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A Gauss–Kuzmin-type theorem for θ-expansions

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Abstract. A generalization of the regular continued fractions was given by S. Chakraborty and B. V. Rao in 2004. For the transformation which generates this expansion and its invariant measure, the Perron–Frobenius operator is given and studied. For this expansion, we apply the method of Szüsz from 1961 and obtain the solution of its Gauss–Kuzmin-type theorem.

1. Introduction

S. Chakraborty and B. V. Rao [3] considered a continued fraction expansion of a number in terms of an irrational $\theta \in (0, 1)$. This new expansion of positive reals, different from the regular continued fraction expansion, is called the θ-expansion.

The purpose of this paper is to give some ergodic properties, and to solve a Gauss–Kuzmin problem for θ-expansions. In order to solve the Gauss–Kuzmin problem, we apply the method of Szüsz [14], [21]. First, we outline the historical framework of this problem. In Section 1.2, we present the current framework. In Section 1.3, we review known results. In Section 1.4, the main theorem will be shown.

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1.1. Gauss' Problem and its progress. Any irrational $0 \leq x < 1$ can be written as the infinite regular continued fraction

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} := [0; a_1, a_2, a_3, \ldots], \tag{1.1}$$

where $a_n \in \mathbb{N}_+ := \{1, 2, 3, \ldots\}$ [7]. Define the *regular continued fraction* (or *Gauss*) *transformation* τ on the unit interval $I := [0, 1]$ by

$$\tau(x) = \begin{cases} \dfrac{1}{x} - \left\lfloor \dfrac{1}{x} \right\rfloor & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases} \tag{1.2}$$

where $\lfloor \cdot \rfloor$ denotes the floor (or entire) function. With respect to the asymptotic behavior of iterations $\tau^n = \tau \circ \cdots \circ \tau$ (with τ repeated n times) of τ, in 1800 Gauss wrote (in modern notation) that

$$\lim_{n \to \infty} \lambda\left(\tau^n \leq x\right) = \frac{\log(1 + x)}{\log 2}, \quad x \in I, \tag{1.3}$$

where λ denotes the Lebesgue measure on I. In 1812, Gauss asked Laplace [2] to estimate the *n-th error term* $e_n(x)$ defined by

$$e_n(x) := \lambda(\tau^{-n}[0, x]) - \frac{\log(1 + x)}{\log 2}, \quad n \geq 1, \ x \in I. \tag{1.4}$$

This has been called *Gauss' Problem.* In 1928, Kuzmin [9] showed that $e_n(x) = \mathcal{O}(q^{\sqrt{n}})$ as $n \to \infty$, uniformly in x with some (unspecified) $0 < q < 1$. Independently, Lévy [12] proved in 1929 that $|e_n(x)| \leq q^n$ for $n \in \mathbb{N}_+$, $x \in I$, with $q = 0.67157\ldots$. For such historical reasons, the *Gauss–Kuzmin–Lévy theorem* is regarded as the first basic result in the rich metrical theory of continued fractions.

Apart from the regular continued fraction expansion, very many other continued fraction expansions were studied [14], [16]. By such a development, generalizations of these problems for non-regular continued fractions are also called as the *Gauss–Kuzmin problem* and the *Gauss–Kuzmin–Lévy problem* [8], [10], [11], [17], [18], [19], [20].

1.2. θ-expansions as dynamical system. In this paper, we consider a generalization of the Gauss transformation, and prove an analogous result.

This transformation was studied by S. Chakraborty and B. V. Rao, and P. S. Chakraborty and A. Dasgupta in [3] and [4], respectively, and by Sebe and Lascu in [20].

Fix an irrational $\theta \in (0, 1)$. In [3], the authors showed that any $x \in (0, \theta)$ can be written in the form

$$x = \cfrac{1}{a_1\theta + \cfrac{1}{a_2\theta + \cfrac{1}{a_3\theta + \ddots}}} := [0; a_1\theta, a_2\theta, \ldots], \tag{1.5}$$

where a_n's are non-negative integers. We will simply write (1.5)

$$x := [a_1\theta, a_2\theta, \ldots]. \tag{1.6}$$

Such a_n's are called *incomplete quotients* (or *continued fraction digits*) of x with respect to the expansion in (1.5) in this paper.

This continued fraction is treated as the following dynamical systems.

Definition 1.1. Fix an irrational $\theta \in (0, 1)$ and $m \in \mathbb{N}_+$ such that $\theta^2 = 1/m$.

(i) The measure-theoretical dynamical system $([0, \theta], \mathcal{B}_{[0,\theta]}, T_\theta)$ is defined as follows: $\mathcal{B}_{[0,\theta]}$ denotes the σ-algebra of all Borel subsets of $[0, \theta]$, and T_θ is the transformation

$$T_\theta : [0, \theta] \to [0, \theta]; \qquad T_\theta(x) := \begin{cases} \dfrac{1}{x} - \theta \left\lfloor \dfrac{1}{x\theta} \right\rfloor & \text{if } x \in (0, \theta], \\ 0 & \text{if } x = 0. \end{cases} \tag{1.7}$$

(ii) In addition to (i), we write $([0, \theta], \mathcal{B}_{[0,\theta]}, \gamma_\theta, T_\theta)$ as $([0, \theta], \mathcal{B}_{[0,\theta]}, T_\theta)$, with the following probability measure γ_θ on $([0, \theta], \mathcal{B}_{[0,\theta]})$:

$$\gamma_\theta(A) := \frac{1}{\log(1 + \theta^2)} \int_A \frac{\theta dx}{1 + \theta x}, \quad A \in \mathcal{B}_{[0,\theta]}. \tag{1.8}$$

Define the *quantized index map* $\eta : [0, \theta] \to \mathbb{N}$ by

$$\eta(x) := \begin{cases} \left\lfloor \dfrac{1}{x\theta} \right\rfloor & \text{if } x \neq 0, \\ \infty & \text{if } x = 0. \end{cases} \tag{1.9}$$

By using T_θ and η, the sequence $(a_n)_{n\in\mathbb{N}_+}$ in (1.5) is obtained as follows:

$$a_n(x) = \eta(T_\theta^{n-1}(x)), \quad n \geq 1, \tag{1.10}$$

with $T_\theta^0(x) = x$.

In this way, T_θ algorithmically generates the θ-expansion.

Proposition 1.1. *Let $([0,\theta], \mathcal{B}_{[0,\theta]}, \gamma_\theta, T_\theta)$ be as in Definition 1.1* (ii).

(i) *$([0,\theta], \mathcal{B}_{[0,\theta]}, \gamma_\theta, T_\theta)$ is ergodic.*

(ii) *The measure γ_θ is invariant under T_θ, that is, $\gamma_\theta(A) = \gamma_\theta(T_\theta^{-1}(A))$ for any $A \in \mathcal{B}_{[0,\theta]}$.*

Proof. See Section 8 in [3]. □

By Proposition 1.1 (ii), $([0,\theta], \mathcal{B}_{[0,\theta]}, \gamma_\theta, T_\theta)$ is a "dynamical system" in the sense of [1, Definition 3.1.3].

1.3. Known results and applications. For θ-expansions, we show known results and their applications in this subsection.

Let $0 < \theta < 1$ and $m \in \mathbb{N}_+$ such that $\theta^2 = 1/m$. In what follows, the stated identities hold for all n in case x has an infinite θ-expansion, and they hold for $n \leq k$ in case x has a finite θ-expansion terminating at the k-th stage [3].

To this end, define real functions $p_n(x)$ and $q_n(x)$, for $n \in \mathbb{N}_+$, by

$$p_n(x) := a_n(x)\theta p_{n-1}(x) + p_{n-2}(x), \tag{1.11}$$
$$q_n(x) := a_n(x)\theta q_{n-1}(x) + q_{n-2}(x), \tag{1.12}$$

with $p_{-1}(x) := 1$, $p_0(x) := 0$, $q_{-1}(x) := 0$ and $q_0(x) := 1$. By using (1.11) and (1.12), we can verify that

$$x = \frac{p_n(x) + T_\theta^n(x)p_{n-1}(x)}{q_n(x) + T_\theta^n(x)q_{n-1}(x)}, \quad n \geq 0, \tag{1.13}$$

and

$$x - \frac{p_n(x)}{q_n(x)} = \frac{(-1)^{n+1}T_\theta^n(x)}{q_n(x)(q_n(x) + T_\theta^n(x)q_{n-1}(x))}, \quad n \geq 0. \tag{1.14}$$

Putting $\mathbb{N}_m := \{m, m+1, \ldots\}$, $m \in \mathbb{N}_+$, the incomplete quotients a_n, $n \in \mathbb{N}_+$, take positive integer values in $\mathbb{N}_m$.

We now introduce a partition of the interval $[0,\theta]$ which is natural to the θ-expansions. Such a partition is generated by the *fundamental intervals* (or *cylinders*) of rank n. For any $n \in \mathbb{N}_+$ and $i^{(n)} = (i_1, \ldots, i_n) \in \mathbb{N}_m^n$, define the *fundamental interval associated with* $i^{(n)}$ by

$$I\left(i^{(n)}\right) = \{x \in [0, \theta] : a_k(x) = i_k \text{ for } k = 1, \ldots, n\}, \tag{1.15}$$

where $I\left(i^{(0)}\right) = [0, \theta]$.

Using the ergodicity of T_θ and Birkhoff's ergodic theorem [5], a number of results were obtained.

For q_n in (1.11) it was shown that its asymptotic growth rate β is defined as

$$\beta = \lim_{n \to \infty} \frac{1}{n} \log q_n. \tag{1.16}$$

This is a Lévy-type result. Although the calculation algorithm is correct, Chakraborty and Rao [3] misspelled the expression of β. They should write that for almost all $x \in [0, \theta)$

$$\beta = \frac{-1}{\log(1+\theta^2)} \int_0^\theta \frac{\theta \log x}{1+\theta x} \mathrm{d}x. \tag{1.17}$$

They also give a Khintchin-type result, i.e., the asymptotic value of the arithmetic mean of $a_1, a_2, \ldots, a_n$, where a_n's are given in (1.10). We have

$$\lim_{n \to \infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = \infty. \tag{1.18}$$

In [20], Sebe and Lascu proved a Gauss–Kuzmin theorem for the transformation T_θ. In order to solve the problem, they applied the theory of random systems with complete connections (RSCC) by Iosifescu [6]. We recall that a random system with complete connections is a quadruple

$$\left\{\left([0, \theta], \mathcal{B}_{[0,\theta]}\right), \left(\mathbb{N}_m, \mathcal{P}(\mathbb{N}_m)\right), u, P\right\}, \tag{1.19}$$

where $u : [0, \theta] \to [0, \theta]$,

$$u(x) = u_i(x) := \frac{1}{i\theta + x}, \tag{1.20}$$

and P is a transition probability function from $\left([0, \theta], \mathcal{B}_{[0,\theta]}\right)$ to $(\mathbb{N}_m, \mathcal{P}(\mathbb{N}_m))$ given by

$$P(x) = P_i(x) := \frac{\theta x + 1}{(x + i\theta)(x + (i+1)\theta)}. \tag{1.21}$$

Here $\mathcal{P}(\mathbb{N}_m)$ denotes the power set of $\mathbb{N}_m$. Also, the associated Markov operator of RSCC (1.19) is denoted by U and has the transition probability function

$$Q(x, A) = \sum_{i \in W(x,A)} P_i(x), \quad x \in [0, \theta],\ A \in \mathcal{B}_{[0,\theta]}, \tag{1.22}$$

where $W(x, A) = \{i \in \mathbb{N}_m : u_i(x) \in A\}$.

Using the asymptotic and ergodic properties of operators associated with RSCC (1.19), i.e., the ergodicity of RSCC, they obtained a convergence rate result for the Gauss–Kuzmin-type theorem.

For more details about using RSCC in solving Gauss–Kuzmin–Lévy-type theorems, see [10], [17], [18], [19], [20].

1.4. Main theorem. We show our main theorem in this subsection. We mention that applying the Szüsz method, we obtain a better rate of convergence than that obtained in [20].

If $\theta \in (0,1)$ and $m \in \mathbb{N}_+$ such that $\theta^2 = 1/m$, the measure γ_θ in (1.8) is the unique absolutely continuous invariant measure for the map T_θ in (1.7). In particular, if one iterates any other absolutely continuous invariant measure repeatedly by T_θ, it will converge exponentially to γ_θ.

Let μ be a non-atomic probability measure on $\mathcal{B}_{[0,\theta]}$, and define

$$F_n(x) := \mu(T_\theta^n \leq x), \quad x \in [0,\theta], \ n \in \mathbb{N}_+, \tag{1.23}$$

$$F(x) := \lim_{n\to\infty} F_n(x), \quad x \in I, \tag{1.24}$$

with $F_0(x) = \mu([0,x])$.

Then the following holds.

Theorem 1.1 (A Gauss–Kuzmin-type theorem)**.** *Let T_θ and F_n be as in (1.7) and (1.23), respectively. Then there exists a constant $0 < q < \theta$ such that F_n can be written as*

$$F_n(x) = \frac{\log(1+\theta x)}{\log(1+\theta^2)} + \mathcal{O}(q^n) \tag{1.25}$$

uniformly with respect to $x \in [0,\theta]$.

Remark 1.1. From (1.25), we see that

$$F(x) = \gamma_\theta([0,x]). \tag{1.26}$$

In fact, the Gauss–Kuzmin theorem estimates the error

$$e_\theta(x) := e_\theta(x,\mu) = \mu(T_\theta^n \leq x) - \gamma_\theta([0,x]), \quad x \in [0,\theta]. \tag{1.27}$$

The rest of the paper is organized as follows. In Section 2, we derive the associated Perron–Frobenius operator under different probability measures on $([0,\theta], \mathcal{B}_{[0,\theta]})$. We treat the Perron–Frobenius operator of $([0,\theta], \mathcal{B}_{[0,\theta]}, \gamma_\theta, T_\theta)$, and derive its asymptotic behavior. In Section 3, we prove Theorem 1.1 for θ-expansions. In Section 3.1, we give the necessary results used to prove the Gauss–Kuzmin theorem. The essential argument of the proof is the Gauss–Kuzmin-type equation. Using some properties of the Perron–Frobenius operator of T_θ under γ_θ, we give some results concerning the behavior of the derivative of $\{F_n\}$ in (1.23), which will allow us to complete the proof of Theorem 1.1 in Section 3.2.

2. The Perron–Frobenius operator of T_θ under γ_θ

Let $([0,\theta], \mathcal{B}_{[0,\theta]}, \gamma_\theta, T_\theta)$ be as in Definition 1.1. In this section, we derive its Perron–Frobenius operator.

Let μ be a probability measure on $([0,\theta], \mathcal{B}_{[0,\theta]})$ such that $\mu((T_\theta)^{-1}(A)) = 0$, whenever $\mu(A) = 0$ for $A \in \mathcal{B}_{[0,\theta]}$. Since μ is non-atomic and T_θ is ergodic, For example, this condition is satisfied if T_θ is μ-preserving, that is, $\mu(T_\theta)^{-1} = \mu$. Let $L^1([0,\theta],\mu) := \{f : [0,\theta] \to \mathbb{C} : \int_0^\theta |f| \mathrm{d}\mu < \infty\}$. The *Perron–Frobenius operator* U of $([0,\theta], \mathcal{B}_{[0,\theta]}, \mu, T_\theta)$ is defined as the bounded linear operator on the Banach space $L^1([0,\theta],\mu)$ such that the following holds:

$$\int_A Uf \,\mathrm{d}\mu = \int_{(T_\theta)^{-1}(A)} f \,\mathrm{d}\mu \quad \text{for all} \quad A \in \mathcal{B}_{[0,\theta]},\ f \in L^1([0,\theta],\mu). \tag{2.1}$$

For more details, see [1], [7] or Appendix A in [10].

Proposition 2.1. *Let $([0,\theta], \mathcal{B}_{[0,\theta]}, \gamma_\theta, T_\theta)$ be as in Definition 1.1, and let U denote its Perron–Frobenius operator.*

(i) *The following equation holds:*

$$Uf(x) = \sum_{i \geq m} P_i(x)\, f(u_i(x)), \quad m \in \mathbb{N}_+,\ f \in L^1([0,\theta],\gamma_\theta), \tag{2.2}$$

where P_i and u_i, $i \geq m$, are as in (1.21) and (1.20), respectively.

(ii) *Let μ be a probability measure on $\mathcal{B}_{[0,\theta]}$. Assume that μ is absolutely continuous with respect to the Lebesgue measure λ_θ (and denote $\mu \ll \lambda_\theta$, i.e., if $\mu(A) = 0$ for every set A with $\lambda_\theta(A) = 0$), and let $h = d\mu/d\lambda_\theta$ a.e. in $[0,\theta]$. Then the following holds:*

(a) *The Perron–Frobenius operator S of T_θ under μ is given a.e. in $[0,\theta]$ by the equation*

$$Sf(x) = \frac{1}{h(x)} \sum_{i \geq m} \frac{1}{(i\theta + x)} f(u_i(x))h(u_i(x)) \tag{2.3}$$

$$Sf(x) = \frac{Ug(x)}{(1+\theta x)h(x)}, \quad f \in L^1([0,\theta],\mu), \tag{2.4}$$

where $g(x) := (1+\theta x)f(x)h(x)$, $x \in [0,\theta]$. *In addition, the* n*-th power* S^n *of* S *is given as follows:*

$$S^n f(x) = \frac{U^n g(x)}{(1+\theta x)h(x)}, \tag{2.5}$$

for any $f \in L^1([0,\theta],\mu)$ *and any* $n \in \mathbb{N}_+$.

(b) *The Perron–Frobenius operator* V *of* T_θ *under* λ_θ *is given a.e. in* $[0,\theta]$ *by the equation*

$$Vf(x) = \sum_{i \geq m} \frac{1}{(i\theta + x)^2} f(u_i(x)), \;\; f \in L^1([0,\theta],\lambda_\theta). \tag{2.6}$$

The powers of V *are given a.e. in* $[0,\theta]$ *by the equation*

$$V^n f(x) = \frac{U^n g(x)}{1+\theta x}, \quad f \in L^1([0,\theta],\lambda_\theta),\; n \in \mathbb{N}_+, \tag{2.7}$$

where $g(x) := (1+\theta x)f(x)$, $x \in [0,\theta]$.

(c) *For any* $n \in \mathbb{N}_+$ *and* $A \in \mathcal{B}_{[0,\theta]}$, *we have*

$$\mu\left((T_\theta)^{-n}(A)\right) = \int_A U^n f(x) \mathrm{d}\gamma_\theta(x), \tag{2.8}$$

where $f(x) := (\log(1+\theta^2))\frac{1+x\theta}{\theta^2}h(x)$, $x \in [0,\theta]$.

Proof. See Appendix. $\square$

For a function $f : [0,\theta] \to \mathbb{C}$, define the *variation* $\mathrm{var}_A f$ of f on a subset A of $[0,\theta]$ by

$$\mathrm{var}_A f := \sup \sum_{i=1}^{k-1} |f(t_{i+1}) - f(t_i)|, \tag{2.9}$$

where the supremum being taken over $t_1 < \cdots < t_k$, $t_i \in A$, $1 \leq i \leq k$, and $k \geq 2$. We write simply $\mathrm{var} f$ for $\mathrm{var}_{[0,\theta]} f$. Let $L^\infty([0,\theta])$ denote the collection of all bounded measurable functions $f : [0,\theta] \to \mathbb{C}$. It is known that $L^\infty([0,\theta]) \subset L^1([0,\theta])$. Let $L([0,\theta])$ denote the Banach space of all complex-valued Lipschitz continuous functions on $[0,\theta]$ with the following norm $\|\cdot\|_L$:

$$\|f\|_L := \sup_{x\in[0,\theta]} |f(x)| + s(f), \tag{2.10}$$

with

$$s(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}, \quad f \in L([0,\theta]). \tag{2.11}$$

In the following proposition, we show that the operator U in (2.2) preserves monotonicity and enjoys a contraction property for Lipschitz continuous functions.

Proposition 2.2. *Let U be as in* (2.2)*.*

(i) *Let $f \in L^{\infty}([0,\theta])$. Then the following holds:*

  (a) *If f is non-decreasing (non-increasing), then Uf is non-increasing (non-decreasing).*

  (b) *If f is monotone, then*

$$\mathrm{var}\,(Uf) \leq k_m \cdot \mathrm{var} f \quad \textit{where} \quad k_m := \frac{1}{m+1}. \tag{2.12}$$

(ii) *For any $f \in L([0,\theta])$, we have*

$$s(Uf) \leq q \cdot s(f), \tag{2.13}$$

where

$$q := m \left(\sum_{i \geq m} \left(\frac{m}{i^3(i+1)} + \frac{i+1-m}{i(i+1)^3} \right) \right) \tag{2.14}$$

Proof. See Appendix. □

3. Proof of Theorem 1.1

In this section, we prove our main theorem applying the method of Szüsz [21]. Let $\theta \in (0,1)$ and $m \in \mathbb{N}_+$ such that $\theta^2 = 1/m$.

3.1. Necessary lemmas. In this subsection, we show some lemmas. First, we show that $\{F_n\}$ in (1.23) satisfy a Gauss–Kuzmin-type equation.

Lemma 3.1. *For functions $\{F_n\}$ in* (1.23)*, the following Gauss–Kuzmin-type equation holds:*

$$F_{n+1}(x) = \sum_{i \geq m} \left\{ F_n\left(\frac{1}{i\theta}\right) - F_n\left(\frac{1}{i\theta + x}\right) \right\}, \tag{3.1}$$

for $x \in [0,\theta]$ and $n \in \mathbb{N}$.

Proof. Let $I_n = \{x \in [0, \theta] : T_\theta^n(x) \leq x\}$ and

$$I_{n,i} = \left\{ x \in I_n : \frac{1}{i\theta + x} < T_\theta^n(x) < \frac{1}{i\theta} \right\}. \tag{3.2}$$

From (1.7) and (1.10), we see that

$$T_\theta^n(x) = \frac{1}{a_{n+1}\theta + T_\theta^{n+1}(x)}, \quad n \in \mathbb{N}_+. \tag{3.3}$$

From the definition of $I_{n,i}$ and (3.3) it follows that for any $n \in \mathbb{N}$, $I_{n+1} = \bigcup_{i \geq m} I_{n,i}$. Then (3.1) holds because $F_{n+1}(x) = \mu(I_{n+1})$ and

$$\mu(I_{n,i}) = F_n\left(\frac{1}{i\theta}\right) - F_n\left(\frac{1}{i\theta + x}\right). \tag{3.4}$$

$\square$

Remark 3.1. Suppose that F_0' exists everywhere in $[0, \theta]$ and is bounded (μ has bounded density). Then by induction, we have that F_n' exists and it is bounded for any $n \in \mathbb{N}_+$. This allows us to differentiate (3.1) term by term, obtaining

$$F_{n+1}'(x) = \sum_{i \geq m} \frac{1}{(i\theta + x)^2} F_n'\left(\frac{1}{i\theta + x}\right). \tag{3.5}$$

We introduce functions $\{f_n\}$ as follows:

$$f_n(x) := (1 + \theta x) F_n'(x), \quad x \in [0, \theta], \ n \in \mathbb{N}. \tag{3.6}$$

Then (3.5) is

$$f_{n+1}(x) = \sum_{i \geq m} P_i(x) f_n(u_i(x)), \tag{3.7}$$

where $P_i(x)$ and $u_i(x)$ are given in (1.21) and (1.20), respectively. By Proposition 2.1 (i), we have that $f_{n+1}(x) = U f_n(x)$.

Lemma 3.2. *For $\{f_n\}$ in (3.6), define $M_n := \max_{x \in [0,\theta]} |f_n'(x)|$. Then*

$$M_{n+1} \leq q \cdot M_n, \tag{3.8}$$

where q is the constant in (2.14).

Proof. Since

$$P_i(x) = \frac{1}{\theta}\left[\frac{1-i\theta^2}{x+i\theta} - \frac{1-(i+1)\theta^2}{x+(i+1)\theta}\right], \tag{3.9}$$

we have

$$f'_{n+1}(x) = \sum_{i\geq m} \frac{1-(i+1)\theta^2}{(x+i\theta)(x+(i+1)\theta)^3} f'_n(\alpha_i) - \sum_{i\geq m} \frac{P_i(x)}{(x+i\theta)^2} f'_n(u_i(x)), \tag{3.10}$$

where $u_{i+1}(x) < \alpha_i < u_i(x)$. Now (3.10) implies

$$M_{n+1} \leq M_n \cdot \max_{x\in[0,\theta]} \left(\sum_{i\geq m} \frac{(i+1)\theta^2-1}{(x+i\theta)(x+(i+1)\theta)^3} + \sum_{i\geq m} \frac{P_i(x)}{(x+i\theta)^2}\right). \tag{3.11}$$

We now must calculate the maximum value of the sums in this expression. First, we note that

$$\frac{(i+1)\theta^2-1}{(x+i\theta)(x+(i+1)\theta)^3} \leq m^2\frac{(i+1)\theta^2-1}{i(i+1)^3}, \tag{3.12}$$

where we used that $0 \leq x \leq \theta$. Next, let

$$h(x) := \sum_{i\geq m} \frac{P_i(x)}{(x+i\theta)^2}. \tag{3.13}$$

By Proposition 2.1 (i) and Proposition 2.2 (i)(a), we have that function h is decreasing for $x \in [0,\theta]$. Hence, $h(x) \leq h(0)$. This leads to

$$\sum_{i\geq m} \frac{P_i(x)}{(x+i\theta)^2} \leq \cdot \sum_{i\geq m} \frac{1}{i^3(i+1)}. \tag{3.14}$$

The relations (3.11), (3.12) and (3.14) imply (3.8) and that q is as in (2.14). $\square$

3.2. Proof of Theorem 1.1. For $\{F_n\}$ in (1.23), we introduce a function $R_n(x)$ such that

$$F_n(x) = \frac{\log(1+\theta x)}{\log(1+\theta^2)} + R_n(x). \tag{3.15}$$

Because $F_n(0) = 0$ and $F_n(\theta) = 1$, we have $R_n(0) = R_n(\theta) = 0$. To prove Theorem 1.1, we have to show the existence of a constant $0 < q < \theta$ such that

$$R_n(x) = \mathcal{O}(q^n). \tag{3.16}$$

For $\{f_n\}$ in (3.6), if we can show that $f_n(x) = \frac{\theta}{\log(1+\theta^2)} + \mathcal{O}(q^n)$, then its integration will show the equation (1.25).

To demonstrate that $f_n(x)$ has this desired form, it suffices to prove the following lemma.

Lemma 3.3. *For any $x \in [0,\theta]$, there exists a constant $q := q(x)$ with $0 < q < \theta$ such that*

$$f_n'(x) = \mathcal{O}(q^n). \tag{3.17}$$

Proof. Let q be as in (2.14). Using Lemma 3.2, to show (3.17), it is enough to prove that $q < \theta$. Since in the particular cases studied for $m := 10$ we have $\theta = 0.316228$ and $q = 0.0533201$, and for $m := 17$ we have $\theta = 0.242536$ and $q = 0.0305636$, we may assume that $q < \theta$ for any $m \in \mathbb{N}$. □

Appendix A. Proofs of propositions

We prove Propositions 2.1 and 2.2 in this section.

Proof of Proposition 2.1. (i) See Proposition 14 (i) in [20].

(ii)(a) Let $T_{\theta,i}$ denote the restriction of T_θ to the subinterval $I_i := \left(\frac{1}{\theta(i+1)}, \frac{1}{\theta i}\right]$, $i \geq m$, $m \in \mathbb{N}$, that is,

$$T_{\theta,i}(x) = \frac{1}{x} - \theta i, \quad x \in I_i. \tag{A.1}$$

Let $C(A) := T_\theta^{-1}(A)$ and $C_i(A) := (T_{\theta,i})^{-1}(A)$ for $A \in \mathcal{B}_{[0,\theta]}$. Since $C(A) = \bigcup_i C_i(A)$, and $C_i \cap C_j$ is a null set when $i \neq j$, we have

$$\int_{C(A)} f \,\mathrm{d}\gamma_\theta = \sum_{i\geq m} \int_{C_i(A)} f \,\mathrm{d}\gamma_\theta, \quad f \in L^1([0,\theta],\gamma_\theta),\ A \in \mathcal{B}_{[0,\theta]}. \tag{A.2}$$

From (A.2), for any $f \in L^1([0,\theta],\gamma_\theta)$ and $A \in \mathcal{B}_{[0,\theta]}$, we have

$$\begin{aligned}
&\int_{C(A)} f(x)\,\mathrm{d}\mu(x)\\
&\quad = \sum_{i\geq m} \int_{C_i(A)} f(x)\,\mathrm{d}\mu(x) = \sum_{i\geq m} \int_{C_i(A)} f(x)h(x)\,\mathrm{d}x\\
&\quad = \sum_{i\geq m} \int_A \frac{f(u_i(y))\,h(u_i(y))}{(\theta i + y)^2}\,\mathrm{d}y = \int_A \sum_{i\geq m} \frac{f(u_i(x))\,h(u_i(x))}{(\theta i + x)^2}\,\mathrm{d}x.
\end{aligned} \tag{A.3}$$

Since $\mathrm{d}\mu = h\mathrm{d}\lambda_\theta$, (2.3) follows from (A.3).

Now, since $g(x) = (\theta x + 1)f(x)h(x)$, from (2.2) we have

$$Ug(x) = (\theta x + 1)\sum_{i\geq m} \frac{h(u_i(x))}{(\theta i + x)^2}\, f(u_i(x)). \tag{A.4}$$

Now, (2.4) follows immediately from (2.3) and (A.4). Using mathematical induction, (2.5) follows easily.

(ii)(b) The formula (2.6) is a consequence of (2.4) and follows immediately.

(ii)(c) See [20]. □

Proof of Proposition 2.2. (i)(a) We will assume that f is non-decreasing. The proof for non-increasing f will be analogous. Let $x < y$, $x, y \in [0, \theta]$. We have $Uf(y) - Uf(x) = S_1 + S_2$, where

$$S_1 = \sum_{i \geq m} P_i(y) \left(f(u_i(y)) - f(u_i(x)) \right), \tag{A.5}$$

$$S_2 = \sum_{i \geq m} \left(P_i(y) - P_i(x) \right) f(u_i(x)). \tag{A.6}$$

Clearly, $S_1 \leq 0$. Now, since $\sum_{i \geq m} P_i(x) = 1$ for any $x \in [0, \theta]$, we can write

$$S_2 = -\sum_{i \geq m} \left(f(u_m(x)) - f(u_i(x)) \right) \left(P_i(y) - P_i(x) \right). \tag{A.7}$$

It can be seen easily that the functions P_i are increasing for all $i \geq m$. Also, using that $f(u_m(x)) \geq f(u_i(x))$, we have that $S_2 \leq 0$. Thus $Uf(y) - Uf(x) \leq 0$, and the proof is complete.

(i)(b) We will assume that f is non-decreasing. The proof for non-increasing f will be analogous. Then by (a) we have

$$\operatorname{var} Uf = Uf(0) - Uf(\theta) = \sum_{i \geq m} \left(P_i(0) f(u_i(0)) - P_i(\theta) f(u_i(\theta)) \right). \tag{A.8}$$

By calculus, we have

$$\begin{aligned}
\operatorname{var} Uf &= \sum_{i \geq m} \left(\frac{m}{i(i+1)} f\left(\frac{1}{\theta i} \right) - \frac{m+1}{(i+1)(i+2)} f\left(\frac{1}{\theta(i+1)} \right) \right) \\
&= \frac{1}{m+1} f(\theta) - \sum_{i \geq m} \frac{1}{(i+1)(i+2)} f\left(\frac{1}{\theta(i+1)} \right) \\
&\leq \frac{1}{m+1} f(\theta) - \sum_{i \geq m} \frac{1}{(i+1)(i+2)} f(0) = \frac{1}{m+1} \left(f(\theta) - f(0) \right) = \frac{1}{m+1} \operatorname{var} f.
\end{aligned}$$

(ii) For $x \neq y$, $x, y \in [0, \theta]$, we have

$$\begin{aligned}
\frac{Uf(y) - Uf(x)}{y - x} &= \sum_{i \geq m} \frac{P_i(y) - P_i(x)}{y - x} f(u_i(x)) \\
&\quad - \sum_{i \geq m} P_i(y) \frac{f(u_i(y)) - f(u_i(x))}{u_i(y) - u_i(x)} \cdot u_i(x) u_i(y).
\end{aligned} \tag{A.9}$$

We remark that

$$P_i(x) = \frac{\theta(i+1-m)}{\theta(i+1)+x} + \frac{\theta(m-i)}{\theta i + x}, \quad i \geq m, \tag{A.10}$$

and then

$$\sum_{i\geq m} \frac{P_i(y)-P_i(x)}{y-x} f(u_i(x)) = \sum_{i\geq m} \frac{\theta(i+1-m)}{(y+\theta(i+1))(x+\theta(i+1))} \left(f(u_{i+1}(x)) - f(u_i(x))\right). \tag{A.11}$$

Assume that $x > y$. It then follows from (A.9) and (A.11) that

$$\left|\frac{Uf(y)-Uf(x)}{y-x}\right| \leq s(f) \sum_{i\geq m} \left(\frac{\theta^2(i+1-m)}{(y+\theta i)(y+\theta(i+1))^3} + \frac{P_i(y)}{(y+\theta i)^2}\right). \tag{A.12}$$

Since the sum of the right side of (A.12) is bounded by q (see (3.11)), and since

$$s(Uf) = \sup_{x\neq y} \left|\frac{Uf(y)-Uf(x)}{y-x}\right|, \tag{A.13}$$

the proof is complete. □

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