=1}^{n} \left(p_k^{\alpha}\, q_k^{1-\alpha} + q_k^{\alpha}\, p_k^{1-\alpha}\right) - 2}{2^{1-\alpha} - 1}, \tag{1.4}$$

where $\alpha \neq 1$. The J-divergence of degree α is a one parameter generalization of (1.3) since (1.4) tends to (1.3) as $\alpha \to 1$. This measure satisfies the composition law

$$\begin{aligned} &J_{nm,\alpha}(P \star R,\, Q \star S) + J_{nm,\alpha}(P \star S,\, Q \star R) \\ &\qquad = 2\, J_{n,\alpha}(P, Q) + 2\, J_{m,\alpha}(R, S) + \lambda\, J_{n,\alpha}(P, Q)\, J_{m,\alpha}(R, S) \end{aligned} \tag{1.5}$$

for all $P,\, Q \in \Gamma_n^o$ and $R,\, S \in \Gamma_m^o$ where

$$P \star R = (p_1 r_1, \cdots, p_1 r_m, p_2 r_1, \cdots, p_2 r_m, \cdots, p_n r_1, \cdots, p_n r_m)$$

and $\lambda = 2^{\alpha-1} - 1$. If $\alpha \to 1$, then (1.5) tends to

$$J_{nm}(P \star R,\, Q \star S) + J_{nm}(P \star S,\, Q \star R) = 2\, J_n(P, Q) + 2\, J_m(R, S). \tag{1.6}$$

The measures (1.3) and (1.4) were characterized in [3] through the sum property and the composition laws (1.6) and (1.5). The functional equations

$$f(pr, qs) + f(ps, qr) = (r + s)\, f(p, q) + (p + q)\, f(r, s) \qquad (\, p, q, r, s \in\,]0,\, 1[\,) \tag{1.7}$$

$$f(pr, qs) + f(ps, qr) = f(p, q)\, f(r, s) \qquad (\, p, q, r, s \in\,]0,\, 1[\,) \tag{1.8}$$

were instrumental in the characterization of (1.3) and (1.4), respectively. In this paper, we solve the functional equation

$$f(pr, qs) + f(ps, qr) = g(p, q)\, f(r, s) + g(r, s)\, f(p, q) \qquad (p, q, r, s \in\,]0,\, 1]\,), \tag{FE}$$

where f and g are complex-valued functions on the open-closed unit interval $]0,1]$. The equation (FE) is a generalization of (1.7) and (1.8). For some other functional equations and inequalities related to characterization of distance measures between probabilities distributions see [3], [5], [6] and [7].

2. Notation and terminology

Let I denote the open-closed unit interval $]0,\,1]$. Let $\mathbb{R}$ and $\mathbb{C}$ denote the set of real numbers and the set of complex numbers, respectively. A map $L : I \to \mathbb{C}$ is called *logarithmic* if and only if $L(xy) = L(x) + L(y)$ for all $x, y \in I$. A function $\ell : I^2 \to \mathbb{C}$ is called *bilogarithmic* if and only if it is logarithmic in each variable. A function M on I is called *multiplicative* if and only if $M(xy) = M(x)\,M(y)$ for all $x,\, y \in I$. For regular solutions of multiplicative or logarithmic Cauchy functional equation the interested reader should refer to [1]. The capital letters M and L along with their subscripts are used exclusively for multiplicative and logarithmic maps, respectively. For a map $f : I \to \mathbb{C}$, the notation $f \neq 0$ means that f is not identically zero on I; "f is nonzero" means $f \neq 0$.

3. Some preliminary results

The following lemmas are needed to establish the main results of this paper.

Lemma 1. *The function $f : I^2 \to \mathbb{C}$ satisfies the functional equation*

$$f(pr, qs) + f(ps, qr) = 2\,f(p,q) + 2\,f(r,s) \tag{3.1}$$

if and only if

$$f(p,q) = L(p) + L(q) + \ell\left(\frac{p}{q},\ \frac{p}{q}\right), \tag{3.2}$$

where $L : I \to \mathbb{C}$ is an arbitrary logarithmic map and $\ell : I^2 \to \mathbb{C}$ is a bilogarithmic function.

Proof. It is easy to check that (3.2) satisfies (3.1). Now we prove the converse. Letting $q = s = 1$ in (3.1), we see that

$$f(p,r) = 2g(p) + 2g(r) - g(pr), \tag{3.3}$$

where

$$g(p) := f(p,1). \tag{3.4}$$

Letting $s = 1$ in (3.1) and then using (3.3) in the resulting equation, we obtain

$$g(pqr) + g(p) + g(q) + g(r) = g(pr) + g(qr) + g(pq)\,. \tag{3.5}$$

For fixed r, defining

$$\phi(p) := g(pr) - g(p) - g(r) \tag{3.6}$$

we see that (3.5) reduces to

$$\phi(pq) = \phi(p) + \phi(q). \tag{3.7}$$

Hence by (3.7) and (3.6), we get

$$\phi(p) = 2\,l(p,r), \tag{3.8}$$

where $\ell : I^2 \to \mathbb{C}$ is a logarithmic function in the first variable. Using (3.8) in (3.6), we obtain

$$g(pr) - g(p) - g(r) = 2\,\ell(p,r). \tag{3.9}$$

Since the left side of (3.9) is symmetric with respect to r and s, so also the right side. Thus $\ell(p,r) = \ell(r,p)$ and ℓ is a complex-valued bilogarithmic function on I. Again, defining

$$G(p) := g(p) - \ell(p,p) \tag{3.10}$$

and using it in (3.9), we obtain

$$G(pr) = G(p) + G(r). \tag{3.11}$$

Thus

$$G(p) = L(p), \tag{3.12}$$

where $L : I \to \mathbb{C}$ is an arbitrary logarithmic function. Now using (3.12) in (3.10), we get

$$g(p) = L(p) + \ell(p,p) \tag{3.13}$$

and (3.13) in (3.3) yields the asserted form (3.2) of f. This completes the proof of the lemma.

The following lemma easily follows from Lemma 1.

Lemma 2. *Let $M : I \to \mathbb{C}$ be a given nonzero multiplicative function. The function $f : I^2 \to \mathbb{C}$ satisfies the functional equation*

$$f(pr, qs) + f(ps, qr) = 2M(rs)\, f(p, q) + 2M(pq)\, f(r, s) \tag{3.14}$$

if and only if

$$f(p, q) = M(p)\, M(q) \left[L(p) + L(q) + \ell\left(\frac{p}{q}, \frac{p}{q}\right)\right], \tag{3.15}$$

where $L : I \to \mathbb{C}$ is an arbitrary logarithmic map and $\ell : I^2 \to \mathbb{C}$ is a bilogarithmic function.

The following result is contained in [7].

Lemma 3 [7]**.** *The functions $f, g : I^2 \to \mathbb{C}$ satisfy the functional equation*

$$f(pr, qs) + f(ps, qr) = f(p, q)\, g(r, s) \tag{3.16}$$

if and only if

$$\begin{cases} f = 0 \\ g \quad \text{arbitrary;} \end{cases} \tag{3.17}$$

$$\begin{cases} f(p, q) = M(p)\, M(q)\, [\alpha + L(q) - L(p)] \\ g(r, s) = 2\, M(r)\, M(s); \end{cases} \tag{3.18}$$

$$\begin{cases} f(p, q) = \alpha M_1(p) M_2(q) + \beta M_1(q) M_2(p) \\ g(r, s) = M_1(r) M_2(s) + M_1(s) M_2(r), \end{cases} \tag{3.19}$$

where α, β are arbitrary complex constants, $L : I \to \mathbb{C}$ is an arbitrary logarithmic map, and $M, M_1, M_2 : I \to \mathbb{C}$ are multiplicative functions.

Lemma 4. *Let $M_1, M_2 : I \to \mathbb{C}$ be any two nonzero multiplicative maps with $M_1 \neq M_2$. Then the function $f : I \to \mathbb{C}$ satisfies the functional equation*

$$\begin{aligned} f(pr, qs) + f(ps, qr) = {} & [M_1(r) M_2(s) + M_1(s) M_2(r)]\; f(p, q) \\ & + [M_1(p) M_2(q) + M_1(q) M_2(p)]\; f(r, s) \end{aligned} \tag{3.20}$$

if and only if

$$\begin{aligned} f(p, q) = {} & M_1(p) M_2(q)\, [L_1(p) + L_2(q)] \\ & + M_1(q) M_2(p)\, [L_1(q) + L_2(p)], \end{aligned} \tag{3.21}$$

where $L_1, L_2 : I \to \mathbb{C}$ are logarithmic functions.

Proof. It is easy to verify that f given by (3.21) satisfies (3.20). Obviously, $f = 0$ is a solution of (3.20) and is of the form (3.21). We now suppose that $f \neq 0$. Setting $q = s = 1$ in (3.20), we get

$$f(p,r) = [M_1(r) + M_2(r)]\, g(p) + [M_1(p) + M_2(p)]\, g(r) - g(pr), \tag{3.22}$$

where

$$g(p) := f(p,1). \tag{3.23}$$

With $q = p$ and $s = r$, the equation (3.20) yields

$$f(pr,pr) = f(p,p)M_1(r)M_2(r) + f(r,r)M_1(p)M_2(p). \tag{3.24}$$

From this it follows that

$$f(p,p) = 2\, L(p)\, M_1(p)\, M_2(p), \tag{3.25}$$

where $L : I \to \mathbb{C}$ is a logarithmic function. Letting $p = r$, $q = s$ in (3.20), we have

$$f\left(p^2, q^2\right) + f(pq,pq) = 2f(p,q)\ [M_1(p)M_2(q) + M_1(q)M_2(p)]\,. \tag{3.26}$$

Now (3.25) in (3.2) gives

$$g\left(p^2\right) = 2g(p)\ [M_1(p) + M_2(p)] - L(p)\, M_1(p)M_2(p). \tag{3.27}$$

Putting (3.25), (3.22) and (3.27) into (3.26), we have

$$\begin{aligned}
&\{2g(p)\ [M_1(p) + M_2(p)] - 2L(p)M_1(p)M_2(p)\}\ \left[M_1(q)^2 + M_2(q)^2\right] \\
&\quad + \{2g(q)\ [M_1(q) + M_2(q)] - 2L(q)M_1(q)M_2(q)\}\ \left[M_1(p)^2 + M_2(p)^2\right] \\
&\quad - 2g(pq)\,[M_1(pq) + M_2(pq)] + 4L(pq)M_1(pq)M_2(pq) \\
&= 2\,\{g(p)\,[M_1(q) + M_2(q)] + g(q)\,[M_1(p) + M_2(p)] - g(pq)\} \\
&\quad \{M_1(p)M_2(q) + M_1(q)M_2(p)\}
\end{aligned}$$

which can be rewritten as

$$\left\{\begin{aligned}
&[M_1(p) - M_2(p)]\ [M_1(q) - M_2(q)] \\
&[2g(pq) - \{M_1(pq) + M_2(pq)\}\, L(pq)] \\
&= [M_1(p) - M_2(p)]\ [M_1(pq) - M_2(pq)] \\
&[2g(q) - \{M_1(q) + M_2(q)\}\, L(q)] \\
&+ [M_1(q) - M_2(q)]\ [M_1(pq) - M_2(pq)] \\
&[2g(p) - \{M_1(p) + M_2(p)\}\, L(p)]\,.
\end{aligned}\right. \tag{3.28}$$

Defining

$$L_0(p) := \begin{cases} \dfrac{2g(p) - [M_1(p) + M_2(p)]\, L(p)}{M_1(p) - M_2(p)} & \text{if } \ p \neq 1 \\ 0 & \text{if } \ p = 1, \end{cases}$$

we obtain from this definition and $g(1) = 0$ that

$$g(p) = [M_1(p) - M_2(p)]\, L_0(p) + [M_1(p) + M_2(p)]\, L(p) \tag{3.29}$$

for all $p \in I$. Further, from (3.28) it follows that

$$L_0(pq) = L_0(p) + L_0(q) \tag{3.30}$$

whenever $p \neq 1$ and $q \neq 1$. The function L_0 evidently satisfies (3.30) when $p = 1$ or $q = 1$. Now, using (3.29) in (3.22), we obtain (3.21), where $2L_1 := L + L_0$ and $2L_2 := L - L_0$. This completes the proof of the lemma.

4. The solution of (FE)

In this section, we display all general solution of the functional equation (FE) without assuming any regularity conditions on the unknown functions.

Theorem. *The functions $f, g : I^2 \to \mathbb{C}$ satisfy the functional equation*

$$f(pr, qs) + f(ps, qr) = f(p, q)\, g(r, s) + f(r, s)\, g(p, q) \quad (p, q, r, s \in I) \tag{FE}$$

if and only if

$$\begin{cases} f = 0 \\ g \quad \textit{arbitrary;} \end{cases} \tag{4.1}$$

$$\begin{cases} f(p, q) = M(p)\, M(q) \left[L(p) + L(q) + \ell\left(\dfrac{p}{q}, \ \dfrac{p}{q}\right)\right] \\ g(r, s) = 2M(r)\, M(s); \end{cases} \tag{4.2}$$

$$\begin{cases} f(p, q) = M_1(p)M_2(q)\, [L_1(p) + L_2(q)] + \\ \qquad\qquad + M_1(q)M_2(p)\, [L_1(q) + L_2(p)] \\ g(r, s) = M_1(r)M_2(s) + M_1(s)M_2(r); \end{cases} \tag{4.3}$$

$$\begin{cases} f(p, q) = \frac{a}{2}\, [M_3(p)M_4(q) + M_3(q)M_4(p)] \\ g(r, s) = \frac{1}{2}\, [M_3(r)M_4(s) + M_3(s)M_4(r)]\,, \end{cases} \tag{4.4}$$

where $M, M_1, M_2, M_3, M_4 : I \to \mathbb{C}$ are multiplicative functions, $L, L_1, L_2 : I \to \mathbb{C}$ are logarithmic functions, $\ell : I^2 \to \mathbb{C}$ is a bilogarithmic function, and a is an arbitrary nonzero complex constant.

Proof. It is easy to note that $f = 0$ and any arbitrary function g satisfy (FE). Hence, we get (4.1). From now on we assume that $f \neq 0$.

Interchanging p with r and q with s in (FE), we see that

$$f(pr, qs) + f(qr, ps) = f(r, s)g(p, q) + f(p, q)g(r, s). \tag{4.5}$$

From (4.5) and (FE), we get

$$f(ps, qr) = f(qr, ps). \tag{4.6}$$

Letting $r = s = 1$ in (4.6), we see that f is symmetric, that is

$$f(p, q) = f(q, p). \tag{4.7}$$

Interchanging r with s in (FE), we get

$$f(ps, qr) + f(pr, qs) = f(p, q)g(s, r) + f(s, r)g(p, q). \tag{4.8}$$

Using the symmetry of f and (FE) in (4.8), we obtain

$$f(p, q)g(r, s) = f(p, q)g(s, r). \tag{4.9}$$

Since $f \neq 0$, we see that g is symmetric, that is

$$g(r, s) = g(s, r). \tag{4.10}$$

Using (FE), we have

$$f(pxr, qys) + f(pys, qxr) = f(p, q)g(xr, ys) + g(p, q)f(xr, ys). \tag{4.11}$$

Similarly

$$f(pxs, qyr) + f(pyr, qxs) = f(p, q)g(xs, yr) + g(p, q)f(xs, yr). \tag{4.12}$$

Adding (4.11) to (4.12), we get

$$\begin{aligned} f(pxr, qys) + f(pys, qxr) + f(pxs, qyr) + f(pyr, qxs) \\ = f(p, q)g(xr, ys) + f(xr, ys)g(p, q) \\ + f(p, q)g(xs, yr) + f(xs, yr)g(p, q). \end{aligned} \tag{4.13}$$

Using (FE) in (4.13), we see that

$$\begin{aligned} f(pxr, qys) + f(pys, qxr) + f(pxs, qyr) + f(pyr, qxs) \\ = f(p, q)\left[g(xr, ys) + g(xs, yr)\right] \\ + g(p, q)\left[f(x, y)g(r, s) + g(x, y)f(r, s)\right]. \end{aligned} \tag{4.14}$$

Similarly, we have

$$
f(pxr, qys) + f(pyr, qxs) + f(pxs, qyr) + f(pys, qxr) \\
= f(x, y)\,[g(pr, qs) + g(ps, qr)] \\
+ g(x, y)\,[f(p, q)g(r, s) + g(p, q)f(r, s)]\,. \tag{4.15}
$$

Comparing (4.15) with (4.14), we see that

$$
f(p, q)\,[g(xr, ys) + g(xs, yr) - g(x, y)g(r, s)] \\
= f(x, y)\,[g(pr, qs) + g(ps, qr) - g(p, q)g(r, s)]\,. \tag{4.16}
$$

Since $f \neq 0$, there exists a pair of x_0, y_0 such that $f(x_0, y_0) \neq 0$. Letting $x = x_0$, $y = y_0$ and temporarily fixing r and s in (4.16), we get

$$
g(pr, qs) + g(ps, qr) = g(p, q)g(r, s) + A(r, s)f(p, q), \tag{4.17}
$$

where $A : I^2 \to \mathbb{C}$ is some complex-valued function. Interchanging p with r and q with s, we get

$$
g(pr, qs) + g(ps, qr) = g(p, q)g(r, s) + A(p, q)f(r, s), \tag{4.18}
$$

since g is symmetric. From the above two equations, we get

$$
A(r, s)f(p, q) = A(p, q)f(r, s).
$$

Hence, since $f \neq 0$, we get

$$
A(p, q) = \alpha^2\, f(p, q), \tag{4.19}
$$

where α is a complex constant. Thus, letting (4.19) into (4.17), we have

$$
g(pr, qs) + g(ps, qr) = g(p, q)g(r, s) + \alpha^2 f(p, q)f(r, s). \tag{4.20}
$$

Now, using (4.20) along with (FE), we determine the solution of (FE). We consider two cases depending on whether $\alpha = 0$ or $\alpha \neq 0$.

Case 1. Suppose $\alpha \neq 0$. First multiplying (FE) by α and then adding it to (4.20), we see that

$$
F(pr, qs) + F(ps, qr) = F(p, q)\,F(r, s), \tag{4.21}
$$

where

$$
F(p, q) := \alpha\, f(p, q) + g(p, q). \tag{4.22}
$$

Similarly, subtracting (4.20) from α times (FE), we get

$$
G(pr, ps) + G(ps, qr) = -G(p, q)\,G(r, s), \tag{4.23}
$$

where

$$G(p,q) := \alpha\, f(p,q) - g(p,q). \tag{4.24}$$

The solutions of (4.21) and (4.23) can be determined from Lemma 3. Hence, we have

$$F(p,q) = M_1(p)M_2(q) + M_1(q)M_2(p), \tag{4.25}$$

and

$$G(p,q) = -M_3(p)M_4(q) - M_3(q)M_4(p), \tag{4.26}$$

where $M_i : I \to \mathbb{C}$ $(i = 1,2,3,4)$ are multiplicative functions. From (4.22), (4.24), (4.25) and (4.26) and then using the form of f and g in (FE), we obtain the asserted solution (4.4) with $a = \frac{1}{\alpha}$.

Case 2. Next, we suppose $\alpha = 0$. Then (4.20) with $\alpha = 0$ yields

$$g(p,q) = M_1(p)M_2(q) + M_1(q)M_2(p), \tag{4.27}$$

where $M_1, M_2 : I \to \mathbb{C}$ are nonzero multiplicative functions.

Subcase 2.1. Suppose $M_1 = M_2 = M$ (say). Then, we have

$$g(p,q) = 2M(p)\, M(q). \tag{4.28}$$

Inserting (4.28) into (FE), we get

$$f(pr,qs) + f(ps,qr) = 2M(rs)\, f(p,q) + 2M(pq)\, f(r,s). \tag{4.29}$$

Using Lemma 2, we get the asserted solution (4.2).

Subcase 2.2. Suppose $M_1 \neq M_2$. Then using (4.27) in (FE) and using Lemma 4, we get the solution (4.3).

Since, no more cases are left, now the proof of the theorem is complete.

Remark. In the characterization of distance measures (1.3) and (1.4), one requires the real-valued solutions. Since, reals are not quadratically closed under multiplication, from our theorem one can not extract directly the real-valued solution of (FE). However, the real-valued solution can be extracted by a simple screening process. We leave this to the interested readers.

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References

[1] J. ACZÉL and J. DHOMBRES, Functional Equations in Several Variables, *Cambridge University Press*, *Cambridge*, 1989.

[2] J. K. CHUNG, PL. KANNAPPAN and C. T. NG, A generalization of the cosine-sine functional equation on groups, *Linear Algebra and Its Applications* **66** (1985), 259–277.

[3] J. K. CHUNG, PL. KANNAPPAN, C. T. NG and P. K. SAHOO, Measures of distance between probability distributions, *J. Math. Anal. Appl.* **139** (1989), 280–292.

[4] H. JEFFREYS, An invariant form for the prior probability in estimation problems, *Proc. Roy. Soc. London Ser. A* **186** (1946), 453–461.

[5] PL. KANNAPPAN and P. K. SAHOO, Kullback-Leibler type distance measures between probability distributions, *Jour. Math. Phy. Sci.* **26** (1992), 443–454.

[6] PL. KANNAPPAN, P. K. SAHOO and J. K. CHUNG, On a functional equation associated with the symmetric divergence measures, *Utilitas Math.* **44** (1993), 75–83.

[7] PL. KANNAPPAN, P. K. SAHOO and J. K. CHUNG, An equation associated with the distance between probability distributions, *Annales Mathematical Silesianal* (*to appear*).

[8] S. KULLBACK, Information theory and statistics, *Peter Smith*, *Gloucester, MA*, 1978.

[9] S. KULLBACK and R. A. LEIBLER, On information and sufficiency, *Annals. Math. Statist* **22** (1951), 79–86.

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