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Almost periodic functions valued in a convex bornological space

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Abstract. This paper deals for the first time with the theory of almost periodicity for functions valued in a vector space endowed with a convex bornology (i.e., a convex bornological space). New results in the topological setting are then obtained. A new extension of Bochner's Criterion is given. It is shown that the space of almost periodic functions valued in a complete convex bornological space inherits the same structure for a natural bornology. Finally, integration, derivation and nonlinear operations in spaces of almost periodic functions are introduced in the perspective of the study of differential problems in this new setting of almost periodicity.

1. Introduction

Almost periodic functions valued in a Banach space were widely studied. Since the original work of H. Bohr in 1925 (see [5]), the concept was introduced by Bochner in [3]. Actually, Bochner extended the Favard theory of almost-periodic functions (see [12]) to Banach spaces. Later, Levitan (see [18]) extended in turn the result due to Favard to functions taking values in a Banach space. We may also cite Kadets [16] and Zaidman [22] who extended the Bohl–Bohr Theorem on the integration of almost periodic functions. We can also mention [1], [11], [19], [22] for more details on this matter.

Almost periodicity in the setting of topological vector spaces (TVS) was introduced by Bochner and von Neumann in [4]. Among mathematicians who contributed to this field, with emphasis on Fréchet spaces, we may cite

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N'Guérékata [20], Bugajewski and N'Guérékata [9]. Almost periodic functions and representation theory with range in barreled spaces were studied by Shtern in [21]. The theory of almost periodicity without local convexity can be found in the paper of Gal and N'Guérékata [13], where p-Fréchet spaces are considered, and in the more recent paper of Khan and Alsulami (see [17]).

The purpose of this paper is to extend the concept of almost periodicity to real variable functions with range in a bornological vector space and to investigate their properties. By a bornological space we mean a space endowed with a bornology; here we are concerned especially with convex bornological spaces (CBS). Details can be found in [14], [15]. This work comes originally from the idea that bornologies should play a role in the study of almost periodic functions since their ranges in a TVS are totally bounded. More precisely, this paper presents a twofold interest: not only it brings to light the topological aspect of almost periodicity, but also enlarges its range of applications. It is subdivided in five sections as follows:

In Section 2, we recall basic properties of disks and Minkowski functionals. Elements on the theory of CBS are presented with usual notations, and some results on almost periodic functions valued in a TVS are also given. Section 3 is devoted to the statement and the proof of a new extension of Bochner's Criterion which is valid in non-locally convex spaces, too. Section 4 is concerned with our main subject: the *bornological almost periodicity.* This concept leads to the notion of *bornologically almost periodic* functions valued in a locally convex space (LCS). It is shown that in a large class of LCS which contains Fréchet spaces, the two notions of almost periodicity coincide. Similarly, the notion of *topologically almost periodic* functions valued in a CBS is defined. A last section deals with integration, derivation and nonlinear operations of Nemetskii type on almost periodic functions valued in a CBS. It provides the basic tools for the study of differential equations in our new setting.

2. Preliminaries and notations

2.1. Disks and Minkowski functionals. Throughout this section, E denotes a $\mathbb{K}$-vector space with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We recall some basic properties of Minkowski functionals and disks. We refer to [8] and [15].

Definition 2.1. A subset A of E is called a *disk* if it is absolutely convex, that is, balanced and convex. Its absolutely convex envelope, denoted by $\Gamma(A)$, is the smaller disk of E which contains it.

Proposition 2.2.

(i) *A subset $A \subset E$ is a disk if and only if:* $\forall \lambda, \mu \in \mathbb{K}, |\lambda| + |\mu| \leq 1, \lambda A + \mu A \subset A$.

(ii) *An intersection of disks is a disk, and the direct or inverse image of a disk by a linear map is a disk.*

The balanced envelope of A is equal to $\bigcup_{|\lambda| \leq 1} \lambda A$. Hence, the convex envelope of a balanced subset is a disk. The closed or open unit ball of a semi-norm is an absorbent disk. Recall that the *Minkowski functional* $\| \cdot \|_A$ of an absorbent disk A in E is defined by

$$\|x\|_A = \inf\{\alpha \in \mathbb{R}_+, x \in \alpha A\}, \quad \forall x \in E.$$

Let $A \subset E$ be a disk. It is absorbent in the spanned subspace $E_A = \sum_{\lambda > 0} \lambda A$, and E_A can be endowed with the semi-norm $\| \cdot \|_A$. The disk A is called *norming* (resp. *completing* or *Banach*) if $(E_A, \| \cdot \|_A)$ is a normed (resp. Banach) space. If A and B are two disks in E, then we have:

Proposition 2.3. *Let A and B denote two disks in E.*

(i) $E_{A+B} = E_A + E_B = E_{\Gamma(A \cup B)}$ *($\Gamma(\cdot)$ denoting the absolutely convex envelope). Furthermore, the norms $\| \cdot \|_{A+B}$ and $\| \cdot \|_{\Gamma(A \cup B)}$ are equivalent.*

(ii) *For every $x \in E_{A+B}$, we have*

$$\|x\|_{A+B} = \inf\{\max(\|y\|_A, \|z\|_B), x = y + z, (y, z) \in E_A \times E_B\}.$$

The image of a disk under a linear map is a disk. Moreover, we have:

Proposition 2.4. *Let F denote a $\mathbb{K}$-vector space, and $u : E \longrightarrow F$ a linear map. For every disk A of E, $u(A)$ is a disk in F; moreover, $F_{u(A)}$ and E_A/N are isometric, where $N = E_A \cap \ker u$.*

2.2. Convex bornological vector spaces. For the materials developed in this section, we refer to [8], [14], [15].

2.2.1. *Generalities on bornological spaces.*

Definition 2.5. A bornology on a set X is a family $\mathcal{B}$ of subsets which is a covering of X, stable under finite union and hereditary with respect to the inclusion. The couple $(X, \mathcal{B})$ is called a *bornological set*; the elements of $\mathcal{B}$ are then called *bounded sets*. A basis of $\mathcal{B}$ is any subfamily $\mathcal{B}_0$ such that every element of $\mathcal{B}$ is in some element of $\mathcal{B}_0$.

The bornology $\mathcal{B}$ is said to be *finer* than the bornology $\mathcal{B}'$ if $\mathcal{B}' \subset \mathcal{B}$. A map between two bornological spaces is said to be *bounded* if it transforms bounded sets into bounded sets. If $(X_i, \mathcal{B}_i)_{i\in I}$ is a family of bornological sets, the *product bornology* on $X = \prod_{i\in I} X_i$ is given by the sets $\prod_{i\in I} B_i$, where $B_i \in \mathcal{B}_i$, $i \in I$. The field $\mathbb{K}$ will be endowed with the bornology given by its natural topology.

Definition 2.6. A bornology $\mathcal{B}$ on X is a *vector space bornology* if the maps $(x, y) \mapsto x + y$ from X^2 to X, and $(\lambda, x) \mapsto \lambda x$ from $\mathbb{K} \times X$ to X are bounded. Then $(X, \mathcal{B})$ is called a *bornological vector space* (BVS). It is said to be *separated* if $\{0\}$ is the only bounded subspace of X.

Definition 2.7. A vector bornology $\mathcal{B}$ on X is *convex* if it admits a basis consisting of convex sets, which may be chosen to be absolutely convex. Then $(X, \mathcal{B})$ is called a *convex bornological vector space* (CBS).

A product of CBS is a CBS. If A and B are two bounded disks in X, $\pi_{BA} : X_A \longrightarrow X_B$ denotes the canonical injection. It it shown that X can be represented as the bornological inductive limit $X = \varinjlim(X_A, \pi_{AB})$. Moreover, X is separated if and only if the spaces $(X_A, \|\cdot\|_A)$ are normed.

Definition 2.8. A convex bornology on a vector space is said to be complete if it admits a basis consisting of completing disks. A convex bornological vector space (CBS) is said to be complete if its bornology is complete.

Definition 2.9. A sequence (x_n) of elements of the CBS X *converges in the sense of Mackey* or *converges bornologically* to $x \in X$ if there exists a bounded disk B such that X_B contains x and (x_n), and $x_n \to x$ in X_B, which is denoted by $x_n \overset{M}{\to} x$. A subset F of X is said to be *b-closed* (or *M-closed*) if $x \in F$, whenever $(x_n) \subset E$ and $x_n \overset{M}{\to} x$ in X.

Definition 2.10. Let X and Y denote two convex bornological spaces. An application $f : X \longrightarrow Y$ is said to be sequentially M-continuous if for any sequence (x_n) in X, $x_n \overset{M}{\to} x$ implies $f(x_n) \overset{M}{\to} f(x)$.

2.2.2. *Bornological dual, t-topology and canonical bornology.* We denote by $X^\times$ the *bornological dual* of X, that is, the vector space of bounded linear forms on X. This means that for every bounded set B in X and every $\varphi \in X^\times$, $\varphi(B)$ is a bounded subset of $\mathbb{R}$ for its usual topology.

We call *t-topology* on X the locally convex topology defined by the bornivorous disks in X. Endowed with this topology, X is denoted by ${}^tX$ and $({}^tX)^*$ stands for its topological dual. Algebraically, we have $X^\times = ({}^tX)^*$, then the

t-topology is separated if and only if for any $x \in X \setminus \{0\}$ there exists $\varphi \in X^\times$ such that $\varphi(x) \neq 0$. In this case, X is said to be *regular*, and the CBS X is also separated.

The *canonical bornology* on an LCS E is the one defined by its topology; endowed with this bornology, E is denoted by ${}^bE$. If $X = {}^{bt}X = {}^b({}^tX)$, then X is called a *topological CBS*; similarly, E is said to be a *bornological LCS* (this is the classical sense) if $E = {}^{tb}E = {}^t({}^bE)$.

2.3. Almost periodic functions valued in a separated TVS. We assume known the basic results on almost periodic functions valued in a Banach space. More account can be found in [1], [11], [19], [20], [22].

Definition 2.11 ([22], p. 20). Let $P \subset \mathbb{R}$, and l be a positive real number. Then P is said to be l-dense in $\mathbb{R}$ if $P \cap [a, a+l] \neq \emptyset$ for every $a \in \mathbb{R}$. The set P is said to be relatively dense in $\mathbb{R}$ if it is l-dense for some $l > 0$.

Theorem 2.12 ([22], Proposition 2, p. 73)**.** *A set $P \subset \mathbb{R}$ is relatively dense if and only if there exists a compact set $K \subset \mathbb{R}$ such that $\mathbb{R} = K + P$.*

Definition 2.13. Let E denote a separated topological vector space. A continuous map $f : \mathbb{R} \longrightarrow E$ is said to be almost periodic if for any neighborhood U of the origin in E there exists a relatively dense set P in $\mathbb{R}$ such that

$$f(t+\tau) - f(t) \in U, \quad \forall \tau \in P, \quad \forall t \in \mathbb{R}.$$

Theorem 2.14 ([20], Theorems 3.1.4, p. 52)**.** *Let E denote a separated locally convex vector space.*

(i) *If $f : \mathbb{R} \longrightarrow E$ is an almost periodic function, then f is uniformly continuous and its range $\{f(t) : t \in \mathbb{R}\}$ is totally bounded.*

(ii) *If (f_n) is a sequence of almost periodic functions, $f_n : \mathbb{R} \longrightarrow E$ such that (f_n) converges uniformly to f on $\mathbb{R}$, then f is almost periodic.*

Theorem 2.15 ([20], Theorems 3.1.5, p. 52)**.** *Let E denote a separated locally convex vector space. If $f : \mathbb{R} \longrightarrow E$ is an almost periodic function, then its range $\{f(t) : t \in \mathbb{R}\}$ is totally bounded.*

Theorems 2.14 and 2.15 enable to prove Bochner's Criterion:

Theorem 2.16 (Bochner's Criterion [20], Theorem 3.1.8, p. 55)**.** *Let E be a Fréchet space. Then $f : \mathbb{R} \longrightarrow E$ is almost periodic if and only if for every sequence of real numbers (s'_n), there exists a subsequence (s_n) such that $(f(\cdot+s_n))$ converges uniformly.*

3. An extension of Bochner's Criterion

The analysis of the proof of Theorem 2.15 ([20], Theorem 3.1.5) shows that it is valid even if E is not complete nor locally convex. In the following, we extend Bochner's Criterion to a class of TVS not necessarily locally convex.

Theorem 3.1. *Let E be a separated topological vector space in which every totally bounded closed set is a metrizable compact set. A continuous function $f : \mathbb{R} \longrightarrow E$ is almost periodic if and only if for every sequence (s'_n) of real numbers there exists a subsequence (s_n) such that $(f(\cdot + s_n))$ converges uniformly.*

Proof. Assume that f satisfies the sequential property of the theorem. We have to show that it is almost periodic.

Step 1. From Theorem 2.15, the set $K = \overline{f(\mathbb{R})}$ is totally bounded and closed, it follows from the hypothesis that it is a metrizable compact set. Let (ξ_n) denote a dense sequence in $\mathbb{R}$, and (s'_n) an arbitrary real sequence. For each $k \in \mathbb{N}^*$, $(f(\xi_k + s'_n))$ is a sequence in K which is metrizable, hence we may apply the diagonal sequence technique to get a subsequence (s_n) of (s'_n) for which $(f(\xi_k + s_n))$ converges in K for every $k \in \mathbb{N}^*$.

Step 2. We show that for each fixed $t \in \mathbb{R}$, $(f(t + s_n))$ is a Cauchy sequence in K. From the density of (ξ_n), there exists a subsequence $(\xi_{k,t})$ such that $\xi_{k,t} \to t$ as $k \to \infty$. Let U be a neighborhood of zero; since f is uniformly continuous, there exists a balanced neighborhood V of zero such that $V + V + V \subset U$, and

$$\exists \eta > 0, \quad \forall t, t' \in \mathbb{R}, \quad |t - t'| < \eta, \quad f(t) - f(t') \in V.$$

Since $\xi_{k,t}$ converges to t, there exists $l \in \mathbb{N}^*$ such that $|\xi_{l,t} - t| < \eta$ for $k \geq l$. The convergent sequence $(f(\xi_{l,t} + s_n))$ is a Cauchy sequence, and then

$$\exists n_0 \in \mathbb{N}^*, \quad \forall m, n \geq n_0 : f(\xi_{l,t} + s_n) - f(\xi_{l,t} + s_m) \in V.$$

Writing

$$\begin{aligned} f(t + s_n) - f(t + s_m) = [f(t + s_n) - f(\xi_{l,t} + s_m)] + [f(\xi_{l,t} + s_n) - f(\xi_{l,t} + s_m)] \\ + [f(\xi_{l,t} + s_n) - f(t + s_m)], \end{aligned}$$

we find that $f(t+s_n) - f(t+s_m) \in U$ for $m, n \geq n_0$, which shows that $(f(t+s_n))$ is a Cauchy sequence in K. Since K is sequentially complete, $(f(t + s_n))$ is convergent; we set $g(t) = \lim_{n\to\infty} f(t + s_n)$. From the uniform continuity of f, it follows that g is also uniformly continuous.

Step 3. We show that $(f(\cdot + s_n))$ converges to g uniformly on $\mathbb{R}$. Let U denote a neighborhood of zero. We keep the notation of Step 2, and take V to be closed. Since f is almost periodic, it follows from Theorem 2.12 that there exist a compact set Q and a relatively dense set P in $\mathbb{R}$ such that $\mathbb{R} = Q + P$, and

$$f(t+\tau) - f(t) \in V, \quad \forall (t,\tau) \in \mathbb{R} \times P. \tag{3.1}$$

If we set $t = \alpha + \tau$ with $(\alpha, \tau) \in Q \times P$, then we have

$$\begin{aligned} f(t+s_n) - g(t) = {} & [f(\alpha + s_n + \tau) - f(\alpha + s_n)] \\ & + [f(\alpha + s_n) - g(\alpha)] + [g(\alpha) - g(\alpha + \tau)]. \end{aligned}$$

Using (3.1), we have that $f(\alpha + s_n + \tau) - f(\alpha + s_n) \in V$. Moreover, V being closed and $g(\alpha) - g(\alpha + \tau) = \lim_{n\to\infty}[f(\alpha + s_n + \tau) - f(\alpha + s_n)]$, we get that $g(\alpha) - g(\alpha + \tau) \in V$. At this stage, it must be shown that there exists $n_0 \in \mathbb{N}^*$ such that $f(\alpha + s_n) - g(\alpha) \in V$ for all $\alpha \in Q$ and $n \geq n_0$. To do that, assume that the contrary holds:

$$\forall k \in \mathbb{N}^*, \quad \exists \alpha_k \in Q, \quad \exists n_k \in \mathbb{N}^*, \quad f(\alpha_k + s_{n_k}) - g(\alpha_k) \notin V. \tag{3.2}$$

From the compactness of Q, assume that (α_k) converges to α_0 in Q; then we have

$$\begin{aligned} f(\alpha_k + s_{n_k}) - g(\alpha_k) = {} & [f(\alpha_k + s_{n_k}) - f(\alpha_0 + s_{n_k})] \\ & + [f(\alpha_0 + s_{n_k}) - g(\alpha_0)] + [g(\alpha_0) - g(\alpha_k)]. \end{aligned}$$

The uniform continuity of f and its almost periodicity show that the two first brackets of the right-hand member tend to zero as $k \to \infty$, and the last term also tends to zero because of the continuity of g. It follows that $f(\alpha_k + s_{n_k}) - g(\alpha_k) \to 0$ as $k \to \infty$, contradicting (3.2) and proving the necessity part.

The converse result follows from the sufficiency part of the proof of [20, Theorem 3.1.8, p. 55]. In fact, it is easily seen that this proof works even for a general separated topological vector space. □

Remark 3.2. Theorem 3.1 can be applied to complete metrizable topological spaces, and then to Fréchet spaces. Let E be a separated *quasi-complete* TVS (i.e., an LCS in which every separated closed bounded set in E is complete). If every compact set in E is a compact set in E_B for some bounded disk B, then the conclusion of Theorem 3.1 holds true. It is also the case if E is the strict inductive limit of a sequence (E_n) of Fréchet spaces (in this case E is non-metrizable).

4. Almost periodic functions valued in a CBS

4.1. Definitions and elementary properties. Let $(X, \mathcal{B})$ be a separated convex bornological space; then $\mathcal{B}$ has a basis consisting of norming elements.

Definition 4.1. A function $f : \mathbb{R} \longrightarrow X$ is said to be almost periodic if there exists a norming bounded disk $B \subset X$ such that $f(\mathbb{R}) \subset X_B$ and $f : \mathbb{R} \longrightarrow X_B$ is almost periodic. The set of such functions will be denoted by $AP(X)$.

The definition means that f is continuous from $\mathbb{R}$ to the normed space $(X_B, \|\cdot\|_B)$, and for every $\varepsilon > 0$ there exist $l(\varepsilon) > 0$ and an $l(\varepsilon)$-dense set P_ε such that

$$\|f(t+\tau) - f(t)\|_B < \varepsilon, \quad \forall (t, \tau) \in \mathbb{R} \times P_\varepsilon.$$

Definition 4.2. Let X be a separated CBS. A subset K of X is said to be *strictly totally bounded* (resp. a *strictly compact set*, resp. a *relatively strictly compact set*) if there exists a bounded disk B such that K is totally bounded (resp. a compact set, resp. a relatively compact set) in X_B.

Proposition 4.3. *Let $f : \mathbb{R} \longrightarrow X$ be an almost periodic function. Then the range of f is strictly totally bounded. If, moreover, X is a complete CBS, then the range of f is relatively strictly compact.*

Proof. Since f is almost periodic, there exists a bounded disk B in X such that $f(\mathbb{R}) \subset X_B$ and $f : \mathbb{R} \longrightarrow X_B$ is almost periodic. Then, the range of f is totally bounded in the normed space X_B. Moreover, if X is a complete CBS, we may assume that B is a Banach disk, and then the range of f is a relatively compact set in the Banach space X_B. □

Remark 4.4. It is easily seen that if $f : \mathbb{R} \longrightarrow X$ is almost periodic, then $f(\cdot + s)$ and $\check{f}$ are almost periodic, where s is any real number and $\check{f}(t) = f(-t)$.

We have the Bochner type criterion:

Theorem 4.5. *Let X be a complete convex bornological space, and $f : \mathbb{R} \longrightarrow X$ a function. The following properties are equivalent:*

(i) *the function f is almost periodic;*

(ii) *there exists a bounded disk B such that $f : \mathbb{R} \longrightarrow X_B$ is continuous, and from any sequence $(r'_n)_n$ of real numbers, one can extract a subsequence $(r_n)_n$ such that the sequence $(f(t + r_n))_n$ converges in X_B uniformly in $t \in \mathbb{R}$.*

Proof. The proof results from the definition and the Bochner criterion applied to almost periodic functions valued in a Banach space. □

Corollary 4.6. *Let X be a complete convex bornological space. Then the set of almost periodic functions $f : \mathbb{R} \longrightarrow X$ is an $\mathbb{R}$-vector space.*

Proof. The non-trivial part of the proof is the almost periodicity of $f + g$, where $f : \mathbb{R} \longrightarrow X$ and $g : \mathbb{R} \longrightarrow X$ are two almost periodic functions. This follows immediately from the relation

$$\|(f+g)(s) - (f+g)(t)\|_{A+B} \leq \max(\|f(s) - f(t)\|_A, \|g(s) - g(t)\|_B) \qquad (4.1)$$

and the above Bochner criterion. □

4.2. Bornologically almost periodic functions. We consider the relationship between the almost periodicity of a function valued in an LCS E and its almost periodicity as a function valued in the CBS ${}^bE$. We introduce the following:

Definition 4.7. Let E be a separated topological vector space. A function $f : \mathbb{R} \longrightarrow E$ is said to be *bornologically almost periodic* if $f : \mathbb{R} \longrightarrow {}^bE$ is almost periodic.

Proposition 4.8. *Let E be a separated topological vector space, and let $f : \mathbb{R} \longrightarrow E$ be a bornologically almost periodic function. Then f is almost periodic, and moreover, if E is sequentially complete, its range is a relatively compact set.*

Proof. Assume that $f : \mathbb{R} \longrightarrow {}^bE$ is almost periodic. Note that a bornological vector space is separated if and only if any Mackey convergent sequence has a unique limit. Since a Mackey convergent sequence in E is also convergent for its topology and E is separated, it follows that ${}^bE$ is a separated BVS. Hence there exists a norming disk B in E such that $f(\mathbb{R}) \subset E_B$, and $f : \mathbb{R} \longrightarrow E_B$ is almost periodic. Since the topology of $(E_B, \|\cdot\|_B)$ is finer than the one induced by E, it follows that $f : \mathbb{R} \longrightarrow E$ is almost periodic. In addition, if E is sequentially complete, it is bornologically complete (cf. [15], Corollary p. 41), that is, ${}^bE$ is a complete CBS. In this case, we may choose B to be a Banach disk, and then the range of f becomes a relatively compact set in E_B and also in E. □

Theorem 4.9. *Let E be a separated quasi-complete topological vector space such that every compact set in E is a compact set in E_B for some bounded disk B. Then a function $f : \mathbb{R} \longrightarrow E$ is almost periodic if and only if it is bornologically almost periodic.*

Proof. Let $\Gamma(\overline{f(\mathbb{R})})$ denote the balanced convex envelope of $\overline{f(\mathbb{R})}$; since E is separated and quasi-complete, it is a compact set, and so is $K = 2\Gamma(\overline{f(\mathbb{R})})$.

It follows from the compactness condition that there exists a bounded disk B in E for which K is a compact set in E_B. Let $\mathcal{T}$ and $\mathcal{T}'$ denote the respective topologies of E and E_B (for the norm $\|\cdot\|_B$). Since the embedding $E_B \longrightarrow E$ is continuous, it induces a homeomorphism between the compact spaces $(K, \mathcal{T}')$ and $(K, \mathcal{T})$. This proves the continuity of $f : \mathbb{R} \longrightarrow E_B$ as composition of two continuous functions. Now let V denote a neighborhood of zero in E_B; then $K \cap V$ is a neighborhood of zero in K for the topology induced by $\mathcal{T}'$. Since the two topologies $\mathcal{T}$ and $\mathcal{T}'$ coincide on K, there exists a neigborhood U of zero in E for which $K \cap V = K \cap U$ is a neighborhood of zero in K for the topology induced by $\mathcal{T}$. Using the almost periodicity of $f : \mathbb{R} \longrightarrow E$, there is a relatively dense set P in $\mathbb{R}$ such that

$$f(t+\tau) - f(t) \in U, \quad \forall(\tau, P) \in P \times \mathbb{R}.$$

Since $f(t+\tau)$ and $f(t)$ are in $\Gamma(\overline{f(\mathbb{R})})$, which is absolutely convex, it follows that

$$f(t+\tau) - f(t) \in K, \quad \forall(\tau, P) \in P \times \mathbb{R}.$$

The two above relations show that $f(t+\tau) - f(t) \in K \cap U$ for all $(\tau, P) \in P \times \mathbb{R}$, and finally

$$f(t+\tau) - f(t) \in V, \quad \forall(\tau, P) \in P \times \mathbb{R},$$

showing that $f : \mathbb{R} \longrightarrow E_B$ is almost periodic.
The converse follows from Proposition 4.8. □

Corollary 4.10. *If E is a Fréchet space, a function $f : \mathbb{R} \longrightarrow E$ is almost periodic if and only if it is bornologically almost periodic.*

Proof. It is a fact that if E is a Fréchet space and A is a compact set in E, then there exists a compact disk B such that A is a compact set in the Banach space E_B. Since a Fréchet space is obviously separated and quasi-complete, the hypotheses of Theorem 4.9 are fully satisfied, which concludes the proof. □

Remark 4.11. The above results show that a family of E-valued almost periodic functions with range in the same subspace E_B, where E satisfies the hypotheses of Theorem 4.9 and B is a Banach disk, can be considered as a subset of the space of almost periodic functions with range in the Banach space E_B.

4.3. Topologically almost periodic functions. Let X be a regular CBS, that is, ${}^tX$ is a separated LCS. Every function $f : \mathbb{R} \longrightarrow X$ may be considered as a function with range in ${}^tX$ or in ${}^{bt}X$.

Definition 4.12. Let X be a regular convex bornological space. A function $f : \mathbb{R} \longrightarrow X$ is said to be *topologically almost periodic* if $f : \mathbb{R} \longrightarrow {}^tX$ is almost periodic.

Proposition 4.13. *Let X be a regular CBS, and $f : \mathbb{R} \longrightarrow X$ be an almost periodic function. Then $f : \mathbb{R} \longrightarrow {}^tX$ is almost periodic. Moreover, if X is complete, the range of f is a relatively compact set in ${}^tX$.*

Proof. Assume that $f : \mathbb{R} \longrightarrow X$ is almost periodic. The bornology of X being finer than the one of ${}^{bt}X$, it follows that $f : \mathbb{R} \longrightarrow {}^{bt}X$ is also almost periodic. Actually, the immediate application of Proposition 4.8 gives the almost periodicity of $f : \mathbb{R} \longrightarrow {}^tX$. Furthermore, if X is a complete CBS, it follows from Proposition 4.3 that $f(\mathbb{R})$ is a strictly compact set in X. Thus there is a bounded Banach disk B in X such that $f(\mathbb{R})$ is a relatively compact set in the Banach space X_B. Moreover, if V is any bornivorous disk in X, it absorbs B, and then the embedding $i : X_B \longrightarrow X_V$ is continuous, which means that $i : X_B \longrightarrow {}^tX$ is continuous. Since ${}^tX$ is separated, $\overline{f(\mathbb{R})}^{X_B}$ is a compact set in ${}^tX$, hence $f(\mathbb{R})$ is a relatively compact set in ${}^tX$. □

4.4. Spaces of almost periodic functions valued in a CBS.

Theorem 4.14. *Let X and Y denote two separated CBS, and let $u : X \longrightarrow Y$ be a bounded linear map. If $f : \mathbb{R} \longrightarrow X$ is an almost periodic function, so is $u \circ f : \mathbb{R} \longrightarrow Y$.*

Proof. Let B be a norming bounded disk for which $f : \mathbb{R} \longrightarrow X_B$ is almost periodic, and denote by ${}^bX_B$ the space $(X_B, \|\cdot\|_B)$ endowed with its canonical bornology. A bounded set B' in ${}^bX_B$ being a bounded set in X_B, there is $\lambda > 0$ such that $B' \subset \lambda B$. It follows that $u(B') \subset \lambda u(B)$, which implies that $u(B')$ is a bounded set in Y. Furthermore, $u : X_B \longrightarrow {}^tY$ is bounded (in the topological sense). Since X_B is a metrizable locally convex space, it is bornological. Then, ${}^tY$ being locally convex, $u : X_B \longrightarrow {}^tY$ is a continuous map. It follows that its kernel $N = (\ker u) \cap X_B$ is a closed set, and then the quotient space X_B/N is separated. From Proposition 2.4, $A = u(B)$ is a norming disk. On the other hand, $u(X_B) = Y_A$, and $u(\lambda B) = \lambda u(B)$ is a bounded set in Y_A, then $u : X_B \longrightarrow Y_A$ is a bounded linear map which proves that it is continuous. It follows that $u \circ f : \mathbb{R} \longrightarrow Y_A$ is a continuous function. Now we show that it is almost periodic. To do that, let V denote a neighborhood of zero in Y_A. From the continuity of $u : X_B \longrightarrow Y_A$, there exists a neighborhood W of zero in X_B such that $u(W) \subset V$. The almost periodicity of f gives a relatively dense set P in $\mathbb{R}$ such that

$$f(t+\tau) - f(t) \in W, \quad \forall (\tau, t) \in P \times \mathbb{R}.$$

Since $u(W) \subset V$; it follows that

$$u \circ f(t+\tau) - u \circ f(t) \in V, \quad \forall (\tau, t) \in P \times \mathbb{R},$$

showing that $u \circ f$ is almost periodic. □

Now we introduce a complete convex bornological vector space of functions valued in X for which $AP(X)$ is a complete subspace.

Definition 4.15. A function $f : \mathbb{R} \longrightarrow X$ is said to be *globally bounded* if $f(\mathbb{R})$ is a bounded set of X. It is said to be *globally M-continuous* if there exists a bounded set B such that $f : \mathbb{R} \longrightarrow X_B$ is continuous. The set of all globally M-continuous and globally bounded functions will be denoted by $M_{gb}(X)$.

Clearly, $AP(X)$ is a subspace of $M_{gb}(X)$. It is easy to see that if $f \in M_{gb}(X)$, there exists a bounded disk B such that $f(\mathbb{R}) \subset B$ and $f : \mathbb{R} \longrightarrow X_B$ is continuous. For a bounded disk $B \subset X$, we denote by $C_b(X_B)$ the space of bounded continuous X_B-valued functions defined on $\mathbb{R}$.

Definition 4.16. A subset $H \subset M_{gb}(X)$ will be said to be *bounded* if there exists a bounded disk B in X such that $H(\mathbb{R}) := \cup_{u \in H} u(\mathbb{R}) \subset B$ and $H \subset C_b(X_B)$ (i.e., $f \in C(X_B)$ as a X_B-valued function for every $f \in H$). The set of all such subsets of $M_{gb}(X)$ defines a convex vector bornology on $M_{gb}(X)$, called the *natural bornology* of $M_{gb}(X)$.

For a bounded disk $B \subset X$, we set

$$H^B := \{f \in M_{gb}(X), f(\mathbb{R}) \subset B\} \cap C_b(X_B).$$

If B is a disk, it is easily seen that H^B is also a disk. Moreover, if $\mathcal{B}_0$ is a basis of bounded disks of the bornology of X, then the sets H^B form a basis of the natural bornology of $M_{gb}(X)$ when B runs over $\mathcal{B}_0$.

Theorem 4.17. *If X is a complete CBS, then $M_{gb}(X)$ is also a complete CBS and $AP(X)$ is a complete subspace of $M_{gb}(X)$. Moreover, if X is regular, $M_{gb}(X)$ is also regular.*

Proof. Suppose that X is complete, and let B be a completing bounded disk in X, we show that H^B is a completing disk in $M_{gb}(X)$. For the sake of simplicity, we set $H^B = H$. Let $(f_n)_n$ be a Cauchy sequence in $(M_{gb}(X)_H, \|\cdot\|_H)$, where $\|\cdot\|_H$ denotes the Minkowski functional of H, we have

$$\forall \varepsilon > 0, \quad \exists n_0 \in \mathbb{N}, \quad \forall (m,n) \in \mathbb{N}^2, \quad m \geq n_0, n \geq n_0, \quad \|f_n - f_m\|_H \leq \varepsilon/2,$$

and then $f_n - f_m \in \varepsilon H$ if $(m,n) \in \mathbb{N}^2$ satisfies $m \geq n_0, n \geq n_0$. It follows that

$$\forall (m,n) \in \mathbb{N}^2, \quad m \geq n_0,\, n \geq n_0, \quad f_n(t) - f_m(t) \in \varepsilon B, \quad \forall t \in \mathbb{R}.$$

Hence we have proved that

$$\forall \varepsilon, \quad \exists n_0 \in \mathbb{N}, \quad \forall (m,n) \in \mathbb{N}^2, \quad m \geq n_0,\, n \geq n_0, \quad \sup_{t \in \mathbb{R}} \|f_n(t) - f_m(t)\|_B \leq \varepsilon.$$

This means that (f_n) is a Cauchy sequence in the Banach space $C_b(X_B)$ of bounded continuous functions endowed with the sup norm, thus it converges to a continuous function f. Since $f_n(\mathbb{R}) \subset B$ for every n, it follows that $\|f_n(t)\|_B \leq 1$ for every t and every n. Thus, $\|f(t)\|_B \leq 1$ for all $t \in \mathbb{R}$, which implies that f is globally bounded.

To show that $AP(X)$ is a complete subspace of $M_{gb}(X)$, it suffices (cf. [14], Proposition 1, p. 35) to prove that $AP(X)$ is b-closed in $M_{gb}(X)$. This means that if (f_n) is a sequence in $AP(X)$ which b-converges to a function f in $M_{gb}(X)$, it must be shown to be almost periodic. But the first part of the proof shows that the sequence of X_B-valued functions (f_n) converges uniformly to f. Then it follows from Theorem 2.14, that $f \in AP(X_B)$, and then $f \in AP(X)$. Hence $AP(X)$ is b-closed in $M_{gb}(X)$ proving that $AP(X)$ is complete.

Now, suppose that X is regular, and let $f \in M_{gb}(X) \setminus \{0\}$; there exists $t_0 \in \mathbb{R}$ such that $f(t_0) \neq 0$. Consider the map h from $M_{gb}(X)$ to X defined by $h(f) = f(t_0)$. Clearly, h is linear and if B is any bounded disk, we have $h(H^B) = \{f(t_0); f \in H^B\} \subset B$; hence h is bounded. Since X is regular and $f(t_0) \neq 0$, it follows that there exists $\psi \in X^\times$ such that $\psi(f(t_0)) \neq 0$. Then we have $\varphi = \psi \circ h \in M_{gb}(X)^\times$ and $\varphi(f) \neq 0$, proving the regularity of $M_{gb}(X)$ and the theorem. □

5. Operations in spaces of almost periodic functions

5.1. Integration of almost periodic functions valued in a CBS. For a general theory of integration of functions valued in a CBS, we refer to [2] and [7]. Here we give a definition in the spirit of [2, Section 3], adapted to our setting. Let X denote a complete CBS, and let $f : \mathbb{R} \longrightarrow X$ be an almost periodic function. Thus, there exists a Banach disk B such that $f : \mathbb{R} \longrightarrow X_B$ is almost periodic. Then, for every $(a,b) \in \mathbb{R}^2$, one can define the Bochner integral of f from a to b in the Banach space X_B. We denote by $\int_a^b (X_B; f)$ this integral. In fact, it is easy to see that this integral is independent of the chosen Banach disk B.

Definition 5.1. Let $f : \mathbb{R} \longrightarrow X$ be an almost periodic function. For $a, b \in \mathbb{R}$, the Bochner integral of f from a to b is defined by $\int_a^b f(t)dt := \int_a^b (X_B; f)$, where B is any Banach disk such that $f : \mathbb{R} \longrightarrow X_B$ is almost periodic.

We have the following:

Theorem 5.2. *Let X be a complete CBS, and let $f : \mathbb{R} \longrightarrow X$ be an almost periodic function such that $F(\mathbb{R})$ is relatively strictly compact in X, where $F(t) = \int_0^t f(s)ds$. Then $F : \mathbb{R} \longrightarrow X$ is almost periodic.*

Proof. Let B be a bounded Banach disk such that $f : \mathbb{R} \longrightarrow X_B$ is almost periodic. Then f is Bochner integrable as a continuous X_B-valued function. It follows that $F(t) \in X_B$ for every $t \in \mathbb{R}$. Let C be a second bounded Banach disk such that $F(\mathbb{R})$ is a relatively compact set in X_C, and consider the Banach disk $A = B + C$. Then $X_A = X_B + X_C$. The injections $X_B \longrightarrow X_A$ and $X_C \longrightarrow X_A$ being continuous, $f : \mathbb{R} \longrightarrow X_A$ is an almost periodic function, and then $F(\mathbb{R})$ is relatively compact in X_A. Since X_A is a Banach space, it follows (see [20, Theorem 3.2.6], or [22, Theorem 1, p. 58]), that $F : \mathbb{R} \longrightarrow X_A$ is almost periodic. This means that $F : \mathbb{R} \longrightarrow X$ is almost periodic. □

5.2. Derivation of almost periodic functions valued in a CBS. There are many possible definitions for differentiability of functions defined between LCS or CBS, see e.g. [10] and references therein. Our choice of definition is suggested by the boundedness of the range of an almost periodic function. It is in fact a particular case of the notion of locally differentiable functions between normed spaces, see [10, Section 1.5].

Definition 5.3. Let X be a separated CBS. A function $f : \mathbb{R} \longrightarrow X$ will be said to be *globally b-derivable* or shortly *gb-derivable* if there exists a norming disk B such that $f(\mathbb{R}) \subset X_B$ and $f : \mathbb{R} \longrightarrow X_B$ is derivable on $\mathbb{R}$.

If $f : \mathbb{R} \longrightarrow X$ is a function valued in a regular CBS X, it may be interesting to consider its derivability as a function valued in the LCS ${}^tX$.

Definition 5.4. Let X be a regular CBS. A function $f : \mathbb{R} \longrightarrow X$ will be said to be t-derivable at a point $t_0 \in \mathbb{R}$ if the function $f : \mathbb{R} \longrightarrow {}^tX$ is a derivable function at t_0.

The following proposition is a direct consequence of the continuity of the canonical inclusion $X_B \longrightarrow {}^tX$ for a bounded disk B.

Proposition 5.5. *Let X be a regular CBS. Then every gb-derivable function $f : \mathbb{R} \longrightarrow X$ is t-derivable on $\mathbb{R}$.*

Remark 5.6. It is easy to see that if an almost periodic function $f : \mathbb{R} \longrightarrow X$ is *gb*-derivable, then there exists a norming disk B such that $f : \mathbb{R} \longrightarrow X_B$ is a derivable almost periodic function.

Definition 5.7. Let X be a separated CBS. A function $f : \mathbb{R} \to X$ is said to be *uniformly M-continuous* if there exists a norming disk B in X such that $f(\mathbb{R}) \subset X_B$ and $f : \mathbb{R} \to X_B$ is uniformly continuous.

We can state the following version of Bochner's Theorem:

Theorem 5.8 (Bochner's Theorem)**.** *Let X be a completed CBS and f : $\mathbb{R} \to X$ a derivable almost periodic function. If the derivative f' is* uniformly M-continuous, *then it is also almost periodic.*

Proof. It is easy to see that we can find a Banach disk B such that $f : \mathbb{R} \to X_B$ is a derivable almost periodic function with $f(\mathbb{R}) \subset B$, and $f' : \mathbb{R} \to X_B$ is uniformly continuous. Then, the proof will follow from Bochner's Theorem for almost periodic functions valued in a Banach space (see [22, Bochner's Theorem, p. 25]). But we give a direct proof by proving that $f' \in M_{gb}(X)$, and that there exists $(g_n) \subset AP(X)$ such that $g_n \overset{M}{\to} f'$ in $M_{gb}(X)$. To do this, following the proof of the above-mentioned theorem, we set $g_n = f_{\frac{1}{n}} - f$, and for every $t \in \mathbb{R}$

$$g_n(t) - f'(t) = n \int_t^{t+\frac{1}{n}} f'(s)ds - n \int_t^{t+\frac{1}{n}} f'(t)ds,$$

$$g_n(t) - f'(t) = n \int_t^{t+\frac{1}{n}} (f'(s) - f'(t))ds.$$

Let $\varepsilon \in (0, 1)$. Since $f' : \mathbb{R} \to X_B$ is uniformly continuous, there exists $\eta > 0$ such that for every $s, s' \in \mathbb{R}$,

$$|s - s'| < \eta \Rightarrow \|f'(s) - f'(s')\|_B \leq \varepsilon.$$

Fix $m \in \mathbb{N}^*$ such that $m\eta > 1$. It follows that for every $t \in \mathbb{R}$,

$$\|g_n(t) - f'(t)\|_B \leq n \int_t^{t+\frac{1}{n}} \|f'(s) - f'(t)\|_B ds, \quad n \geq m,$$

and then, for every $t \in \mathbb{R}$, we have

$$\|g_n(t) - f'(t)\|_B \leq \varepsilon, \quad n \geq m. \tag{5.1}$$

On the other hand, since $f(\mathbb{R}) \subset B$, we have $g_m(\mathbb{R}) \subset 2mB$, and consequently $f'(\mathbb{R}) \subset (2m+\varepsilon)B$. It follows that $f' \in M_{gb}(X)$. Furthermore, $g_n(t) - f'(t) \in \varepsilon B$ for every $t \in \mathbb{R}$ and $n \geq m$. Thus, $g_n(\mathbb{R}) \subset 2(m+\varepsilon)B$ for $n \geq m$. Since $\varepsilon < 1$, it follows that $f'(\mathbb{R}) \subset 2(m+1)B$ and $g_n(\mathbb{R}) \subset 2(m+1)B$, for every $n \in \mathbb{N}^*$. Hence, f' and $g_n, n \in \mathbb{N}^*$ are in $M_{gb}(X)$. We show that (g_n) converges to f' in the Banach space generated by $H^{2(m+1)B}$ in $M_{gb}(X)$, where the norm is the Minkowski functional of $H^{2(m+1)B}$. We notice that $H^{2(m+1)B} = 2(m+1)H^B$; then, the two disks generate the same subspace in $M_{gb}(X)$, and their Minkowski functionals are equivalent. Hence, it suffices to show that (g_n) converges to f' in the Banach space generated by H^B, endowed with the Minkowski functional of H^B. In fact, this follows from (5.1), which means that $g_n - f' \in \varepsilon H^B$ for $n \geq m$, since $g_n - f' \in C_b(X_B)$. Thus, $g_n \overset{M}{\to} f'$ in $M_{gb}(X)$. Since $g_n \in AP(X)$ and $AP(X)$ is b-closed in $M_{gb}(X)$, it follows that $f' \in AP(X)$. □

5.3. Nonlinear operations on almost periodic functions. In the sequel, X and Y denote convex bornological spaces. For a topological vector space E, we denote by $\mathcal{V}_E(0)$ the set of neighborhoods of the zero. We introduce the following two definitions.

Definition 5.9. Let E be a TVS, and M be an arbitrary set. A function $f : \mathbb{R} \times M \longrightarrow E$ is said to be *almost periodic with parameters in M* if the following conditions are satisfied:

(i) $\forall x \in M, f_x : \mathbb{R} \longrightarrow E$ is continuous, where $f_x(t) = f(t, x)$;

(ii) $\forall U$, a neighborhood of zero in E, $\forall x \in M$, $\exists P_x$, a relatively dense set in $\mathbb{R}$ such that $f(t+\tau, x) - f(t, x) \in U, \forall (t, \tau) \in \mathbb{R} \times P_x$.

Definition 5.10. Let X be a separated CBS, and M an arbitrary set. A function $f : \mathbb{R} \times M \longrightarrow X$ is said to be *almost periodic with parameters in M* if there exists a bounded norming disk B in X such that $f(\mathbb{R} \times M) \subset X_B$, and $f : \mathbb{R} \times M \longrightarrow X_B$ is almost periodic with parameters in M.

We have the following fundamental result:

Theorem 5.11. *Let E and F be two separated topological vector spaces. Let $\phi : \mathbb{R} \longrightarrow E$ be almost periodic, and $f : \mathbb{R} \times \phi(\mathbb{R}) \longrightarrow F$ be almost periodic with parameters in $\phi(\mathbb{R})$. Assume that*

$$\begin{aligned} &\forall V \in \mathcal{V}_F(0), \quad \exists U \in \mathcal{V}_E(0), \quad f(t,x) - f(t,y) \in V, \\ &\forall t \in \mathbb{R}, \quad \forall x, y \in \phi(\mathbb{R}), \quad x - y \in U. \end{aligned} \tag{5.2}$$

Then, the function $\psi : \mathbb{R} \longrightarrow F$ defined by $\psi(t) = f(t, \phi(t))$ is almost periodic.

PROOF. First, we show that ψ is continuous. For this, let $t_0 \in \mathbb{R}$, and let $V \in \mathcal{V}_F(0)$. Take a balanced element W of $\mathcal{V}_F(0)$ such that $W + W \subset V$. From (5.2), there exists a balanced element U of $\mathcal{V}_E(0)$ such that

$$f(t,x) - f(t,y) \in W, \quad \forall x, y \in \phi(\mathbb{R}), \quad x - y \in U. \tag{5.3}$$

We have

$$\psi(t) - \psi(t_0) = [f(t,\phi(t)) - f(t,\phi(t_0))] + [f(t,\phi(t_0)) - f(t_0,\phi(t_0))]. \tag{5.4}$$

From the continuity of $f_{\phi(t_0)}$ and ϕ at t_0, there exists $\eta > 0$ such that if $|t-t_0| < \eta$, then

$$f(t,\phi(t_0)) - f(t_0,\phi(t_0)) \in W \quad \text{and} \quad \phi(t) - \phi(t_0) \in U. \tag{5.5}$$

It follows from (5.3) and (5.5) that $f(t,\phi(t)) - f(t,\phi(t_0)) \in W$. Hence, using (5.4), we get $\psi(t) - \psi(t_0) \in W + W \subset V$, proving that F is continuous.
We prove the almost periodicity of ψ. For this, assume that the conditions of the theorem are fulfilled. Let $V \in \mathcal{V}_F(0)$. Choose a balanced element W of $\mathcal{V}_F(0)$ such that

$$W + W + W + W \subset V.$$

From (5.2), there exists a balanced element U of $\mathcal{V}_E(0)$ such that

$$f(t,x) - f(t,y) \in W, \quad \forall t \in \mathbb{R}, \quad \forall x, y \in \phi(\mathbb{R}), \quad x - y \in U. \tag{5.6}$$

On the other hand, since ϕ is almost periodic, there exists a relatively dense set P_0 in $\mathbb{R}$ such that

$$\phi(t) - \phi(t+\tau) \in U, \quad \forall (t,\tau) \in \mathbb{R} \times P_0. \tag{5.7}$$

Since $\phi(\mathbb{R})$ is totally bounded in E, there exist $x_1, \ldots, x_p$ in $\phi(\mathbb{R})$ such that

$$\phi(\mathbb{R}) \subset \bigcup_{i=1}^{p} (x_i + U).$$

Hence, for every $s \in \mathbb{R}$, there exists $j \in \{1, \ldots, p\}$ such that $\phi(s) \in x_j + U$, i.e., $\phi(s) - x_j \in U$. Then, we deduce from (5.6) that

$$f(t,\phi(s)) - f(t,x_j) \in W, \quad \forall t \in \mathbb{R}. \tag{5.8}$$

We have

$$\psi(t) - \psi(s) = [f(t,\phi(t)) - f(t,\phi(s))] + [f(t,\phi(s)) - f(s,\phi(s))], \tag{5.9}$$

and

$$\begin{aligned} f(t,\phi(s)) - f(s,\phi(s)) = {} & [f(t,\phi(s)) - f(t,x_j)] \\ & + [f(t,x_j) - f(s,x_j)] + [f(s,x_j) - f(s,\phi(s))]. \end{aligned}$$

Since $\phi(t)-x_j$ (and also $x_j-\phi(t)$) is in U, it follows that there exists a relatively dense set P_j in $\mathbb{R}$ such that

$$[f(t,\phi(s))-f(t,x_j)]+[f(s,x_j)-f(s,\phi(s))]\in W+W. \tag{5.10}$$

From Definition 5.9, (ii), for each $j\in\{1,\dots,p\}$ there exists a relatively dense set P_j in $\mathbb{R}$ such that

$$f(t,x_j)-f(t+\tau,x_j)\in W,\quad \forall (t,\tau)\in\mathbb{R}\times P_j. \tag{5.11}$$

Denote by P a relatively dense set in $\mathbb{R}$ such that $P\subset\bigcap_{i=0}^{p}P_i$. We have that $\phi(t)-\phi(t+\tau)\in U$, and then $f(t,\phi(t))-f(t,\phi(t+\tau))\in W$ for every $(t,\tau)\in\mathbb{R}\times P$. It follows from (5.9), (5.10), (5.11) with $s=t+\tau$ that

$$\psi(t)-\psi(t+\tau)\in W+W+W+W\subset V,\quad \forall (t,\tau)\in\mathbb{R}\times P. \tag{5.12}$$

Hence ψ is almost periodic. □

Now, we can state the following two results on nonlinear operations on almost periodic functions valued in a CBS.

Corollary 5.12. *Let X and Y be two separated CBS, where X is regular. Let $\phi:\mathbb{R}\longrightarrow X$ be topologically almost periodic, and $f:\mathbb{R}\times\phi(\mathbb{R})\longrightarrow Y$ be almost periodic with parameters in $\phi(\mathbb{R})$. Assume that there exists a bounded norming disk C in Y such that $f(\mathbb{R}\times\phi(\mathbb{R}))\subset Y_C$, and*

$$\forall\varepsilon>0,\ \exists U\in\mathcal{V}_{{}^tX}(0),\quad \|f(t,x)-f(t,y)\|_C\le\varepsilon,\quad \forall t\in\mathbb{R},\ \forall x,y\in\phi(\mathbb{R}),\ x-y\in U.$$

Then, the function $F:\mathbb{R}\longrightarrow Y$ defined by $F(t)=f(t,\phi(t))$ is almost periodic.

Proof. Since X is regular, ${}^tX$ is a separated LCS. Taking $E={}^tX$ and $F=Y_A$, where $A=B+C$, it is easy to see from the continuity of the embeddings $Y_B\longrightarrow X_A$ and $X_C\longrightarrow X_A$ that the hypotheses of Theorem 5.11 are satisfied, which concludes the proof. □

Corollary 5.13. *Let X and Y be two complete convex bornological spaces. Let $\phi:\mathbb{R}\longrightarrow X$ be almost periodic, and $f:\mathbb{R}\times\phi(\mathbb{R})\longrightarrow Y$ be almost periodic with parameters in $\phi(\mathbb{R})$. Let A be a bounded disk in X such $\phi:\mathbb{R}\longrightarrow X_A$ is almost periodic, and let B be a bounded disk in Y as in Definition 5.10. Assume that*

$$\forall\varepsilon>0,\ \exists r>0,\quad \|f(t,x)-f(t,y)\|_B\le\varepsilon,\quad \forall t\in\mathbb{R},\ \forall x,y\in\phi(\mathbb{R}),\ \|x-y\|_A\le r.$$

Then, the function $\psi:\mathbb{R}\longrightarrow Y$ defined by $\psi(t)=f(t,\phi(t))$ is almost periodic.

Proof. If we take $E = X_A$ and $F = Y_B$, then ϕ and f satisfy all the hypotheses of Theorem 5.11. Thus, we can conclude that function ψ is almost periodic. □

Remark 5.14. To conclude, we want to point out that this last section enables one to deal with differential equations of the form $u'(t) = Tu(t)+f(t, u(t)) = v(t)$. Here T is a bounded linear operator on $AP(X)$, $v \in AP(X)$, and $f : \mathbb{R}\times X \longrightarrow X$ with X, a complete CBS.

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