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De Branges–Rovnyak spaces and generalized Dirichlet spaces

By BARTOSZ ŁANUCHA (Lublin) and MARIA NOWAK (Lublin)

Abstract. We consider the relations between the generalized Dirichlet spaces $\mathcal{D}(\mu)$ and de Branges–Rovnyak spaces $\mathcal{H}(b)$. Such relations were studied in the papers [10], [11], and more recently, in [2] and [3]. Here we obtain further results in this direction.

1. Introduction

Let H^2 be the standard Hardy space of the open unit disk $\mathbb{D}$. For μ a finite positive Borel measure on $\mathbb{T} = \partial\mathbb{D}$ and f a holomorphic function in $\mathbb{D}$, the Dirichlet integral of f with respect to μ is defined by

$$D_\mu(f) = \int_{\mathbb{D}} |f'(z)|^2 P\mu(z) dA(z),$$

where $P\mu$ is the Poisson integral of μ, and dA denotes area measure on $\mathbb{D}$, normalized to have unit total mass.

For $\lambda \in \mathbb{T}$, we define the local Dirichlet integral of f at λ by

$$D_\lambda(f) = \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f(\lambda) - f(e^{it})}{\lambda - e^{it}} \right|^2 dt,$$

where $f(\lambda)$ is the nontangential limit of f at λ. If $f(\lambda)$ does not exist, then we set $D_\lambda(f) = \infty$. It is known that if $f \in H^2$, then

$$D_\mu(f) = \int_{\mathbb{T}} D_\lambda(f) d\mu(\lambda) \quad \text{(see, e.g., [8])}.$$

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The generalized Dirichlet space $\mathcal{D}(\mu)$ consists of those functions $f \in H^2$ for which $D_\mu(f)$ is finite. The space $\mathcal{D}(\mu)$ is a Hilbert space with the norm

$$\|f\|^2_{\mathcal{D}(\mu)} = \|f\|_2^2 + D_\mu(f).$$

If $\mu = \delta_\lambda$, $\lambda \in \mathbb{T}$, then $D_\mu(f) = D_\lambda(f)$ and $\mathcal{D}(\delta_\lambda)$ is called *the local Dirichlet space at* λ. The more general case when μ is a finitely atomic measure, that is, $\mu = \sum_{j=1}^n \mu_j \delta_{\lambda_j}$, where $\lambda_1, \ldots, \lambda_n$ are distinct points of $\mathbb{T}$ and $\mu_1, \ldots, \mu_n$ are positive numbers, was considered by Sarason in [11]. In this case,

$$\|f\|^2_{\mathcal{D}(\mu)} = \|f\|_2^2 + \sum_{j=1}^n \mu_j \left\| \frac{f(\lambda_j) - f(z)}{\lambda_j - z} \right\|_2^2.$$

It has been shown in [8] that if $f(z) = \sum_{k=0}^\infty \hat{f}(k) z^k$ belongs to $\mathcal{D}(\delta_\lambda)$, then the series $\sum_{k=0}^\infty \hat{f}(k)\lambda^k$ converges. Consequently, if f is in $\mathcal{D}(\mu)$, with μ as above, then all the series $\sum_{k=0}^\infty \hat{f}(k)\lambda_j^k$ converge.

In [11], the author considered the function

$$K_\mu(z) = 1 - \sum_{j=1}^n \mu_j \frac{\overline{\lambda}_j z}{(1 - \overline{\lambda}_j z)^2}, \tag{1.1}$$

and proved that if $w_1, \ldots, w_n$ are the zeros of K_μ in $\mathbb{D}$, and

$$\widetilde{a}(z) = \prod_{j=1}^n (1 - \overline{\lambda}_j z), \tag{1.2}$$

then

$$\mathcal{D}(\mu) = \mathcal{M}(\widetilde{a}) \oplus_{\mathcal{D}(\mu)} K_{\tilde{B}}, \tag{1.3}$$

where $K_{\tilde{B}} = H^2 \ominus \tilde{B}H^2$ is the model space corresponding to the finite Blaschke product $\tilde{B}$ with zero sequence $w_1, \ldots, w_n$ and $\mathcal{M}(\widetilde{a}) = \widetilde{a}H^2$, see [11, Corollary 1].

For $\chi \in L^\infty(\mathbb{T})$, let T_χ denote the bounded Toeplitz operator on H^2, that is, $T_\chi f = P(\chi f)$, where P is the orthogonal projection of $L^2(\mathbb{T})$ onto H^2. Given a function b in the unit ball of H^∞, the *de Branges–Rovnyak space* $\mathcal{H}(b)$ is the image of H^2 under the operator $(I - T_b T_{\overline{b}})^{1/2}$. The space $\mathcal{H}(b)$ is given the Hilbert space structure that makes the operator $(I - T_b T_{\overline{b}})^{1/2}$ a coisometry of H^2 onto $\mathcal{H}(b)$, namely,

$$\langle (I - T_b T_{\overline{b}})^{1/2} f, (I - T_b T_{\overline{b}})^{1/2} g \rangle_b = \langle f, g \rangle_2 \quad (f, g \in (\ker(I - T_b T_{\overline{b}})^{1/2})^\perp).$$

It is known [12, p. 10] that $\mathcal{H}(b)$ is a Hilbert space with reproducing kernel

$$k_w^b(z) = \frac{1 - \overline{b(w)}b(z)}{1 - \overline{w}z} \quad (z, w \in \mathbb{D}).$$

Here, we are interested in the case when the function b is not an extreme point of the unit ball of H^∞, that is, the case when the function $\log(1 - |b|)$ is integrable on $\mathbb{T}$ ([7, p. 138]). Then, there exists an outer function $a \in H^\infty$ for which $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$. Moreover, if we suppose that $a(0) > 0$, then a is uniquely determined, and we say that (b, a) is a pair.

Recall now that the Smirnov class $\mathcal{N}^+$ consists of the holomorphic functions in $\mathbb{D}$ that are quotients of functions in H^∞ in which the denominators are outer functions. If (b, a) is a pair, then the quotient $\varphi = \frac{b}{a}$ is in $\mathcal{N}^+$. Conversely, for every nonzero function $\varphi \in \mathcal{N}^+$, there exists a unique pair (b, a) such that $\varphi = \frac{b}{a}$. It is worth noting here that if φ is rational, then the functions a and b in the representation of φ are also rational (see [13]).

For a function φ that is holomorphic on $\mathbb{D}$, we define T_φ to be the operator of multiplication by φ on the domain $\mathcal{D}(T_\varphi) = \{f \in H^2 \colon \ \varphi f \in H^2\}$. We note that T_φ is bounded on H^2 if and only if $\varphi \in H^\infty$. It was proved in [13] that $\mathcal{D}(T_\varphi)$ is dense in H^2 if and only if $\varphi \in \mathcal{N}^+$. Moreover, if φ is a nonzero function in $\mathcal{N}^+$ with canonical representation $\varphi = \frac{b}{a}$, then $\mathcal{D}(T_\varphi) = aH^2$. In this case, T_φ has a unique adjoint T_φ^*. Following Sarason [13, p. 286], we define $T_{\overline{\varphi}} = T_\varphi^*$. The next theorem says that the de Branges–Rovnyak space $\mathcal{H}(b)$ is the domain of $T_{\overline{\varphi}}$.

Theorem 1.1 ([13]). *Let (b, a) be a pair, and let $\varphi = \frac{b}{a}$. Then the domain of $T_{\overline{\varphi}}$ is $\mathcal{H}(b)$ and for $f \in \mathcal{H}(b)$,*

$$\|f\|_b^2 = \|f\|_2^2 + \|T_{\overline{\varphi}} f\|_2^2.$$

In 1997, D. Sarason [10] proved that for $\lambda \in \mathbb{T}$ and

$$b_\lambda(z) = \frac{(1 - \alpha)\overline{\lambda} z}{1 - \alpha \overline{\lambda} z}, \quad \alpha = \frac{3 - \sqrt{5}}{2},$$

the space $\mathcal{D}(\delta_\lambda)$ coincides with $\mathcal{H}(b_\lambda)$, with equality of norms. In 2010, N. Chevrot, D. Guillot and T. Ransford [2] identified all the functions b and the measures μ for which $\mathcal{D}(\mu) = \mathcal{H}(b)$ with equality of norms.

In their recent paper [3], C. Costara and T. Ransford proved that if (b, a) is a rational pair, then $\mathcal{D}(\mu) = \mathcal{H}(b)$, without supposing equality of norms, if and only if:

(i) the zeros of the function a on $\mathbb{T}$ are all simple, and

(ii) the support of μ is exactly equal to this set of zeros.

In the proof of this result, the following theorem due to Ball and Kriete [1] was used.

Theorem 1.2 ([1], [3]). *Let (b_1, a_1) and (b_2, a_2) be pairs. Then $\mathcal{H}(b_1) \subset \mathcal{H}(b_2)$ if and only if the following two conditions both hold:*

(i) *there exist $g, h \in H^\infty$ such that $b_2 = gb_1 + ha_2$,*

(ii) *there exists $\gamma > 0$ such that $|a_1| \leq \gamma |a_2|$ m-a.e. on $\mathbb{T}$.*

In this paper, we will use also the following description of $\mathcal{H}(b)$ obtained in [3]:

Theorem 1.3 ([3]). *Let (b, a) be a rational pair, and let $\lambda_1, \ldots, \lambda_n$ be the zeros of a on $\mathbb{T}$, listed according to multiplicity. Then*

$$\mathcal{H}(b) = \left\{ p + \prod_{j=1}^{n} (z - \lambda_j) g : \ p \in P_{n-1}, \ g \in H^2 \right\}, \tag{1.4}$$

where P_{n-1} denotes the set of polynomials of degree at most $n - 1$.

Here, we mainly concentrate on pairs (b, a) for which $\varphi = b/a = \prod_{j=1}^{n} (1 - \overline{\lambda}_j z)^{-1}$, where $\lambda_1, \ldots, \lambda_n$ are distinct points from $\mathbb{T}$. We find an explicit formula for $T_{\overline{\varphi}} f$, $f \in H(\overline{\mathbb{D}})$, which implies the equality of the spaces $\mathcal{D}(\mu) = \mathcal{H}(b)$ and some inequalities between their norms. Moreover, in Theorem 2.3, we obtain the following result on the structure of $\mathcal{H}(b)$:

$$\mathcal{H}(b) = \mathcal{M}(a) \oplus_{\mathcal{H}(b)} \operatorname{span}\{k^b_{\lambda_1}, \ldots, k^b_{\lambda_n}\},$$

where $k^b_{\lambda_j}$ are the corresponding kernel functions. Next, we show that

$$\operatorname{span}\{k^b_{\lambda_1}, \ldots, k^b_{\lambda_n}\} = K_B,$$

where B is a Blaschke product of order n. From this, we get a description of $\mathcal{H}(b)$ analogous to that obtained by Sarason for $\mathcal{D}(\mu)$, when μ is a finitely atomic measure on $\mathbb{T}$. It turns out that in the special case when $\varphi(z) = (1 - z^n)^{-1}$ and $\mu = \frac{1}{n^2} \sum_{j=1}^{n} \delta_{e_j}$, where e_j are the n-th roots of 1, the model space K_B equals to the model space $K_{\tilde{B}}$ in (1.3).

We remark that in view of Theorem 1.2, the same argument as in [3] can be used to show that some of the results obtained for these special $\mathcal{H}(b)$ can be extended to the case when (b, a) are rational pairs such that $\lambda_1, \ldots, \lambda_n$ are the simple zeros of a on $\mathbb{T}$.

In Section 4, we apply Theorem 1.3 to show a relation between two spaces $\mathcal{H}(b)$ and $\mathcal{H}(b_1)$ in the case when the sets of zeros on $\mathbb{T}$ of the corresponding functions a and a_1 differ by a single point. In particular, we show that if λ is a zero of the function a of order $k \geq 2$ and $f \in \mathcal{H}(b)$, then the derivative $f^{(k-1)}$ has a nontangential limit at λ. We would like to mention that the existence of the nontangential limits of derivatives of the functions in $\mathcal{H}(b)$ is discussed in [5] and [6].

After preparing this manuscript, we found that our Theorem 4.1 is contained in Theorem 1.2 in the recent work [4]. However, our approach and proofs are different than those in [4].

2. Main results

In this section, we deal with the function $\varphi \in \mathcal{N}^+$ defined by

$$\varphi(z) = \frac{1}{\prod_{j=1}^{n}(1-\overline{\lambda}_j z)}, \tag{2.1}$$

where $\lambda_1, \dots, \lambda_n$ are distinct points from $\mathbb{T}$. By the Riesz–Fejér theorem (see [9, p. 118]), there is a unique polynomial r of degree n, without zeros in $\overline{\mathbb{D}}$, such that $r(0) > 0$, and

$$1 + \Big|\prod_{j=1}^{n}(1-\overline{\lambda}_j z)\Big|^2 = |r(z)|^2 \quad \text{on } \mathbb{T}.$$

Then the functions in the corresponding pair (b, a) are given by

$$a(z) = \frac{\prod_{j=1}^{n}(1-\overline{\lambda}_j z)}{r(z)}, \qquad b(z) = \frac{1}{r(z)}.$$

We observe that $b \in H(\overline{\mathbb{D}})$, and since λ_j, $1 \leq j \leq n$, are the zeros of a on $\mathbb{T}$, we have $|b(\lambda_j)| = 1$ for every $1 \leq j \leq n$. It then follows (see [12, p. 48]) that b has an angular derivative in the sense of Carathéodory at every λ_j, and consequently, every function $f \in \mathcal{H}(b)$ has a nontangential limit at λ_j, $1 \leq j \leq n$. Moreover,

$$f(\lambda_j) = \langle f, k^b_{\lambda_j} \rangle_b,$$

where

$$k^b_{\lambda_j}(z) = \frac{1 - \overline{b(\lambda_j)}b(z)}{1-\overline{\lambda}_j z}.$$

We first prove the following:

Lemma 2.1. *Let φ be given by* (2.1). *Then, for every f in $H(\overline{\mathbb{D}})$,*

$$T_{\overline{\varphi}} f(z) = f(z) + \sum_{j=1}^{n} \overline{a}_j \lambda_j \frac{f(\lambda_j) - f(z)}{\lambda_j - z}, \tag{2.2}$$

where $a_j = \left(\prod_{l=1, l\neq j}^{n} (1 - \overline{\lambda}_l \lambda_j)\right)^{-1}$.

Proof. Clearly,

$$\varphi(z) = \sum_{j=1}^{n} \frac{a_j}{1 - \overline{\lambda}_j z},$$

where a_j is defined in the Lemma. It is also known that $T_{\overline{\varphi}}$ is well defined on $H(\overline{\mathbb{D}})$. Moreover, using the formula

$$T_{\overline{\varphi}} f(z) = \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \overline{\widehat{\varphi}(l)} \hat{f}(l+m) z^m \quad \text{(see, e.g., [13, Lemma 6.1])}, \tag{2.3}$$

one can easily check that

$$T_{\overline{\varphi}} f(z) = \sum_{j=1}^{n} \overline{a}_j T_{\overline{k}_{\lambda_j}} f(z),$$

where $k_{\lambda_j}(z) = (1 - \overline{\lambda}_j z)^{-1}$. Since $k_{\lambda_j}(z) = \sum_{l=0}^{\infty} \overline{\lambda}_j^l z^l$, we get, by (2.3),

$$T_{\overline{k}_{\lambda_j}} f(z) = \sum_{m=0}^{\infty} \left(\sum_{l=0}^{\infty} \lambda_j^l \hat{f}(l+m) \right) z^m = \sum_{m=0}^{\infty} r_m z^m,$$

where $r_m = \hat{f}(m) + \lambda_j \hat{f}(m+1) + \lambda_j^2 \hat{f}(m+2) + \cdots$. Consequently,

$$\begin{aligned}
(\lambda_j - z) T_{\overline{k}_{\lambda_j}} f(z) &= \sum_{m=0}^{\infty} \lambda_j r_m z^m - \sum_{m=0}^{\infty} r_m z^{m+1} \\
&= \lambda_j f(\lambda_j) + \sum_{m=1}^{\infty} (\lambda_j r_m - r_{m-1}) z^m = \lambda_j f(\lambda_j) - \sum_{m=1}^{\infty} \hat{f}(m-1) z^m \\
&= \lambda_j f(\lambda_j) - z \sum_{m=0}^{\infty} \hat{f}(m) z^m = \lambda_j (f(\lambda_j) - f(z)) + (\lambda_j - z) f(z).
\end{aligned}$$

Hence,

$$T_{\overline{k}_{\lambda_j}} f(z) = f(z) + \lambda_j \frac{f(\lambda_j) - f(z)}{\lambda_j - z}.$$

Since $\sum_{j=1}^{n} a_j = 1$, we get formula (2.2). □

Theorem 2.2. *Let (b, a) be a pair such that $b/a = \varphi$, where φ is given by (2.1). If $f \in \mathcal{H}(b)$, then*

$$T_{\overline{\varphi}} f(z) = f(z) + \sum_{j=1}^{n} \overline{a}_j \lambda_j \frac{f(\lambda_j) - f(z)}{\lambda_j - z}, \tag{2.4}$$

where $f(\lambda_j)$ is the nontangential limit of f at λ_j, and a_j are such as in Lemma 2.1. In particular, the sum on the right side of (2.4) belongs to H^2.

Proof. Let $f \in \mathcal{H}(b)$. It follows from Theorem 1.1 that $T_{\overline{\varphi}} f \in H^2$. Since polynomials are dense in $\mathcal{H}(b)$ (see, e.g., [12, p. 25]), we can choose a sequence of polynomials $\{p_m\}$ such that $p_m \to f$ in $\mathcal{H}(b)$. Then $p_m \to f$ and $T_{\overline{\varphi}} p_m \to T_{\overline{\varphi}} f$ in H^2. This implies that $p_m(z) \to f(z)$ and $T_{\overline{\varphi}} p_m(z) \to T_{\overline{\varphi}} f(z)$ for every $z \in \mathbb{D}$. Moreover, since the functionals $f \mapsto f(\lambda_j)$ are bounded on $\mathcal{H}(b)$ (see [12, pp. 48–49]), we see that $p_m(\lambda_j) \to f(\lambda_j)$ for each $1 \leq j \leq n$. From this and Lemma 2.1, for every $z \in \mathbb{D}$,

$$\begin{aligned} T_{\overline{\varphi}} f(z) &= \lim_{m\to\infty} T_{\overline{\varphi}} p_m(z) = \lim_{m\to\infty} \left(p_m(z) + \sum_{j=1}^{n} \overline{a}_j \lambda_j \frac{p_m(\lambda_j) - p_m(z)}{\lambda_j - z} \right) \\ &= f(z) + \sum_{j=1}^{n} \overline{a}_j \lambda_j \frac{f(\lambda_j) - f(z)}{\lambda_j - z}. \qquad \square \end{aligned}$$

Theorem 2.3. *Let (b, a) be a pair such that $b/a = \varphi$, where φ is given by (2.1). We have*

$$\mathcal{H}(b) = \mathcal{M}(a) \oplus_{\mathcal{H}(b)} \operatorname{span}\{k^b_{\lambda_1}, \dots, k^b_{\lambda_n}\}.$$

In particular, $\mathcal{M}(a)$ is a closed subspace of $\mathcal{H}(b)$.

Proof. Let $V = \operatorname{span}\{k^b_{\lambda_1}, \dots, k^b_{\lambda_n}\} \subset \mathcal{H}(b)$. Since $\mathcal{M}(a) \subset \mathcal{H}(b)$ (see, e.g., [12, p. 24]), it is enough to show that $\mathcal{M}(a) = V^{\perp}$ in $\mathcal{H}(b)$.

Note that if $f \in \mathcal{M}(a)$, then $f(\lambda_j) = \langle f, k^b_{\lambda_j} \rangle_b = 0$ for every $1 \leq j \leq n$. So $\mathcal{M}(a) \subset V^{\perp}$. On the other hand, if $f \in V^{\perp}$, then $f(\lambda_j) = 0$ for each $1 \leq j \leq n$. Thus, by Theorem 2.2,

$$T_{\overline{\varphi}} f(z) = \sum_{j=1}^{n} \overline{a}_j \left(1 - \frac{1}{1 - \overline{\lambda}_j z}\right) f(z) = -\left(\sum_{j=1}^{n} \overline{a}_j \frac{\overline{\lambda}_j z}{1 - \overline{\lambda}_j z}\right) f(z) = g(z) \in H^2.$$

Moreover, for $|z| = 1$ we have

$$\overline{a}_1 \frac{\overline{\lambda}_1 z}{1 - \overline{\lambda}_1 z} + \dots + \overline{a}_n \frac{\overline{\lambda}_n z}{1 - \overline{\lambda}_n z} = \frac{-z^n}{\prod_{j=1}^{n} (z - \lambda_j)},$$

and consequently,

$$z^n f(z) = \prod_{j=1}^{n} (z - \lambda_j) g(z),$$

which shows that $f \in \mathcal{M}(a)$. $\square$

As a corollary from the last Theorem we get the following result, due to Costara and Ransford [2].

Corollary 2.4. *Let (b, a) be a pair such that $b/a = \varphi$, where φ is given by* (2.1)*. Then*

$$\mathcal{H}(b) = \mathcal{D}(\mu),$$

where $\mu = \sum_{j=1}^{n} \mu_j \delta_{\lambda_j}$ and $\mu_j > 0$.

Proof. To see that $\mathcal{H}(b) \subset \mathcal{D}(\mu)$, it is enough to observe that $\mathcal{M}(a) = \mathcal{M}(\widetilde{a}) \subset \mathcal{D}(\mu)$, where $\widetilde{a}$ is given by (1.2) and each of the functions $k^b_{\lambda_j} \in H(\overline{\mathbb{D}}) \subset \mathcal{D}(\mu)$. The other inclusion is an immediate consequence of Theorem 2.2. $\square$

It is known that if $\mathcal{H}(b) = \mathcal{D}(\mu)$, then the norms $\|\cdot\|_b$ and $\|\cdot\|_{\mathcal{D}(\mu)}$ are equivalent ([3]). Moreover, the authors in [2] gave necessary and sufficient conditions for equality of these norms.

In our case, we get the following:

Proposition 2.5. *If $\mu = \sum_{j=1}^{n} |a_j|^2 \delta_{\lambda_j}$, where $a_j = \left(\prod_{l=1, l\neq j}^{n} (1 - \overline{\lambda}_l \lambda_j)\right)^{-1}$ and b is as above, then*

$$\|f\|_b \le \sqrt{n+2}\, \|f\|_{\mathcal{D}(\mu)}.$$

Proof. We have

$$\|f\|_b^2 = \|f\|_2^2 + \|T_{\overline{\varphi}} f\|_2^2 \le \|f\|_2^2 + \left(\|f\|_2 + \sum_{j=1}^{n} |a_j| \left\| \frac{f(\lambda_j) - f(z)}{\lambda_j - z} \right\|_2 \right)^2$$

$$\le \|f\|_2^2 + (n+1) \left(\|f\|_2^2 + \sum_{j=1}^{n} |a_j|^2 \left\| \frac{f(\lambda_j) - f(z)}{\lambda_j - z} \right\|_2^2 \right) \le (n+2) \|f\|^2_{\mathcal{D}(\mu)}. \square$$

Recall that $b(z) = 1/r(z)$, where $r(z)$ is a polynomial of degree n with zeros $w_1', \dots, w_n' \in \mathbb{C} \setminus \overline{\mathbb{D}}$. We now state a result that is analogous to Sarason's result mentioned in the Introduction.

Corollary 2.6. *Let (b, a) be a pair such that $b/a = \varphi$, where φ is given by* (2.1)*. We have*

$$\mathcal{H}(b) = \mathcal{M}(a) \oplus_{\mathcal{H}(b)} K_B,$$

where K_B is the model space corresponding to the finite Blaschke product with zero sequence $w_k = 1/\overline{w_k'}$, $k = 1, \dots, n$.

Proof. In view of Theorem 2.3, it is enough to show that

$$\operatorname{span}\{k^b_{\lambda_1}, \dots, k^b_{\lambda_n}\} = K_B.$$

To this end, we observe that

$$k^b_{\lambda_j}(z) = \frac{1 - \overline{b(\lambda_j)}b(z)}{1 - \overline{\lambda_j}z} = \frac{1 - \frac{b(z)}{b(\lambda_j)}}{1 - \overline{\lambda_j}z} = \frac{1 - \frac{r(\lambda_j)}{r(z)}}{1 - \overline{\lambda_j}z} = \frac{r(z) - r(\lambda_j)}{(1 - \overline{\lambda_j}z)r(z)} = \frac{P(z)}{r(z)},$$

where $P(z)$ is a polynomial of degree at most $n-1$. Since there are constants c and c' such that

$$r(z) = c\prod_{k=1}^{n}(z - w_k') = c'\prod_{k=1}^{n}(1 - \overline{w}_k z),$$

we see that

$$k^b_{\lambda_j}(z) = \frac{P(z)}{c'\prod_{k=1}^{n}(1 - \overline{w}_k z)} \in K_B.$$

Consequently,

$$\operatorname{span}\{k^b_{\lambda_1}, \dots, k^b_{\lambda_n}\} \subset K_B.$$

Since both the spaces K_B and $\operatorname{span}\{k^b_{\lambda_1}, \dots, k^b_{\lambda_n}\}$ have dimension n, the other inclusion follows. $\square$

3. A special case: $\varphi(z) = \frac{1}{1-z^n}$

If $\varphi(z) = \frac{1}{1-z^n}$, then the poles $\lambda_j = e_j$ of φ, $j = 1, \dots, n$, are the n-th roots of 1. To find the canonical representation $\varphi = b/a$, we observe first that in this case

$$R(e^{it}) = 1 + |1 - e^{int}|^2 = 3 - e^{-int} - e^{int},$$

and consider the polynomial

$$\begin{aligned} W(z) &= z^n(3 - z^{-n} - z^n) = -z^{2n} + 3z^n - 1 \\ &= -\left(z^n - \frac{3-\sqrt{5}}{2}\right)\left(z^n - \frac{3+\sqrt{5}}{2}\right) = -(z^n - \alpha)\left(z^n - \frac{1}{\alpha}\right), \end{aligned}$$

where $\alpha = \frac{3-\sqrt{5}}{2}$. Clearly, $W(z)$ has n distinct zeros in $\mathbb{D}$,

$$w_k = \sqrt[n]{\alpha} e_k,$$

and n distinct zeros outside $\overline{\mathbb{D}}$,

$$w_k^{'} = \frac{1}{\overline{w}_k} = \frac{1}{\sqrt[n]{\alpha}} e_k, \quad k = 1, \dots, n.$$

Hence,

$$W(z) = -\prod_{k=1}^{n} (z - w_k^{'})(z - w_k) = \alpha z^n \prod_{k=1}^{n} (z - w_k^{'}) \left(\frac{1}{z} - \overline{w_k^{'}} \right),$$

and

$$r(z) = -\sqrt{\alpha} \prod_{k=1}^{n} (z - w_k^{'}) = -\sqrt{\alpha} \left(z^n - \frac{1}{\alpha} \right) = \frac{1}{\sqrt{\alpha}} (1 - \alpha z^n)$$

is a polynomial satisfying $r(0) > 0$, and

$$|r(z)|^2 = 1 + |1 - z^n|^2 \quad \text{on } \mathbb{T}.$$

Consequently, see, e.g., [13],

$$a(z) = \frac{(1-\alpha)(1-z^n)}{1 - \alpha z^n}$$

and

$$b(z) = \frac{1-\alpha}{1-\alpha z^n}.$$

We observe that in this case formula (2.4) simplifies to

$$T_{\overline{\varphi}} f(z) = f(z) + \frac{1}{n} \sum_{j=1}^{n} e_j \frac{f(e_j) - f(z)}{e_j - z}.$$

We note that if b is as above, then, by Corollary 2.6,

$$\mathcal{H}(b) = \mathcal{M}(a) \oplus_{\mathcal{H}(b)} K_B,$$

where K_B is the model space corresponding to the finite Blaschke product with zero sequence $w_k = \sqrt[n]{\alpha} e_k$, $k = 1, \dots, n$.

On the other hand, by SARASON's result [11], we know that if $\mu = \sum_{j=1}^{n} \mu_j \delta_{e_j}$, then

$$\mathcal{D}(\mu) = \mathcal{M}(a) \oplus_{\mathcal{D}(\mu)} K_{\tilde{B}},$$

where $K_{\tilde{B}}$ is the model space corresponding to the finite Blaschke product $\tilde{B}$ with zeros $\tilde{w}_1, \dots, \tilde{w}_n$, which are the zeros of the function K_μ defined in the Introduction. We now show that in this case, the coefficients μ_j can be chosen so that $B = \tilde{B}$.

The following Theorem was proved in [14].

Theorem. *Let $w_1, \dots, w_n$ be a finite sequence of points in $\mathbb{D}\backslash\{0\}$ (repetitions allowed). Then there exist distinct points $\lambda_1, \dots, \lambda_n$ on the unit circle and positive numbers $\mu_1, \dots, \mu_n$ such that $w_1, \dots, w_n, 1/\overline{w}_1, \dots, 1/\overline{w}_n$ is the zero sequence of the function $K_\mu(z)$ given by* (1.1).

It follows from the proof of this Theorem that if one can find $\lambda_1, \dots, \lambda_n$ on $\mathbb{T}$ such that

$$\frac{\prod_{l=1}^{n} \lambda_l}{\prod_{j=1}^{n} w_j} > 0 \tag{3.1}$$

and

$$\sum_{m\neq l} \frac{\overline{\lambda}_l \lambda_m - \lambda_l \overline{\lambda}_m}{|\lambda_l - \lambda_m|^2} = \sum_{j=1}^{n} \frac{\overline{\lambda}_l w_j - \lambda_l \overline{w}_j}{|\lambda_l - w_j|^2} \quad (l = 1, \dots, n), \tag{3.2}$$

then $\mu_1, \dots, \mu_n$ are uniquely determined and given by

$$\mu_l = \frac{\prod_{j=1}^{n} |\lambda_l - w_j|^2}{\prod_{j=1}^{n} |w_j| \prod_{m\neq l} |\lambda_l - \lambda_m|^2}.$$

We first show that if $w_k = \sqrt[n]{\alpha} e_k$ and $\lambda_k = e_k$, $k = 1, \dots, n$, then conditions (3.1) and (3.2) are fulfilled. Clearly, (3.1) holds. It is enough to prove (3.2) for $e_l = e_1 = 1$, that is

$$\sum_{m=2}^{n} \frac{e_m - \overline{e}_m}{|1 - e_m|^2} = \sum_{j=1}^{n} \frac{w_j - \overline{w}_j}{|1 - w_j|^2}.$$

Symmetry arguments show that both sides of this equality are equal to zero. Now, a calculation gives

$$\mu_l = \frac{1}{n^2}, \quad l = 1, \dots, n.$$

This means that if $\mu = \frac{1}{n^2} \sum_{j=1}^{n} \delta_{e_j}$, then $B = \tilde{B}$, which proves our claim.

4. Characterization of $\mathcal{H}(b)$ in terms of the zeros of a

In this section, using Theorem 1.2, we generalize Theorem 2.3 as follows:

Theorem 4.1. *Let (b, a) be a rational pair, and let $\lambda_1, \dots, \lambda_n$ be the simple zeros of a on $\mathbb{T}$. Then*

$$\mathcal{H}(b) = \mathcal{M}(a) \oplus_{\mathcal{H}(b)} \operatorname{span}\{k^b_{\lambda_1}, \dots, k^b_{\lambda_n}\}.$$

Proof. Let (b_0, a_0) be the rational pair such that the function φ defined by (2.1) has the canonical representation $\varphi = \frac{b_0}{a_0}$. It then follows from the construction of the canonical representation (see, e.g., [13, p. 283]) that

$$a(z) = q(z)a_0(z),$$

where $q(z)$ is a rational holomorphic non-vanishing function in $\overline{\mathbb{D}}$. Consequently,

$$\mathcal{M}(a) = \mathcal{M}(a_0).$$

Moreover, using Theorem 1.2, one can show that

$$\mathcal{H}(b) = \mathcal{H}(b_0).$$

Now, let $V = \operatorname{span}\{k^b_{\lambda_1}, \ldots, k^b_{\lambda_n}\} \subset \mathcal{H}(b)$. As in the proof of Theorem 2.3, we get $\mathcal{M}(a) \subset V^{\perp}$. Now, if $f \in V^{\perp}$, then $f(\lambda_j) = 0$, and since $f \in \mathcal{H}(b_0)$, by Theorem 2.3, $f \in \mathcal{M}(a_0) = \mathcal{M}(a)$. This shows that $V^{\perp} = \mathcal{M}(a)$. □

Our next result is a corollary of Theorem 1.3.

Theorem 4.2. *Let (b, a) and (b_1, a_1) be rational pairs, and let $\lambda, \lambda_1, \ldots, \lambda_n$ and $\lambda_1, \ldots, \lambda_n$ be the zeros of a and a_1 on $\mathbb{T}$, respectively, listed according to multiplicity. If $f \in \mathcal{H}(b)$, then*

$$F(z) = \frac{f(\lambda) - f(z)}{\lambda - z} \in \mathcal{H}(b_1).$$

Conversely, if $f \in H^2$ is such that the nontangential limit $f(\lambda)$ exists and $F \in \mathcal{H}(b_1)$, then $f \in \mathcal{H}(b)$.

Proof. Assume that $f \in \mathcal{H}(b)$. Then, by (1.4),

$$f(z) = p(z) + (z - \lambda)\prod_{j=1}^{n}(z - \lambda_j)g,$$

where p is a polynomial of degree n and $g \in H^2$. So

$$F(z) = \frac{f(\lambda) - f(z)}{\lambda - z} = \frac{p(\lambda) - p(z)}{\lambda - z} + \prod_{j=1}^{n}(z - \lambda_j)g(z) = p_1(z) + \prod_{j=1}^{n}(z - \lambda_j)g(z),$$

where p_1 is a polynomial of degree $n - 1$. By (1.4) again, $F \in \mathcal{H}(b_1)$. The other claim can be proved analogously. □

It follows immediately from Theorem 1.3 that if $f \in \mathcal{H}(b)$ and λ is a zero of a on $\mathbb{T}$, then the nontangential limit of f at λ exists. For the case when λ is a zero of order $k \geq 2$, we get immediately from the above theorem the following:

Corollary 4.3. *If (b, a) is a rational pair and λ is a zero of the function a of order $k \geq 2$ and $f \in \mathcal{H}(b)$, then the derivative f' has a nontangential limit at λ.*

We remark that by the result in [12, p. 46], the derivative f' has a nontangential limit at λ if and only if f has a nontangential limit $f(\lambda)$, and the difference quotient $(f(\lambda) - f(z))/(\lambda - z)$ has a nontangential limit at λ.

Corollary 4.4. *Let (b, a) be a rational pair, and let λ be a zero of the function a of order k. If $f \in \mathcal{H}(b)$, then there is a function h in H^2 such that*

$$f(z) = f(\lambda) + f'(\lambda)(z - \lambda) + \cdots + \frac{f^{(k-1)}(\lambda)}{(k-1)!}(z - \lambda)^{k-1} + (z - \lambda)^k h(z).$$

Proof. By formula (1.4),

$$f(z) = p_{n+k-1}(z) + (z - \lambda)^k \prod_{j=1}^{n} (z - \lambda_j) g(z),$$

where $\lambda_1, \ldots, \lambda_n$ are the other zeros of a and $g \in H^2$. Clearly,

$$f^{(j)}(\lambda) = p_{n+k-1}^{(j)}(\lambda), \quad j = 0, 1, \ldots, k-1.$$

Consequently,

$$\begin{aligned} f(z) &= \sum_{j=0}^{k-1} \frac{f^{(j)}(\lambda)}{j!}(z - \lambda)^j + (z - \lambda)^k \left(p_{n-1}(z) + \prod_{j=1}^{n} (z - \lambda_j) g(z) \right) \\ &= \sum_{j=0}^{k-1} \frac{f^{(j)}(\lambda)}{j!}(z - \lambda)^j + (z - \lambda)^k h(z). \end{aligned}$$

Finally, we remark that the function h is in the space $\mathcal{H}(\widetilde{b})$, where the zeros of the corresponding function $\tilde{a}$ are $\lambda_1, \ldots, \lambda_n$. $\square$

References

[1] J. A. Ball and T. L. Kriete, Operator-valued Nevanlinna–Pick kernels and the functional models for contraction operators, *Integral Equations Operator Theory* **10** (1987), 17–61.

[2] N. Chevrot, D. Guillot and T. Ransford, De Branges–Rovnyak spaces and Dirichlet spaces, *J. Funct. Anal.* **259** (2010), 2366–2383.

[3] C. Costara and T. Ransford, Which de Branges–Rovnyak spaces are Dirichlet spaces (and vice versa)?, *J. Funct. Anal.* **265** (2013), 3204–3218.

[4] E. Fricain, A. Hartmann and W. T. Ross, Concrete examples of $\mathcal{H}(b)$ spaces, *Comput. Methods Funct. Theory* **16** (2016), 287–306.

[5] E. Fricain and J. Mashreghi, Boundary behavior of functions in the de Branges–Rovnyak spaces, *Complex Anal. Oper. Theory* **2** (2008), 87–97.

[6] E. Fricain and J. Mashreghi, Integral representation of the n-th derivative in de Branges–Rovnyak spaces and the norm convergence of its reproducing kernel, *Ann. Inst. Fourier (Grenoble)* **58** (2008), 2113–2135.

[7] K. Hoffman, Banach Spaces of Analytic Functions, Prentice-Hall Series in Modern Analysis, *Prentice-Hall*, *Englewood Cliffs, N.J.*, 1962.

[8] S. Richter and C. Sundberg, A formula for the local Dirichlet integral, *Michigan Math J.* **38** (1991), 355–379.

[9] F. Riesz and B. Sz.-Nagy, Functional Analysis, Dover Books on Advanced Mathematics, *Dover Publications*, *New York*, 1990.

[10] D. Sarason, Local Dirichlet spaces as de Branges–Rovnyak spaces, *Proc. Amer. Math. Soc.* **125** (1997), 2133–2139.

[11] D. Sarason, Harmonically weighted Dirichlet spaces associated with finitely atomic measures, *Integral Equations Operator Theory* **31** (1998), 186–213.

[12] D. Sarason, Sub-Hardy Hilbert spaces in the unit disc, University of Arkansas Lecture Notes in the Mathematical Sciences, Vol. **10**, *John Wiley & Sons*, *New York*, 1994.

[13] D. Sarason, Unbounded Toeplitz operators, *Integral Equations Operator Theory* **61** (2008), 281–298.

[14] D. Sarason and D. Suarez, Inverse problem for zeros of certain Koebe-related functions, *J. Anal. Math.* **71** (1997), 149–158.

BARTOSZ ŁANUCHA
DEPARTMENT OF MATHEMATICS
M. CURIE-SKŁODOWSKA UNIVERSITY
PL. MARII CURIE-SKŁODOWSKIEJ 1
20-031 LUBLIN
POLAND

E-mail: bartosz.lanucha@poczta.umcs.lublin.pl

MARIA NOWAK
DEPARTMENT OF MATHEMATICS
M. CURIE-SKŁODOWSKA UNIVERSITY
PL. MARII CURIE-SKŁODOWSKIEJ 1
20-031 LUBLIN
POLAND

E-mail: mt.nowak@poczta.umcs.lublin.pl

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