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A new characterization of Clifford torus

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Abstract. We extend a previous sharp upper bound of the first strong stability eigenvalue due to ALÍAS *et al.* [1], for the context of a closed submanifold immersed with nonzero parallel mean curvature vector field in the Euclidean sphere, and through this result, we obtain a new characterization for the Clifford torus.

1. Introduction

Given a closed submanifold M^n immersed in the unit Euclidean sphere $\mathbb{S}^{n+p}$ with parallel mean curvature vector field h (which means that h is parallel as a section of the normal bundle of M^n), its *strong stability operator* is defined by

$$J = -\Delta - |\Phi|^2 - n(1 + H^2), \tag{1}$$

where Δ stands for the Laplacian operator on M^n, $|\Phi|$ denotes the length of the traceless second fundamental Φ, and $H = |h|$ is the mean curvature of M^n. We observe that, when $p = 1$, J arises to the classical Jacobi operator established in [2].

We note that J belongs to a class of operators which are usually referred to as Schrödinger operators, that is, operators of the form $\Delta + q$, where q is any continuous function on M^n. The *first strong stability eigenvalue* λ_1^J of M^n is defined as being the smallest real number λ which satisfies the equation $Jf - \lambda f =$

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0 in M^n, for some nonzero $f \in C^\infty(M)$. As it is well known, λ_1^J has the following min–max characterization:

$$\lambda_1^J = \inf \left\{ \frac{\displaystyle\int_M fJf\,dM}{\displaystyle\int_M f^2\,dM}; f \in C^\infty(M), f \neq 0 \right\}, \tag{2}$$

where dM stands for the volume element with respect to the metric induced of M^n.

In his seminal work [9], SIMONS studied the first strong stability eigenvalue of a minimal closed hypersurface M^n immersed in $\mathbb{S}^{n+1}$. In this setting, he proved that either $\lambda_1^J = -n$, and M^n is a totally geodesic sphere, or $\lambda_1^J \leq -2n$, otherwise. Later on, WU in [10] characterized the equality $\lambda_1^J = -2n$ by showing that it holds only for the minimal Clifford torus. Shortly thereafter, PERDOMO [7] provides a new proof of this spectral characterization by the value of λ_1^J. Afterwards, ALÍAS, BARROS and BRASIL JR. [1] extended these results to the case of constant mean curvature hypersurfaces in $\mathbb{S}^{n+1}$, characterizing Clifford torus via the value of λ_1^J.

Proceeding with this picture, we obtain the following extension of the main result of [1] for the context of higher codimension.

Theorem 1.1. *Let M^n be a closed submanifold immersed in $\mathbb{S}^{n+p}$, $n \geq 4$, with nonzero parallel mean curvature vector field. If the normalized scalar curvature of M^n satisfies $R \geq 1$, then*

(i) *either $\lambda_1^J = -n(1 + H^2)$ (and M^n is totally umbilical),*

(ii) *or*

$$\lambda_1^J \leq -2n(1 + H^2) + \frac{n(n-2)}{\sqrt{n(n-1)}} H \max_M |\Phi|.$$

Moreover, the equality occurs if and only if M^n is a Clifford torus $\mathbb{S}^{n-1}(r) \times \mathbb{S}^1(\sqrt{1 - r^2})$, with $r^2 \leq \dfrac{n-2}{n}$.

The proof of Theorem 1.1 is given in Section 3.

2. Some preliminaries and key lemmas

Let M^n be an n-dimensional connected submanifold immersed in a unit Euclidean sphere $\mathbb{S}^{n+p}$. Let $\{\omega_B\}$ be the corresponding dual coframe, and $\{\omega_{BC}\}$ the connection 1-forms on $\mathbb{S}^{n+p}$. We choose a local field of orthonormal frame $\{e_1, \ldots, e_{n+p}\}$ in $\mathbb{S}^{n+p}$, with dual coframe $\{\omega_1, \ldots, \omega_{n+p}\}$, such that, at each point of M^n, $e_1,\ldots,e_n$ are tangent to M^n, and $e_{n+1},\ldots,e_{n+p}$ are normal to M^n. We will use the following convection for indices

$$1\le A,B,C,\ldots\le n+p,\quad 1\le i,j,k,\ldots\le n\quad\text{and}\quad n+1\le\alpha,\beta,\gamma,\ldots n+p.$$

With restricting on M^n, the second fundamental form A, the curvature tensor R and the normal curvature tensor $R^\perp$ of M^n are given by

$$\omega_{i\alpha}=\sum_j h^\alpha_{ij}\omega_j,\quad A=\sum_{i,j,\alpha}h^\alpha_{ij}\omega_i\otimes\omega_j\otimes e_\alpha,$$
$$d\omega_{ij}=\sum_k\omega_{ik}\wedge\omega_{kj}-\frac{1}{2}\sum_{k,l}R_{ijkl}\omega_k\wedge\omega_l,$$
$$d\omega_{\alpha\beta}=\sum_\gamma\omega_{\alpha\gamma}\wedge\omega_{\gamma\alpha}-\frac{1}{2}\sum_{k,l}R^\perp_{\alpha\beta kl}\omega_k\wedge\omega_l.$$

The Gauss equation is

$$R_{ijkl}=(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})+\sum_\alpha(h^\alpha_{ik}h^\alpha_{jl}-h^\alpha_{il}h^\alpha_{jk}).$$

In particular, the components of the Ricci tensor R_{ik} and the normalized scalar curvature R are given, respectively, by

$$R_{ik}=(n-1)\delta_{ik}+n\sum_\alpha H^\alpha h^\alpha_{ik}-\sum_{\alpha,j}h^\alpha_{ij}h^\alpha_{jk}\tag{3}$$

and

$$R=\frac{1}{(n-1)}\sum_i R_{ii}.\tag{4}$$

From (3) and (4), we get the following relation

$$n(n-1)R=n(n-1)+n^2H^2-S,\tag{5}$$

where $S=\sum_{\alpha,i,j}(h^\alpha_{ij})^2$ is the squared norm of the second fundamental form, and, being $h=\sum_\alpha H^\alpha e_\alpha=\frac{1}{n}\sum_\alpha\left(\sum_k h^\alpha_{kk}\right)e_\alpha$ the mean curvature vector field, $H=|h|$ is the mean curvature function of M^n.

The Ricci equation is given by

$$R^\perp_{\alpha\beta ij}=\sum_k(h^\alpha_{ik}h^\beta_{kj}-h^\alpha_{jk}h^\beta_{ki}).\tag{6}$$

From now on, we will deal with submanifolds M^n of $\mathbb{S}^{n+p}$ having *nonzero parallel mean curvature vector field*, which means that the mean curvature function H is, in fact, a positive constant, and that the corresponding mean curvature vector field h is parallel as a section of the normal bundle. In this context, we can choose a local orthonormal frame $\{e_1, \dots, e_{n+p}\}$ such that $e_{n+1} = \dfrac{h}{H}$. Thus,

$$H^{n+1} = \frac{1}{n}\mathrm{tr}(h^{n+1}) = H \quad \text{and} \quad H^\alpha = \frac{1}{n}\mathrm{tr}(h^\alpha) = 0, \alpha \geq n+2. \tag{7}$$

We will also consider the following symmetric tensor

$$\Phi = \sum_{\alpha,i,j} \Phi^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha, \tag{8}$$

where $\Phi^\alpha_{ij} = h^\alpha_{ij} - H^\alpha \delta_{ij}$. Consequently, we have that

$$\Phi^{n+1}_{ij} = h^{n+1}_{ij} - H\delta_{ij} \quad \text{and} \quad \Phi^\alpha_{ij} = h^\alpha_{ij}, \quad n+2 \leq \alpha \leq n+p. \tag{9}$$

Let $|\Phi|^2 = \sum_{\alpha,i,j}(\Phi^\alpha_{ij})^2$ be the square of the length of Φ. From (5), it is not difficult to verify that Φ is traceless with

$$|\Phi|^2 = S - nH^2. \tag{10}$$

From [5, Lemma 4.1] we obtain the following Simons type formula:

Lemma 1. *Let M^n be an n-dimensional ($n \geq 2$) submanifold immersed with nonzero parallel mean curvature vector field in the Euclidean sphere $\mathbb{S}^{n+p}$. Then, we have*

$$\frac{1}{2}\Delta|\Phi|^2 = |\nabla\Phi|^2 + n|\Phi|^2 + n\sum_{\beta,i,j,k} Hh^{n+1}_{ij}h^\beta_{jk}h^\beta_{ki} - \sum_{i,j,k,l}\left(\sum_\alpha h^\alpha_{ij}h^\alpha_{kl}\right)^2 - \sum_{i,j,\alpha,\beta}(R^\perp_{\alpha\beta ij})^2.$$

The next key lemma is due to Barros *et al.* (see [3, Lemma 1]).

Lemma 2. *Let M^n be a Riemannian manifold isometrically immersed into a Riemannian manifold N^{n+p}. Consider $\Psi = \sum_{\alpha,i,j}\Psi^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha$ a traceless symmetric tensor satisfying Codazzi equation. Then the following inequality holds:*

$$|\nabla|\Psi|^2|^2 \leq \frac{4n}{n+2}|\Psi|^2|\nabla\Psi|^2,$$

where $|\Psi|^2 = \sum_{\alpha,i,j}(\Psi^\alpha_{ij})^2$ and $|\nabla\Psi|^2 = \sum_{\alpha,i,j,k}(\Psi^\alpha_{ijk})^2$. In particular, the conclusion holds for the tensor Φ defined in (8).

In order to prove Theorem 1.1, we will also need two algebraic lemmas. The proof of them can be found in [8] and [6], respectively.

Lemma 3. *Let $B, C : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be symmetric linear maps such that $BC - CB = 0$ and $trB = trC = 0$, then*

$$-\frac{n-2}{\sqrt{n(n-1)}}|B|^2|C| \leq \mathrm{tr}(B^2C) \leq \frac{n-2}{\sqrt{n(n-1)}}|B|^2|C|.$$

Lemma 4. *Let $B^1, B^2, \ldots, B^n$ be symmetric $(n \times n)$-matrices. Set $S_{\alpha\beta} = tr(B^\alpha B^\beta)$, $S_\alpha = S_{\alpha\alpha}$, $S = \sum_\alpha S_\alpha$, then*

$$\sum_{\alpha,\beta} |B^\alpha B^\beta - B^\beta B^\alpha|^2 + \sum_{\alpha,\beta} S^2_{\alpha\beta} \leq \frac{3}{2}\left(\sum_\alpha S_\alpha\right)^2.$$

3. Proof of Theorem 1.1

From Lemma 1 we have that

$$\begin{aligned}\frac{1}{2}\Delta|\Phi|^2 &= |\nabla\Phi|^2 + n|\Phi|^2 + n\sum_{\beta,i,j,k} Hh^{n+1}_{ij}h^\beta_{jk}h^\beta_{ki} \\ &\quad - \sum_{i,j,k,l}\left(\sum_\alpha h^\alpha_{ij}h^\alpha_{kl}\right)^2 - \sum_{i,j,\alpha,\beta}(R^\perp_{\alpha\beta ij})^2. \end{aligned} \tag{11}$$

From (7) and (9) we get

$$\begin{aligned}\sum_{i,j,k,\beta} Hh^{n+1}_{ij}h^\beta_{jk}h^\beta_{ki} &= \sum_{i,j,k} Hh^{n+1}_{ij}h^{n+1}_{jk}h^{n+1}_{ki} + \sum_{\beta=n+2}^{n+p}\sum_{i,j,k} Hh^{n+1}_{ij}\Phi^\beta_{jk}\Phi^\beta_{ki} \\ &= H\mathrm{tr}(\Phi^{n+1}+HI)^3 + \sum_{\beta=n+2}^{n+p}\sum_{i,j,k} H\Phi^{n+1}_{ij}\Phi^\beta_{jk}\Phi^\beta_{ki} + \sum_{\beta=n+2}^{n+p} H^2|\Phi^\beta|^2 \\ &= H\mathrm{tr}(\Phi^{n+1})^3 + 3H^2|\Phi^{n+1}|^2 + nH^4 + \sum_{\beta=n+2}^{n+p} H^2|\Phi^\beta|^2 \\ &\quad + \sum_{\beta=n+2}^{n+p}\sum_{i,j,k} H\Phi^{n+1}_{ij}\Phi^\beta_{jk}\Phi^\beta_{ki}. \end{aligned} \tag{12}$$

Since $\operatorname{tr}\Phi^\alpha = 0$ and $\Phi^{n+1}\Phi^\beta - \Phi^\beta\Phi^{n+1} = 0, n+2 \le \beta \le n+p$, from Lemma 3 we obtain

$$\begin{aligned}
&H\operatorname{tr}(\Phi^{n+1})^3 + 3H^2|\Phi^{n+1}|^2 + nH^4 + \sum_{\beta=n+2}^{n+p} H^2|\Phi^\beta|^2 + \sum_{\beta=n+2}^{n+p}\sum_{i,j,k} H\Phi^{n+1}_{ij}\Phi^\beta_{jk}\Phi^\beta_{ki} \\
&\quad\ge -\frac{n-2}{\sqrt{n(n-1)}}H|\Phi^{n+1}|^3 + 2H^2|\Phi^{n+1}|^2 + H^2|\Phi|^2 + nH^4 \\
&\qquad - \frac{n-2}{\sqrt{n(n-1)}}\sum_{\beta=n+2}^{n+p} H|\Phi^{n+1}||\Phi^\beta|^2 \\
&\quad = 2H^2|\Phi^{n+1}|^2 + H^2|\Phi|^2 + nH^4 - \frac{n-2}{\sqrt{n(n-1)}}H|\Phi^{n+1}||\Phi|^2. \qquad (13)
\end{aligned}$$

Hence, from (12) and (13) we have

$$\sum_{\beta,i,j,k} Hh^{n+1}_{ij}h^\beta_{jk}h^\beta_{ki} \ge 2H^2|\Phi^{n+1}|^2 + H^2|\Phi|^2 + nH^4 - \frac{n-2}{\sqrt{n(n-1)}}H|\Phi^{n+1}||\Phi|^2. \quad (14)$$

From Ricci equation (6) we get

$$\begin{aligned}
&\sum_{i,j,k,l}\left(\sum_\alpha h^\alpha_{ij}h^\alpha_{kl}\right)^2 + \sum_{\alpha,\beta,i,j}(R^\perp_{\alpha\beta ij})^2 \\
&\quad = \sum_{\alpha,\beta}(\operatorname{tr}(A^\alpha A^\beta))^2 + \sum_{\alpha\ne n+1,\beta\ne n+1,i,j}(R^\perp_{\alpha\beta ij})^2 \\
&\quad = [\operatorname{tr}(A^{n+1}A^{n+1})]^2 + 2\sum_{\beta\ne n+1}[\operatorname{tr}(A^{n+1}A^\beta)]^2 \\
&\qquad + \sum_{\alpha\ne n+1,\beta\ne n+1}|A^\alpha A^\beta - A^\beta A^\alpha|^2 + \sum_{\alpha\ne n+1,\beta\ne n+1}(\operatorname{tr}(A^\alpha A^\beta))^2. \qquad (15)
\end{aligned}$$

But, using (9) and Lemma 4, we obtain

$$\begin{aligned}
\frac{3}{2}\left(\sum_{\beta\ne n+1}|\Phi^\beta|\right)^2 &\ge \frac{3}{2}\left(\sum_{\beta\ne n+1} tr(A^\beta A^\alpha)\right)^2 \\
&\ge \sum_{\alpha\ne n+1,\beta\ne n+1}[\operatorname{tr}(A^\alpha A^\beta)]^2 + \sum_{\alpha\ne n+1,\beta\ne n+1}|A^\alpha A^\beta - A^\beta A^\alpha|^2. \quad (16)
\end{aligned}$$

Hence, from (15) and (16) we have

$$\sum_{i,j,k,l}\left(\sum_\alpha h^\alpha_{ij}h^\alpha_{kl}\right)^2 + \sum_{\alpha,\beta,i,j}(R^\perp_{\alpha\beta ij})^2$$

$$\le [\mathrm{tr}(A^{n+1}A^{n+1})]^2 + 2\sum_{\beta\neq n+1}[\mathrm{tr}(A^{n+1}A^{\beta})]^2 + \frac{3}{2}\left(\sum_{\beta\neq n+1}|\Phi^{\beta}|^2\right)^2$$
$$= |\Phi^{n+1}|^4 + 2nH^2|\Phi^{n+1}|^2 + n^2H^4 + 2\sum_{\beta\neq n+1}[\mathrm{tr}(\Phi^{n+1}\Phi^{\beta})]^2 + \frac{3}{2}(|\Phi|^2 - |\Phi^{n+1}|^2)^2$$
$$\le \frac{5}{2}|\Phi^{n+1}|^4 + 2nH^2|\Phi^{n+1}|^2 + n^2H^4 + \frac{3}{2}|\Phi|^4 + 2|\Phi^{n+1}|^2(|\Phi|^2 - |\Phi^{n+1}|^2) - 3|\Phi|^2|\Phi^{n+1}|^2$$
$$= \frac{1}{2}|\Phi^{n+1}|^4 + 2nH^2|\Phi^{n+1}|^2 + n^2H^4 - |\Phi|^2|\Phi^{n+1}|^2 + \frac{3}{2}|\Phi|^4. \tag{17}$$

Therefore, from (11), (14) and (17) we get

$$\begin{aligned}
&\frac{1}{2}\Delta|\Phi|^2\\
&\quad\ge n|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi^{n+1}||\Phi|^2 + nH^2|\Phi|^2 - \frac{1}{2}|\Phi^{n+1}|^4 + |\Phi|^2|\Phi^{n+1}|^2 - \frac{3}{2}|\Phi|^4\\
&\quad= (|\Phi| - |\Phi^{n+1}|)\left(\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^2 - \frac{1}{2}(|\Phi| - |\Phi^{n+1}|)(|\Phi| + |\Phi^{n+1}|)^2\right)\\
&\qquad+ |\Phi|^2\left(-|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| + n(1+H^2)\right).
\end{aligned}\tag{18}$$

On the other hand, we note that the following algebraic inequality (3.5) of [4] also holds:

$$(|\Phi| - |\Phi^{n+1}|)(|\Phi| + |\Phi^{n+1}|)^2 \le \frac{32}{27}|\Phi|^3. \tag{19}$$

Moreover, since $R \ge 1$, we use (5) and (10), in order to obtain

$$n^2H^2 = S + n(n-1)(R-1) \ge S = |\Phi|^2 + nH^2,$$

which gives us

$$H \ge \frac{1}{\sqrt{n(n-1)}}|\Phi|. \tag{20}$$

Thus, from (19) and (20) we conclude that

$$\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^2 - \frac{1}{2}(|\Phi| - |\Phi^{n+1}|)(|\Phi| + |\Phi^{n+1}|)^2 \ge \left(\frac{n-2}{n-1} - \frac{16}{27}\right)|\Phi|^3. \tag{21}$$

But, taking into account our assumption that $n \ge 4$, we have that

$$\frac{n-2}{n-1} - \frac{16}{27} > 0. \tag{22}$$

Consequently, inserting (13), (21) and (22) in (18), we get that

$$\begin{aligned}\frac{1}{2}\Delta|\Phi|^2 &\geq |\nabla\Phi|^2 - |\Phi|^2 P_H\left(|\Phi|\right) + \left(|\Phi| - |\Phi^{n+1}|\right)\left(\frac{n-2}{n-1} - \frac{16}{27}\right)|\Phi|^3 \\ &\geq |\nabla\Phi|^2 - |\Phi|^2 P_H\left(|\Phi|\right), \end{aligned} \tag{23}$$

where

$$P_H(x) = |\Phi|^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| - n(1+H^2).$$

If M^n is totally umbilical, then $|\Phi|^2 = 0$ and $J = -\Delta - n(1+H^2)$, where H is constant, so $\lambda_1^J = \lambda_1^{-\Delta} - n(1+H^2) = -n(1+H^2)$, whose corresponding eigenfunctions are the constant functions. On the other hand, following the ideas of [1], when M^n is not umbilical, for an arbitrary $\varepsilon > 0$, we consider the positive smooth function $f_\varepsilon \in C^\infty(M)$ given by

$$f_\varepsilon = \sqrt{\varepsilon + |\Phi|^2}. \tag{24}$$

With a straightforward computation, from (24) we have that

$$f_\varepsilon \Delta f_\varepsilon = \frac{1}{2}\Delta|\Phi|^2 - \frac{1}{4(\varepsilon + |\Phi|^2)}|\nabla|\Phi|^2|^2. \tag{25}$$

Thus, from (25) and (23) we get

$$f_\varepsilon \Delta f_\varepsilon \geq |\nabla\Phi|^2 - |\Phi|^2 P_H(|\Phi|) - \frac{1}{4(\varepsilon + |\Phi|^2)}|\nabla|\Phi|^2|^2. \tag{26}$$

Hence, applying Lemma 2 to Φ, from (26) we obtain

$$\begin{aligned} f_\varepsilon \Delta f_\varepsilon &\geq -|\Phi|^2 P_H(|\Phi|) + |\nabla\Phi|^2 - \frac{n}{(n+2)}\frac{|\Phi|^2}{(\varepsilon + |\Phi|^2)}|\nabla\Phi|^2 \\ &= -|\Phi|^2 P_H(|\Phi|) + \left(1 - \frac{n}{(n+2)}\frac{|\Phi|^2}{(\varepsilon + |\Phi|^2)}\right)|\nabla\Phi|^2 \\ &\geq -|\Phi|^2 P_H(|\Phi|) + \left(1 - \frac{n}{n+2}\right)|\nabla\Phi|^2 \geq -|\Phi|^2 P_H(|\Phi|) + \frac{2}{n+2}|\nabla\Phi|^2. \end{aligned} \tag{27}$$

Then, from (27) we have that

$$\begin{aligned} f_\varepsilon J(f_\varepsilon) &= -f_\varepsilon \Delta f_\varepsilon - \big(|\Phi|^2 + n(1+H^2)\big)\big(\varepsilon + |\Phi|^2\big) \\ &\leq |\Phi|^2 P_H(|\Phi|) - \frac{2}{n+2}|\nabla\Phi|^2 - \big(\varepsilon + |\Phi|^2\big)\big(|\Phi|^2 + n(1+H^2)\big)\big(\varepsilon + |\Phi|^2\big). \end{aligned} \tag{28}$$

From (2) and (28) we obtain

$$\begin{aligned}\lambda_1^J \int_M f_\varepsilon^2\, dM = \lambda_1^J \int_M \left(\varepsilon + |\Phi|^2\right)\, dM &\le \int_M f_\varepsilon J(f_\varepsilon)\, dM \\ &\le \int_M |\Phi|^2 P_H(|\Phi|)\, dM - \frac{2}{n+2}\int_M |\nabla\Phi|^2 dM \\ &\quad - \int_M \left(\varepsilon + |\Phi|^2\right)\left(|\Phi|^2 + n(1+H^2)\right) dM.\end{aligned}$$

Finally, letting $\varepsilon \to 0$ in this last inequality, we have

$$\begin{aligned}\lambda_1^J \int_M |\Phi|^2 dM &\le \int_M \left(|\Phi|^2 P_H(|\Phi|) - |\Phi|^4 - n(H^2+1)|\Phi|^2\right) dM - \frac{2}{n+2}\int_M |\nabla\Phi|^2 dM \\ &\le -2n(H^2+1)\int_M |\Phi|^2 dM + \frac{n(n-2)}{\sqrt{n(n-1)}} H \int_M |\Phi|^3 dM. \end{aligned} \tag{29}$$

Hence, from (29) we get

$$\lambda_1^J \le -2n(H^2+1) + \frac{n(n-2)}{\sqrt{n(n-1)}}\, H \max_M |\Phi|.$$

Now, suppose $\lambda_1^J = -2n(1+H^2) + \frac{n(n-2)}{\sqrt{n(n-1)}} H \max_M |\Phi|$. Thus, from (29) we get that $|\nabla\Phi| \equiv 0$, and using once more Lemma 2, we conclude that $|\Phi|$ must be a positive constant.

On the other hand, from equation (1) it follows that

$$\lambda_1^{-\Delta} - \left(|\Phi|^2 + n(1+H^2)\right) = \lambda_1^J = -2n(1+H^2) + \frac{n(n-2)}{\sqrt{n(n-1)}} H|\Phi|.$$

Thus, since M^n is closed, we obtain

$$0 = \lambda_1^{-\Delta} = |\Phi|^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} H|\Phi| - n(1+H^2) = P_H(|\Phi|).$$

Hence, all the inequalities along this proof must be equalities. In particular, taking into account (22), from (23) we conclude that $|\Phi| = |\Phi^{n+1}|$, and consequently, $\Phi^\alpha = 0$, for all $n+2 \le \alpha \le n+p$. Thus, since e_{n+1} is parallel in the normal bundle of M^n, we are in position to apply [11, Theorem 1] to conclude that M^n is, in fact, isometrically immersed in a $(n+1)$-dimensional totally geodesic submanifold $\mathbb{S}^{n+1}$ of $\mathbb{S}^{n+p}$. Therefore, we can use [1, Theorem 2.2] to finish our proof.

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