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Semigroup operations distributed by the ordinary multiplication or addition on the real numbers

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Dedicated to the memory of Professor Takayuki Furuta

Abstract. We characterize the cancellative and continuous semigroup operations on the real field which are distributed by the ordinary multiplication or addition.

1. Introduction

As usual, $\mathbb{R}$ denotes the ordered field of all real numbers with the ordinary addition $+$ and multiplication $\cdot$. We consider two commutative operations $\star$ and $*$ on $\mathbb{R}$ which satisfy the distributive law

$$x \star (y * z) = (x \star y) * (x \star z) \quad (x, y, z \in \mathbb{R}). \tag{1}$$

If (1) holds, we say that $*$ is distributed by $\star$, or that $\star$ is distributive over $*$. In this paper, we fix an operation $\star$, for instance, $\star = \cdot$ or $+$, and investigate the operations $*$ satisfying (1). By $\mathcal{D}_\star(\mathbb{R})$, we denote the set of all associative, cancellative and continuous operations $*$ on $\mathbb{R}$ satisfying (1). First, we characterize $\mathcal{D}_\cdot(\mathbb{R})$ and $\mathcal{D}_+(\mathbb{R})$.

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To state the result, for each $a > 0$, we define a function φ_a on $\mathbb{R}$ by

$$\varphi_a(x) = (\operatorname{sgn} x)\,|x|^a = \begin{cases} x^a & \text{if } x \geq 0, \\ -|x|^a & \text{if } x < 0. \end{cases}$$

Theorem 1. *Let $*$ be an associative, cancellative and continuous operation on $\mathbb{R}$. Then $*$ is distributed by $\cdot$ if and only if there exists a positive number a such that*

$$x * y = \varphi_{1/a}\bigl(\varphi_a(x) + \varphi_a(y)\bigr) \quad (x, y \in \mathbb{R}).$$

In particular, if $a = 1$, then $* = +$.

Theorem 2. *Let $*$ be an associative, cancellative and continuous operation on $\mathbb{R}$. Then $*$ is distributed by $+$ if and only if there exists a positive number $a \neq 1$ such that*

$$x * y = \log_a(a^x + a^y) \quad (x, y \in \mathbb{R}).$$

As a generalization, we also characterize $\mathcal{D}_\star(\mathbb{R})$ under the assumption that $(\mathbb{R}, \star)$ is homeomorphically isomorphic to $(\mathbb{R}, \cdot)$ or $(\mathbb{R}, +)$. We will describe this characterization in Section 5.

This research is motivated by [7, Theorem 2] and [4, Theorem 2].

2. Preliminaries

We introduce a way to construct a new operation on a set X. Let f be a bijection from X onto another set Y. If Y has an operation $\star$, then $\star$ induces an operation $\star_f$ on X as follows:

$$x \star_f y = f^{-1}\bigl(f(x) \star f(y)\bigr) \quad (x, y \in X).$$

If $(Y, \star)$ is a semigroup, then $(X, \star_f)$ is a semigroup and f is an isomorphism from $(X, \star_f)$ onto $(Y, \star)$. Conversely, if f is an isomorphism from $(X, *)$ onto $(Y, \star)$, then $* = \star_f$.

Let X be a topological space. By $\mathcal{A}(X)$, we denote the set of all associative, cancellative and continuous operations on X. If f is a homeomorphism from X onto another topological space Y, then $\star \in \mathcal{A}(Y)$ implies $\star_f \in \mathcal{A}(X)$.

In case $X = \mathbb{R}$, the operations in $\mathcal{A}(\mathbb{R})$ are well studied. Let

$$\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\} \quad \text{and} \quad \mathbb{R}_1 = \{x \in \mathbb{R} : x > 1\}.$$

Then we have

Theorem A ([6, Theorem 4.4])**.** *If* $* \in \mathcal{A}(\mathbb{R})$*, then the topological semigroup* $(\mathbb{R}, *)$ *is homeomorphically isomorphic to exactly one of*

$$(\mathbb{R}, +),\ (\mathbb{R}_+, +) \quad \textit{and} \quad (\mathbb{R}_1, +).$$

These topological semigroups are not homeomorphically isomorphic to each other.

From this theorem, we see that if $* \in \mathcal{A}(\mathbb{R})$, then $*$ is commutative. The related results may be found in [1], [3], [5]. While the set $\mathcal{D}_+(\mathbb{R}^2)$ is studied in [4].

3. Proof of Theorem 1

Let S be a subset of $\mathbb{R}$ which is closed with respect to $+$, and f a homeomorphism from $\mathbb{R}$ onto S. Then the above argument gives the operation $+_f$ on $\mathbb{R}$. We have

$$\begin{aligned}
&+_f \in \mathcal{D}.(\mathbb{R}) \\
&\quad \iff x \cdot (y +_f z) = (x \cdot y) +_f (x \cdot z) \quad (x, y, z \in \mathbb{R}) \\
&\quad \iff x \cdot f^{-1}\big(f(y) + f(z)\big) = f^{-1}\big(f(x \cdot y) + f(x \cdot z)\big) \quad (x, y, z \in \mathbb{R}) \qquad (2) \\
&\quad \iff f\Big(x \cdot f^{-1}\big(f(y) + f(z)\big)\Big) = f(x \cdot y) + f(x \cdot z) \quad (x, y, z \in \mathbb{R}). \qquad (3)
\end{aligned}$$

Lemma 1. *Let* S *and* f *be as above. If* $+_f \in \mathcal{D}.(\mathbb{R})$*, then* $f(0) = 0$.

Proof. Putting $x = 0$ in (3), we get $f(0) = f(0) + f(0)$, and so $f(0) = 0$. □

Lemma 2. *If* $* \in \mathcal{D}.(\mathbb{R})$*, then* $(\mathbb{R}, *)$ *is homeomorphically isomorphic to* $(\mathbb{R}, +)$.

Proof. By Theorem A, $(\mathbb{R}, *)$ is homeomorphically isomorphic to $(S, +)$, where S is one of $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_1$. Let f be a homeomorphic isomorphism from $(\mathbb{R}, *)$ onto $(S, +)$. Then $+_f = * \in \mathcal{D}.(\mathbb{R})$. By Lemma 1, $0 = f(0) \in S$. Since $0 \in \mathbb{R}$ and $0 \notin \mathbb{R}_+, \mathbb{R}_1$, we conclude that $S = \mathbb{R}$. Thus the lemma was proved. □

Let $H(\mathbb{R})$ be the set of all homeomorphisms from $\mathbb{R}$ onto itself. We put

$$\begin{aligned}
F(\mathbb{R}) &= \Big\{ f \in H(\mathbb{R}) : +_f \in \mathcal{D}.(\mathbb{R}) \Big\}, \\
\Phi(\mathbb{R}) &= \Big\{ f \in H(\mathbb{R}) : f(x \cdot y) = f(x) \cdot f(y) (x, y \in \mathbb{R}) \Big\}.
\end{aligned}$$

For $f \in H(\mathbb{R})$, it is clear that $f \in F(\mathbb{R}) \iff (2) \iff (3)$.

Lemma 3. *If $f \in F(\mathbb{R})$, then $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.*

Proof. Putting $x = -1$ and $z = -y$ in (2), we get

$$-f^{-1}\big(f(y) + f(-y)\big) = f^{-1}\big(f(-y) + f(y)\big) \quad (y \in \mathbb{R}).$$

This equation leads to $f^{-1}\big(f(y) + f(-y)\big) = 0$, and $f(y) + f(-y) = f(0) = 0$ by Lemma 1. □

Lemma 4. *If $f \in H(\mathbb{R})$ and $c \neq 0$, then $+_{cf} = +_f$. In particular, if $f \in F(\mathbb{R})$ and $c \neq 0$, then $cf \in F(\mathbb{R})$.*

Proof. Suppose $f \in H(\mathbb{R})$ and $c \neq 0$. Put $h = cf$. Clearly, $h \in H(\mathbb{R})$. Since

$$h^{-1}(u) = f^{-1}\left(\frac{1}{c}u\right) \quad (u \in \mathbb{R}),$$

it follows that

$$\begin{aligned} x +_{cf} y = x +_h y &= h^{-1}\big(h(x) + h(y)\big) \\ &= f^{-1}\left(\frac{1}{c}\Big(cf(x) + cf(y)\Big)\right) = f^{-1}\big(f(x) + f(y)\big) = x +_f y \end{aligned}$$

for all $x, y \in \mathbb{R}$. Hence $+_{cf} = +_f$. In addition, if $f \in F(\mathbb{R})$, then $+_{cf} = +_f \in \mathcal{D}.(\mathbb{R})$, so that $cf \in F(\mathbb{R})$. □

Lemma 5. $F(\mathbb{R}) = \big\{cf : f \in \Phi(\mathbb{R}), c \neq 0\big\}$.

Proof. We first show that $F(\mathbb{R}) \supset \{cf : f \in \Phi(\mathbb{R}), c \neq 0\}$. Let $f \in \Phi(\mathbb{R})$ and $c \neq 0$. Then

$$\begin{aligned} f\Big(x \cdot f^{-1}\big(f(y) + f(z)\big)\Big) &= f(x) \cdot f\Big(f^{-1}\big(f(y) + f(z)\big)\Big) = f(x) \cdot \big(f(y) + f(z)\big) \\ &= f(x) \cdot f(y) + f(x) \cdot f(z) = f(x \cdot y) + f(x \cdot z). \end{aligned}$$

By (3), $f \in F(\mathbb{R})$, and by Lemma 4, $cf \in F(\mathbb{R})$.

For the opposite inclusion, pick $h \in F(\mathbb{R})$. By Lemma 1, $h(0) = 0$. Since h is injective, $h(1) \neq 0$. Put $c = h(1)$ and $f = (1/c)h$. Then it suffices to show that $f \in \Phi(\mathbb{R})$. Here we remark that $f(1) = 1$ and that $f \in F(\mathbb{R})$ by Lemma 4.

Let $x \in \mathbb{R}$ and $n \in \mathbb{N} = \{1, 2, \ldots\}$. We use (3) to see that

$$\begin{aligned} f\big(x \cdot f^{-1}(n)\big) &= f\Big(x \cdot f^{-1}\big(1 + (n-1)\big)\Big) = f\Big(x \cdot f^{-1}\big(f(1) + f(f^{-1}(n-1))\big)\Big) \\ &= f(x \cdot 1) + f\big(x \cdot f^{-1}(n-1)\big) = f(x) + f\big(x \cdot f^{-1}(n-1)\big). \end{aligned}$$

Repeat this computation and use $f^{-1}(1) = 1$ finally. Then, we get

$$f\big(x \cdot f^{-1}(n)\big) = n \cdot f(x).$$

Substituting $x = f^{-1}(v)$, we have $f\big(f^{-1}(v) \cdot f^{-1}(n)\big) = n \cdot v$, that is,

$$f^{-1}(n) \cdot f^{-1}(v) = f^{-1}\big(n \cdot v\big) \quad (n \in \mathbb{N}, v \in \mathbb{R}). \tag{4}$$

Putting $n = m$ and $v = 1/m$, we have

$$f^{-1}(m) \cdot f^{-1}\left(\frac{1}{m}\right) = f^{-1}(1) = 1 \quad (m \in \mathbb{N}). \tag{5}$$

Hence

$$\begin{aligned} f^{-1}\left(\frac{n}{m} \cdot v\right) &= f^{-1}\left(n \cdot \frac{1}{m} v\right) \\ &= f^{-1}(n) \cdot f^{-1}\left(\frac{1}{m} v\right) && \text{by (4)} \\ &= f^{-1}(n) \cdot f^{-1}\left(\frac{1}{m}\right) \cdot f^{-1}(m) \cdot f^{-1}\left(\frac{1}{m} v\right) && \text{by (5)} \\ &= f^{-1}\left(n \cdot \frac{1}{m}\right) \cdot f^{-1}\left(m \cdot \frac{1}{m} v\right) && \text{by (4)} \\ &= f^{-1}\left(\frac{n}{m}\right) \cdot f^{-1}(v), \end{aligned}$$

for all $m, n \in \mathbb{N}$ and $v \in \mathbb{R}$. In other words,

$$f^{-1}(u \cdot v) = f^{-1}(u) \cdot f^{-1}(v) \quad (v \in \mathbb{R}) \tag{6}$$

for all $u \in \mathbb{Q}_+$; the positive rational numbers. Note that f^{-1} is continuous on $\mathbb{R}$. Then we see that (6) holds for all $u \in \mathbb{R}_+$. Moreover, we use Lemma 3 to see that (6) holds for all $u \in \mathbb{R}$. Putting $u = f(x)$ and $v = f(y)$ in (6), and then applying f to both sides, we arrive at

$$f(x) \cdot f(y) = f(x \cdot y) \quad (x, y \in \mathbb{R}).$$

Hence $f \in \Phi(\mathbb{R})$. □

Lemma 6. $\Phi(\mathbb{R}) = \{\varphi_a : a > 0\}$.

This lemma is essentially proved in [2, §2.1.2, Theorem 3]. For the sake of completeness, we give its proof.

Proof. It is easy to check that $\Phi(\mathbb{R}) \supset \{\varphi_a : a > 0\}$. Conversely, take $f \in \Phi(\mathbb{R})$. Then $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in \mathbb{R}$. This equation and the bijectivity of f yield $f(0) = 0$, $f(1) = 1$, $f(-1) = -1$ and $f(-x) = -f(x)$ for $x \in \mathbb{R}$. Moreover, f is strictly monotone on $\mathbb{R}$, because f is a homeomorphism. From these facts, we see that f is strictly increasing and that $f(x) > 0$ for $x > 0$. Now, define

$$h(t) = \log(f(e^t)) \quad (t \in \mathbb{R}).$$

Then $h(s+t) = h(s) + h(t)$ $(s, t \in \mathbb{R})$, and h is continuous on $\mathbb{R}$. It is known that such a function h is represented by $h(t) = at$ $(t \in \mathbb{R})$ for some $a \in \mathbb{R}$ (see [2, §2.1.1 Theorem 1]). Thus we obtain $f(e^t) = e^{at}$, and hence $f(x) = x^a$ $(x > 0)$. We recall the equation $f(-x) = -f(x)$ to see that $f(x) = (\operatorname{sgn} x)|x|^a$ for all $x \in \mathbb{R}$. Also, we note that f is strictly increasing to see $a > 0$. Therefore, $\Phi(\mathbb{R}) \subset \{\varphi_a : a > 0\}$. □

Proof of Theorem 1. Since $\varphi_a^{-1} = \varphi_{1/a}$, the theorem is restated as follows: $* \in \mathcal{D}.(\mathbb{R})$ if and only if there exists $a > 0$ such that

$$x * y = \varphi_a^{-1}\big(\varphi_a(x) + \varphi_a(y)\big) \quad (x, y \in \mathbb{R}). \tag{7}$$

Assume that $* \in \mathcal{D}.(\mathbb{R})$. Then Lemma 2 says that $(\mathbb{R}, *)$ is homeomorphically isomorphic to $(\mathbb{R}, +)$. Let f be a homeomorphic isomorphism from $(\mathbb{R}, *)$ onto $(\mathbb{R}, +)$. Then $+_f = * \in \mathcal{D}.(\mathbb{R})$, and hence $f \in F(\mathbb{R})$. By Lemmas 5 and 6, there exist $c \neq 0$ and $a > 0$ such that

$$f = c\varphi_a.$$

Therefore, $* = +_f = +_{c\varphi_a} = +_{\varphi_a}$ by Lemma 4. Thus we obtain (7).

Conversely, assume that $*$ satisfies (7) for some $a > 0$. This means $* = +_{\varphi_a}$. While Lemmas 5 and 6 say that $\varphi_a \in F(\mathbb{R})$, that is, $+_{\varphi_a} \in \mathcal{D}.(\mathbb{R})$. Hence $* \in \mathcal{D}.(\mathbb{R})$. □

4. Proof of Theorem 2

Let S be a subset of $\mathbb{R}$ closed with respect to $+$, and g a homeomorphism from $\mathbb{R}$ onto S. For the operation $+_g$ discussed in Section 2, we have

$$\begin{aligned}
&+_g \in \mathcal{D}_+(\mathbb{R}) \\
&\quad \Longleftrightarrow\ x + (y +_g z) = (x + y) +_g (x + z) \quad (x, y, z \in \mathbb{R}) \\
&\quad \Longleftrightarrow\ x + g^{-1}\big(g(y) + g(z)\big) = g^{-1}\big(g(x+y) + g(x+z)\big) \quad (x, y, z \in \mathbb{R}) \qquad (8) \\
&\quad \Longleftrightarrow\ g\Big(x + g^{-1}\big(g(y) + g(z)\big)\Big) = g(x+y) + g(x+z) \quad (x, y, z \in \mathbb{R}). \qquad (9)
\end{aligned}$$

Lemma 7. *If* $* \in \mathcal{D}_+(\mathbb{R})$*, then* $(\mathbb{R}, *)$ *is not homeomorphically isomorphic to* $(\mathbb{R}, +)$.

Proof. Assume that there exists a homeomorphic isomorphism g from $(\mathbb{R}, *)$ onto $(\mathbb{R}, +)$. Then $+_g = * \in \mathcal{D}_+(\mathbb{R})$. Taking $y = z = 0$ in (8), we get

$$x + g^{-1}\big(2g(0)\big) = g^{-1}\big(2g(x)\big) \quad (x \in \mathbb{R}). \tag{10}$$

Letting $x = g^{-1}(0)$ in (10), we get $g^{-1}(0) + g^{-1}\big(2g(0)\big) = g^{-1}\big(2g(g^{-1}(0))\big) = g^{-1}(0)$. Hence $g^{-1}\big(2g(0)\big) = 0$. Thus (10) becomes $x = g^{-1}\big(2g(x)\big)$, that is, $g(x) = 2g(x)$, and so $g(x) = 0$ for all $x \in \mathbb{R}$. This contradicts the fact that g is surjective. Consequently, there is no homeomorphic isomorphism from $(\mathbb{R}, *)$ onto $(\mathbb{R}, +)$. □

Lemma 8. *If* $* \in \mathcal{D}_+(\mathbb{R})$*, then* $(\mathbb{R}, *)$ *is not homeomorphically isomorphic to* $(\mathbb{R}_1, +)$.

Proof. Assume that there exists a homeomorphic isomorphism g from $(\mathbb{R}, *)$ onto $(\mathbb{R}_1, +)$. Then $+_g = * \in \mathcal{D}_+(\mathbb{R})$. Moreover, g is a strictly monotone function which maps $\mathbb{R}$ onto $\mathbb{R}_1$. If g is strictly increasing, then

$$\lim_{x \to -\infty} g(x) = 1.$$

Let y and z tend to $-\infty$ in (8). Then the continuity of g^{-1} shows that

$$x + g^{-1}(2) = g^{-1}(2) \quad (x \in \mathbb{R}).$$

This implies $x = 0$ for all $x \in \mathbb{R}$, which is a contradiction. On the other hand, if g is strictly decreasing, then $\lim_{x\to\infty} g(x) = 1$. By letting $y, z \to \infty$ in (8), we similarly reach a contradiction. Thus the lemma is proved. □

We combine Theorem A with Lemmas 7 and 8 to conclude

Lemma 9. *If* $* \in \mathcal{D}_+(\mathbb{R})$*, then* $(\mathbb{R}, *)$ *is homeomorphically isomorphic to* $(\mathbb{R}_+, +)$.

Let $H(\mathbb{R}, \mathbb{R}_+)$ be the set of all homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}_+$. We put

$$G(\mathbb{R}) = \Big\{g \in H(\mathbb{R}, \mathbb{R}_+) : +_g \in \mathcal{D}_+(\mathbb{R})\Big\},$$
$$\Psi(\mathbb{R}) = \Big\{g \in H(\mathbb{R}, \mathbb{R}_+) : g(x+y) = g(x) \cdot g(y) (x, y \in \mathbb{R})\Big\}.$$

For $g \in H(\mathbb{R}, \mathbb{R}_+)$, it is clear that $g \in G(\mathbb{R}) \iff (8) \iff (9)$.

Lemma 10. *If $g \in H(\mathbb{R}, \mathbb{R}_+)$ and $c > 0$, then $+_{cg} = +_g$. In particular, if $g \in G(\mathbb{R})$ and $c > 0$, then $cg \in G(\mathbb{R})$.*

This can be shown similarly to Lemma 4.

Lemma 11. $G(\mathbb{R}) = \big\{cg : g \in \Psi(\mathbb{R}), c > 0\big\}$.

Proof. We first show that $G(\mathbb{R}) \supset \{cg : g \in \Psi(\mathbb{R}), c > 0\}$. Let $g \in \Psi(\mathbb{R})$ and $c > 0$. Then

$$\begin{aligned} g\Big(x + g^{-1}\big(g(y) + g(z)\big)\Big) &= g(x) \cdot g\Big(g^{-1}\big(g(y) + g(z)\big)\Big) = g(x) \cdot \big(g(y) + g(z)\big) \\ &= g(x) \cdot g(y) + g(x) \cdot g(z) = g(x + y) + g(x + z). \end{aligned}$$

By (9), $g \in G(\mathbb{R})$, and by Lemma 10, $cg \in G(\mathbb{R})$.

For the opposite inclusion, pick $h \in G(\mathbb{R})$. Note that $h(0) > 0$. Put $c = h(0)$ and $g = (1/c)h$. Then it suffices to show that $g \in \Psi(\mathbb{R})$. Here we remark that $g(0) = 1$ and that $g \in G(\mathbb{R})$ by Lemma 10.

Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. We use (9) to see that

$$\begin{aligned} g\big(x + g^{-1}(n)\big) &= g\Big(x + g^{-1}\big(1 + (n - 1)\big)\Big) = g\Big(x + g^{-1}\big(g(0) + g(g^{-1}(n - 1))\big)\Big) \\ &= g(x + 0) + g\big(x + g^{-1}(n - 1)\big) = g(x) + g\big(x + g^{-1}(n - 1)\big). \end{aligned}$$

Hence

$$g\big(x + g^{-1}(n)\big) = n \cdot g(x).$$

Substituting $x = g^{-1}(v)$, we have

$$g^{-1}(n) + g^{-1}(v) = g^{-1}\big(n \cdot v\big) \quad (n \in \mathbb{N}, v \in \mathbb{R}_+).$$

Putting $n = m$ and $v = 1/m$, we have

$$g^{-1}(m) + g^{-1}\left(\frac{1}{m}\right) = g^{-1}(1) = 0 \quad (m \in \mathbb{N}).$$

Using these equations, we obtain

$$\begin{aligned} g^{-1}\left(\frac{n}{m} \cdot v\right) &= g^{-1}\left(n \cdot \frac{1}{m} v\right) = g^{-1}(n) + g^{-1}\left(\frac{1}{m} v\right) \\ &= g^{-1}(n) + g^{-1}\left(\frac{1}{m}\right) + g^{-1}(m) + g^{-1}\left(\frac{1}{m} v\right) \\ &= g^{-1}\left(n \cdot \frac{1}{m}\right) + g^{-1}\left(m \cdot \frac{1}{m} v\right) = g^{-1}\left(\frac{n}{m}\right) + g^{-1}(v), \end{aligned}$$

for all $m, n \in \mathbb{N}$ and $v \in \mathbb{R}_+$. In other words,

$$g^{-1}(u \cdot v) = g^{-1}(u) + g^{-1}(v) \quad (v \in \mathbb{R}_+) \tag{11}$$

for all $u \in \mathbb{Q}_+$. Since g^{-1} is continuous on $\mathbb{R}_+$, (11) holds for all $u \in \mathbb{R}_+$. Thus we obtain

$$g(x) \cdot g(y) = g(x + y) \quad (x, y \in \mathbb{R}).$$

Hence $g \in \Psi(\mathbb{R})$. □

For $a > 0$, we define a function ψ_a on $\mathbb{R}$ by

$$\psi_a(x) = a^x \quad (x \in \mathbb{R}).$$

Lemma 12. $\Psi(\mathbb{R}) = \{\psi_a : a > 0, a \neq 1\}$.

This lemma is known, but we prove it for completeness.

Proof. It is easy to check that $\Psi(\mathbb{R}) \supset \{\psi_a : a > 0, a \neq 1\}$. Conversely, take $g \in \Psi(\mathbb{R})$. Then $g(x + y) = g(x) \cdot g(y)$ for all $x, y \in \mathbb{R}$, and g is continuous on $\mathbb{R}$. It is known that such a function g is represented as

$$g(x) = 0 \quad \text{or} \quad g(x) = e^{cx} \quad (x \in \mathbb{R})$$

for some $c \in \mathbb{R}$ (see [2, §2.1.2 Theorem 1]). Since g is a surjection from $\mathbb{R}$ to $\mathbb{R}_+$, we must exclude the former equation and the latter one with $c = 0$. Putting $a = e^c$ in the latter equation, we obtain $g(x) = a^x$ $(x \in \mathbb{R})$ with $a > 0$ and $a \neq 1$. Hence $\Psi(\mathbb{R}) \subset \{\psi_a : a > 0, a \neq 1\}$. □

Proof of Theorem 2. Since $\psi_a^{-1}(u) = \log_a u$ $(u \in \mathbb{R}_+)$, the theorem is restated as follows: $* \in \mathcal{D}_+(\mathbb{R})$ if and only if there exists $a > 0$ with $a \neq 1$ such that

$$x * y = \psi_a^{-1}\big(\psi_a(x) + \psi_a(y)\big) \quad (x, y \in \mathbb{R}). \tag{12}$$

Assume that $* \in \mathcal{D}_+(\mathbb{R})$. Then Lemma 9 says that $(\mathbb{R}, *)$ is homeomorphically isomorphic to $(\mathbb{R}_+, +)$. Let g be a homeomorphic isomorphism from $(\mathbb{R}, *)$ onto $(\mathbb{R}_+, +)$. Then $+_g = * \in \mathcal{D}_+(\mathbb{R})$, and hence $g \in G(\mathbb{R})$. By Lemmas 11 and 12, there exist $c > 0$ and $a > 0$ with $a \neq 1$ such that

$$g = c\psi_a.$$

Therefore, $* = +_g = +_{c\psi_a} = +_{\psi_a}$ by Lemma 10. Thus we obtain (12).

Conversely, assume that $*$ satisfies (12) for some $a > 0$, $a \neq 1$. This means $* = +_{\psi_a}$. While Lemmas 11 and 12 say that $\psi_a \in G(\mathbb{R})$, that is, $+_{\psi_a} \in \mathcal{D}_+(\mathbb{R})$. Hence $* \in \mathcal{D}_+(\mathbb{R})$. □

5. Generalizations

We generalize Theorems 1 and 2 as follows:

Theorem 3. *Suppose that a topological semigroup* $(\mathbb{R}, \star)$ *is homeomorphically isomorphic to* $(\mathbb{R}, \cdot)$, *and let* ξ *be a homeomorphic isomorphism from* $(\mathbb{R}, \star)$ *onto* $(\mathbb{R}, \cdot)$. *Let* $* \in \mathcal{A}(\mathbb{R})$. *Then* $* \in \mathcal{D}_\star(\mathbb{R})$ *if and only if there exists* $a > 0$ *such that*

$$\xi(x * y) = \varphi_{1/a}\big(\varphi_a(\xi(x)) + \varphi_a(\xi(y))\big) \quad (x, y \in \mathbb{R}). \tag{13}$$

Theorem 4. *Suppose that a topological semigroup* $(\mathbb{R}, \star)$ *is homeomorphically isomorphic to* $(\mathbb{R}, +)$, *and let* ξ *be a homeomorphic isomorphism from* $(\mathbb{R}, \star)$ *onto* $(\mathbb{R}, +)$. *Let* $* \in \mathcal{A}(\mathbb{R})$. *Then* $* \in \mathcal{D}_\star(\mathbb{R})$ *if and only if there exists* $a > 0$ *with* $a \neq 1$ *such that*

$$\xi(x * y) = \log_a\big(a^{\xi(x)} + a^{\xi(y)}\big) \quad (x, y \in \mathbb{R}).$$

Proof of Theorem 3. Let $\eta = \xi^{-1}$. Since η is a homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, we have $*_\eta \in \mathcal{A}(\mathbb{R})$, and η is a homeomorphic isomorphism from $(\mathbb{R}, *_\eta)$ onto $(\mathbb{R}, *)$. At the same time, η is an isomorphism from $(\mathbb{R}, \cdot)$ onto $(\mathbb{R}, \star)$. These facts show that

$$\begin{aligned}
&* \in \mathcal{D}_\star(\mathbb{R}) \\
&\iff x \star (y * z) = (x \star y) * (x \star z) \quad (x, y, z \in \mathbb{R}) \\
&\iff \eta(u) \star \big(\eta(v) * \eta(w)\big) = \big(\eta(u) \star \eta(v)\big) * \big(\eta(u) \star \eta(w)\big) \quad (u, v, w \in \mathbb{R}) \\
&\iff \eta(u) \star \eta(v *_\eta w) = \eta(u \cdot v) * \eta(u \cdot w) \quad (u, v, w \in \mathbb{R}) \\
&\iff \eta\big(u \cdot (v *_\eta w)\big) = \eta\big((u \cdot v) *_\eta (v \cdot w)\big) \quad (u, v, w \in \mathbb{R}) \\
&\iff u \cdot (v *_\eta w) = (u \cdot v) *_\eta (u \cdot w) \quad (u, v, w \in \mathbb{R}) \\
&\iff *_\eta \in \mathcal{D}_\cdot(\mathbb{R}).
\end{aligned}$$

While Theorem 1 says that $*_\eta \in \mathcal{D}_\cdot(\mathbb{R})$ if and only if there exists $a > 0$ such that

$$u *_\eta v = \varphi_{1/a}\big(\varphi_a(u) + \varphi_a(v)\big) \quad (u, v \in \mathbb{R}),$$

that is,

$$\xi(x) *_\eta \xi(y) = \varphi_{1/a}\big(\varphi_a(\xi(x)) + \varphi_a(\xi(y))\big) \quad (x, y \in \mathbb{R}). \tag{14}$$

Since ξ $(= \eta^{-1})$ is an isomorphism from $(\mathbb{R}, *)$ onto $(\mathbb{R}, *_\eta)$, we have

$$\xi(x * y) = \xi(x) *_\eta \xi(y) \quad (x, y \in \mathbb{R}).$$

Hence (14) is equivalent to (13). Thus the theorem was proved. □

Similarly, we can prove Theorem 4 by using Theorem 2.

In this paper, we completely characterize the *cancellative*, associative and continuous operations on $\mathbb{R}$ which are distributed by $\cdot$ or $+$. We want to remove the assumption "cancellative", and do the same for the general associative and continuous operations on $\mathbb{R}$. But it seems to be essentially difficult (cf. [6, Theorem 3.6]).

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References

[1] J. ACZÉL, Sur les opérations définies pour nombres réels, *Bull. Soc. Math. France* **76** (1948), 59–64.

[2] J. ACZÉL, Lectures on Functional Equations and their Applications, *Academic Press*, *New York – London*, 1966.

[3] J. ACZÉL, The state of the second part of Hilbert's fifth problem, *Bull. Amer. Math. Soc. (N.S.)* **20** (1989), 153–163.

[4] S. ANBE, S.-E. TAKAHASI, M. TSUKADA and T. MIURA, Commutative semigroup operations on $\mathbf{R}^2$ compatible with the ordinary additive operation, *Linear Nonlinear Anal.* **1** (2015), 89–93.

[5] R. CRAIGEN and Z. PÁLES, The associativity equation revisited, *Aequationes Math.* **37** (1989), 306–312.

[6] Y. KOBAYASHI, Y. NAKASUJI, S.-E. TAKAHASI and M. TSUKADA, Continuous semigroup structures on $\mathbb{R}$, cancellative semigroups and bands, *Semigroup Forum* **90** (2015), 518–531.

[7] Y. NAKASUJI and S.-E. TAKAHASI, A reconsideration of Jensen's inequality and its applications, *J. Inequal. Appl.* **2013**, 2013:408, 11 pp.

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