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On warped Finslerian gradient Ricci solitons

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Abstract. In this work, we study warped Finslerian gradient Ricci solitons where the base space is Riemannian, and it is showed that the potential function depends only on the base space, or else the warping function is a constant. Also, we prove that a warped product Finsler manifold, when the base space is conformal to a Euclidean space, is a gradient Ricci soliton if and only if the warping and potential functions satisfy some partial differential equations.

1. Introduction

Gradient Ricci solitons have become important tools in Riemannian geometry and general relativity. They have important applications in fluid mechanics, black holes, string theory, and classical and quantum theory of fields. In 1982, Hamilton introduced the notion of Ricci flow in Riemannian geometry, which played a key role in proving the Poincaré conjecture [9]. We recall that a Riemannian manifold $(M, g(t))$ is a gradient Ricci soliton if there exists a smooth function h on M such that

$$\mathrm{Ric}_g + \nabla\nabla h + \lambda g = 0$$

for some constant λ.

The notion of warped product was first introduced by Bishop and O'Neill to construct Riemannian manifolds with negative curvature [6]. It plays an important role in Riemannian geometry and geodesic metric spaces [7]. M. L. de Sousa

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and R. Pina considered gradient Ricci solitons on warped product Riemannian manifolds and proved that the potential function depends only on the base space, and the fiber space is necessarily an Einstein manifold [8].

Some recent works have focused on extending these notions to the Finsler geometry as a natural generalization of Riemannian geometry. Bao in [5] defined Ricci flow on Finsler manifolds by using Akbar-Zadeh's Ricci tensor as follows:

$$\frac{\partial}{\partial t} g_{ij} = -2\,\mathrm{Ric}_{ij}, \qquad g(t=0) = g_0,$$

where 'Ric' denotes the Ricci tensor of the Finsler metric tensor g, and g_0 is an initial Finsler metric tensor. The concept of warped product on Finsler manifolds is defined first by L. Kozma *et al.* [12]. M. M. Rezaii *et al.* generalized some theorems on Riemannian geometry to the warped product Finsler space [2], [3], [4] and [13].

In this paper, we consider warped product Finsler manifold when the base space is Riemannian and prove that the potential function of warped Finslerian gradient Ricci solitons depends only on the base space, or else the warping function is a constant. Also, the fiber space is necessarily an Einstein manifold with regard to an additional condition. Moreover, we find necessary and sufficient conditions for the warped product Finsler manifold, when the base space is conformal to a Euclidean space, to be a gradient Ricci soliton.

2. Preliminaries and notations

Let $\mathbb{F}_1 = (M_1, F_1)$ and $\mathbb{F}_2 = (M_2, F_2)$ be two Finsler manifolds, and $f : M_1 \to \mathbb{R}$ be a non-negative smooth function. We recall that $\mathbb{F} = (M, F)$ is a warped product Finsler manifold where M is a product manifold $M_1 \times M_2$, and the function $F = F_1 \times_f F_2$ is a Finsler metric on $T(M_1 \times M_2)^0$ that is defined as follows:

$$F^2(x_1, x_2, y_1, y_2) = F_1^2(x_1, y_1) + f(x_1)^2 F_2^2(x_2, y_2).$$

Notation. Lowercase Latin letters such as $\{i, j, k, l, \dots\}$, $\{\alpha, \beta, \gamma, \dots\}$ and $\{a, b, c, d, \dots\}$ are used in the upper position for variable indices. They belong to the set $\{1, \dots, m_1\}$, $\{1, \dots, m_2\}$ and $\{1, \dots, m_1 + m_2\}$, respectively, according to the spaces $\mathbb{F}_1$, $\mathbb{F}_2$ or $\mathbb{F} = \mathbb{F}_1 \times_f \mathbb{F}_2$ they represent. Variables of $\mathbb{F}_1$ and $\mathbb{F}_2$ have lower indices 1 and 2, respectively, like x_1^i, y_1^j and x_2^α, y_2^β.

When there is no appropriate position to place indices 1 and 2, objects of $\mathbb{F}_1$ and $\mathbb{F}_2$ will be hat and check, respectively, like $\hat{g}_{ij}$ and $\check{g}_{\alpha\beta}$, to indicate their relevant spaces.

The geodesic spray $G^a = (G^i, G^\alpha)$ of the warped product Finsler manifold is expressed as

$$G^i = \hat{G}^i - \frac{1}{4}\hat{g}^{ih}\frac{\partial f^2}{\partial x_1^h}F_2^2, \qquad G^\alpha = \check{G}^\alpha + \frac{1}{2}\frac{1}{f^2}y_2^\alpha y_1^h\frac{\partial f^2}{\partial x_1^h}. \tag{1}$$

Now, the coefficients of the nonlinear Cartan connection will be

$$G_j^i = \hat{G}_j^i - \frac{1}{4}\frac{\partial \hat{g}^{ih}}{\partial y_1^j}\frac{\partial f^2}{\partial x_1^h}F_2^2, \qquad G_\beta^i = -\frac{1}{4}\hat{g}^{ih}\frac{\partial f^2}{\partial x_1^h}\frac{\partial F_2^2}{\partial y_2^\beta},$$

$$G_j^\alpha = \frac{1}{2}\frac{1}{f^2}y_2^\alpha\frac{\partial f^2}{\partial x_1^j}, \qquad G_\beta^\alpha = \check{G}_\beta^\alpha + \frac{1}{2}\frac{1}{f^2}y_1^h\frac{\partial f^2}{\partial x_1^h}\delta_\beta^\alpha. \tag{2}$$

By means of this non-linear Cartan connection, tangent space can be split into the horizontal and vertical subspaces with the corresponding bases $\left\{\frac{\delta}{\delta x_1^i}, \frac{\delta}{\delta x_2^\alpha}, \frac{\partial}{\partial y_1^i}, \frac{\partial}{\partial y_2^\alpha}\right\}$, where

$$\frac{\delta}{\delta x_1^i} = \frac{\partial}{\partial x_1^i} - G_i^j\frac{\partial}{\partial y_1^j} - G_i^\beta\frac{\partial}{\partial y_2^\beta}, \qquad \frac{\delta}{\delta x_2^\alpha} = \frac{\partial}{\partial x_2^\alpha} - G_\alpha^j\frac{\partial}{\partial y_1^j} - G_\alpha^\beta\frac{\partial}{\partial y_2^\beta}. \tag{3}$$

The local coefficients C_{abc} of the Cartan tensor field of $\mathbb{F}$ are as follows:

$$C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y_1^k} = \hat{C}_{ijk}, \qquad C_{\alpha\beta\gamma} = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial y_2^\gamma} = f^2\check{C}_{\alpha\beta\gamma}, \tag{4}$$

and the other six combinations are zero [4].

In [11], it is shown that the gradient of the warping function f on M_1 is the horizontal gradient which is given by

$$\mathrm{grad}(f) = \hat{\nabla}_h f = \hat{g}^{ij}\frac{\partial f}{\partial x_1^j}\frac{\delta}{\delta x_1^i}, \tag{5}$$

and the h-Laplace operator of f is obtained as

$$\hat{\Delta}_h f = \frac{\delta \hat{g}^{ij}}{\delta x_1^i}\frac{\partial f}{\partial x_1^j} + \hat{g}^{ij}\frac{\partial^2 f}{\partial x_1^i \partial x_1^j} - \hat{P}_{ik}^k\hat{g}^{ij}\frac{\partial f}{\partial x_1^j}, \tag{6}$$

where $\hat{P}^k_{ij} = \hat{G}^k_{ij} - \hat{F}^k_{ij}$. Moreover, we have

$$\hat{G}(\hat{\nabla}_h f^2, \hat{\nabla}_h f^2) = \hat{g}^{ij} \frac{\partial f^2}{\partial x_1^i} \frac{\partial f^2}{\partial x_1^j}, \tag{7}$$

where $\hat{G}$ is the Sasaki Finsler Riemannian metric corresponding to the Finsler manifold $\mathbb{F}_1$. We introduced Ricci tensor on warped product Finsler manifold as

$$\mathrm{Ric}_{bc} := \left(\frac{1}{2} F^2\, \Re\mathrm{ic}\right)_{\mathrm{y}^b \mathrm{y}^c}, \tag{8}$$

where '$\Re$ic' is the Ricci scalar. We also obtained the coefficients of the Ricci tensor in [11].

3. Main results

In this section, we consider that the base space of warped product Finsler manifold is Riemannian and prove that the potential function of the warped gradient Finslerian manifold is independent of the fiber space. Also, if $\check{G}^\gamma_{\alpha\beta}$ is independent of y_2, then the fiber space is necessarily an Einstein manifold. Moreover, we find necessary and sufficient conditions for the warped product Finsler manifold, when the base space is conformal to a Euclidean space, to be a gradient Ricci soliton. These results are generalizations of the work of M. L. de Sousa and R. Pina on warped product Riemannian geometry [8].

Let $(M = M_1 \times M_2, F = F_1 \times_f F_2)$ be a warped product Finsler manifold. The Lie derivative of the warped product Finsler metric tensor $g(x, y)$ with respect to the complete lift, V^c, of a vector field $V = v_1^i(x_1, x_2)\frac{\partial}{\partial x_1^i} + v_2^\alpha(x_1, x_2)\frac{\partial}{\partial x_2^\alpha}$ is defined as

$$\mathcal{L}_{V^c} g_{ab} := \nabla_a v_b + \nabla_b v_a + 2C_{cab} y^d \nabla_d v^c, \tag{9}$$

where $\nabla_a := \nabla_{\frac{\delta}{\delta x^a}}$.

Definition 3.1. Let (M, F) be a warped product Finsler manifold. We call F a Ricci soliton if there exists a smooth vector field $V = v_1^i(x_1, x_2)\frac{\partial}{\partial x_1^i} + v_2^\alpha(x_1, x_2)\frac{\partial}{\partial x_2^\alpha}$ on M and a constant $K \in \mathbb{R}$ such that

$$\mathrm{Ric}_{ab} + \frac{1}{2}\mathcal{L}_{V^c} g_{ab} = K g_{ab}. \tag{10}$$

A Ricci soliton is called a *warped Finslerian gradient Ricci soliton* if there exists some function $h \in C^\infty(M)$ such that $V^c = \nabla h$. In fact, a warped product Finsler space is a gradient Ricci soliton if there exists a smooth function $h \in C^\infty(M)$ (called the potential function) such that

$$\begin{aligned}
&\mathrm{Ric}_{ij} + \nabla_i \nabla_j h + \hat{C}_{lij}\left[y_1^k \nabla_k v_1^l + y_2^\gamma \nabla_\gamma v_1^l\right] = K \hat{g}_{ij},\\
&\mathrm{Ric}_{i\alpha} + \nabla_i \nabla_\alpha h = 0,\\
&\mathrm{Ric}_{\alpha\beta} + \nabla_\alpha \nabla_\beta h + f^2 \check{C}_{\zeta\alpha\beta}\left[y_1^k \nabla_k v_2^\zeta + y_2^\gamma \nabla_\gamma v_2^\zeta\right] = K f^2 \check{g}_{\alpha\beta}. \qquad (11)
\end{aligned}$$

Now, we can state the main result.

Theorem 3.2. *Let (M, F) be a warped product Finsler manifold where the base space is Riemannian. Suppose that F is a warped Finslerian gradient Ricci soliton with h as potential function. Then the warping function f is a constant, or h depends only on the base space.*

Proof. Applying the coefficients of the Ricci tensor that are expressed in [11], when the base space is Riemannian, we have

$$\mathrm{Ric}_{kl} = \hat{\mathrm{Ric}}_{kl} + \hat{G}^j_{kl} \frac{\partial \ln f}{\partial x_1^j} - \frac{1}{2} \frac{\partial^2 \ln f^2}{\partial x_1^k \partial x_1^l}, \qquad \mathrm{Ric}_{k\nu} = 0,$$

$$\mathrm{Ric}_{\mu\nu} = \check{\mathrm{Ric}}_{\mu\nu} + \frac{1}{2}\frac{1}{f^2} y_1^h \frac{\partial f^2}{\partial x_1^h} \check{G}^\gamma_{\mu\nu\gamma} + \check{g}_{\mu\nu}\left[-\frac{1}{2}\hat{\Delta}_h f^2 + \frac{1}{4}\frac{1}{f^2}\hat{G}(\hat{\nabla}_h f^2, \hat{\nabla}_h f^2)\right]. \qquad (12)$$

On the other hand, the second covariant derivatives of h are given by

$$\nabla_k \nabla_l h = \frac{\partial^2 h}{\partial x_1^k \partial x_1^l} - \hat{F}^j_{kl} \frac{\partial h}{\partial x_1^j}, \qquad \nabla_k \nabla_\nu h = \frac{\partial h_\nu}{\partial x_1^k} - \frac{\partial \ln f}{\partial x_1^k} h_\nu,$$

$$\nabla_\mu \nabla_\nu h = \frac{\partial^2 h}{\partial x_2^\mu \partial x_2^\nu} + \frac{1}{2}\frac{\partial f^2}{\partial x_1^h} \hat{g}^{kh} \check{g}_{\mu\nu} \frac{\partial h}{\partial x_1^k} - \check{F}^\gamma_{\mu\nu} \frac{\partial h}{\partial x_2^\gamma} + \frac{1}{2} y_1^t \frac{\partial \ln f}{\partial x_1^t} \frac{\partial \check{g}_{\zeta\nu}}{\partial y_2^\mu} \check{g}^{\zeta\gamma} \frac{\partial h}{\partial x_2^\gamma}. \qquad (13)$$

From assumption, the warped product Finsler manifold (M, F) is a gradient Ricci soliton if the following equations hold:

$$\begin{aligned}
&\mathrm{Ric}_{kl} + \nabla_k \nabla_l h = K \hat{g}_{kl}, \qquad \mathrm{Ric}_{k\nu} + \nabla_k \nabla_\nu h = 0,\\
&\mathrm{Ric}_{\mu\nu} + \nabla_\mu \nabla_\nu h + f^2 \check{C}_{\zeta\mu\nu}\left[y_1^k \nabla_k v_2^\zeta + y_2^\gamma \nabla_\gamma v_2^\zeta\right] = K f^2 \check{g}_{\mu\nu}. \qquad (14)
\end{aligned}$$

By inserting the second equation of (12) and (13) into the second equation of (14), we obtain

$$\frac{\partial h_\nu}{\partial x_1^k} - \frac{\partial \ln f}{\partial x_1^k} h_\nu = 0.$$

Put $H = \frac{\partial h}{\partial x_1^k} - \frac{\partial \ln f}{\partial x_1^k} h$, then by using the above equation, we have $\frac{\partial H}{\partial x_2^\nu} = 0$. Hence the function H is a constant on M_2.

Now, let us differentiate of H with respect to x_1^l:

$$\frac{\partial H}{\partial x_1^l} = \frac{\partial^2 h}{\partial x_1^k \partial x_1^l} - \frac{\partial^2 \ln f}{\partial x_1^k \partial x_1^l} h - \frac{\partial \ln f}{\partial x_1^k} \frac{\partial h}{\partial x_1^l}.$$

Applying this equation to the first equation of (13), we obtain

$$\nabla_k \nabla_l h = \frac{\partial^2 \ln f}{\partial x_1^k \partial x_1^l} h + \frac{\partial \ln f}{\partial x_1^k} \frac{\partial \ln f}{\partial x_1^l} h + \frac{\partial \ln f}{\partial x_1^k} H + \frac{\partial H}{\partial x_1^l} - \hat{F}_{kl}^j \frac{\partial \ln f}{\partial x_1^j} h - \hat{F}_{kl}^j H. \quad (15)$$

By using (15) and the first equation of (12) to the first equation of (14), we have

$$\begin{aligned} K \hat{g}_{kl} = \hat{\mathrm{Ric}}_{kl} + \hat{G}_{kl}^j \frac{\partial \ln f}{\partial x_1^j} - \frac{1}{2} \frac{\partial^2 \ln f^2}{\partial x_1^k \partial x_1^l} + \frac{\partial^2 \ln f}{\partial x_1^k \partial x_1^l} h + \frac{\partial \ln f}{\partial x_1^k} \frac{\partial \ln f}{\partial x_1^l} h \\ + \frac{\partial \ln f}{\partial x_1^k} H + \frac{\partial H}{\partial x_1^l} - \hat{F}_{kl}^j H - \hat{F}_{kl}^j \frac{\partial \ln f}{\partial x_1^j} h. \end{aligned}$$

Differentiating this equation with respect to x_2^γ, we gain

$$\frac{\partial h}{\partial x_2^\gamma} [\frac{\partial^2 \ln f}{\partial x_1^k \partial x_1^l} + \frac{\partial \ln f}{\partial x_1^k} \frac{\partial \ln f}{\partial x_1^l} - \hat{F}_{kl}^j \frac{\partial \ln f}{\partial x_1^j}] = 0.$$

It follows that f is a constant, or h depends only on the base space. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 3.2.

Corollary 3.3. *Let (M, F) be a warped product Finsler manifold, where the base space is Riemannian and the warping function is non-constant. Consider (M, F) a warped Finslerian gradient Ricci soliton with potential function h and $\check{G}_{\mu\nu}^\gamma$ independent of y_2. Then the fiber space is an Einstein.*

Proof. From assumption, (M, F) is a warped Finslerian gradient Ricci soliton with h as potential function and $\check{G}_{\mu\nu\gamma}^\gamma = 0$. By Theorem 3.2, the potential function h is independent of x_2^γ, so the third equation of (14) reduces to

$$\check{\mathrm{Ric}}_{\mu\nu} + \check{g}_{\mu\nu} \left[-\frac{1}{2} \hat{\Delta}_h f^2 + \frac{1}{4} \frac{1}{f^2} \hat{G}(\hat{\nabla}_h f^2, \hat{\nabla}_h f^2) \right] + \frac{1}{2} \frac{\partial f^2}{\partial x_1^h} \hat{g}^{kh} \check{g}_{\mu\nu} \frac{\partial h}{\partial x_1^k} = K f^2 \check{g}_{\mu\nu}.$$

Then the Ricci tensor of $\mathbb{F}_2$ becomes

$$\check{\mathrm{Ric}}_{\mu\nu} = \check{g}_{\mu\nu}\left[Kf^2 + \frac{1}{2}\hat{\Delta}_h f^2 - \frac{1}{4}\frac{1}{f^2}\hat{G}(\hat{\nabla}_h f^2, \hat{\nabla}_h f^2) - \frac{1}{2}\frac{\partial f^2}{\partial x_1^h}\hat{g}^{kh}\frac{\partial h}{\partial x_1^k}\right] \equiv A\check{g}_{\mu\nu}.$$

Since $\frac{\partial h}{\partial x_2^\gamma} = 0$, we imply that $\frac{\partial A}{\partial x_2^\gamma} = 0$. So A is constant on the fiber space, and this means that F_2 is an Einstein metric. □

Now, we prove that a warped product Finsler manifold, when the base space is conformal to a Euclidean space, is a gradient Ricci soliton if and only if the warping and potential functions satisfy some partial differential equations.

Theorem 3.4. *Let $(M, F) = (\mathbb{R}^n \times M_2, F_1 \times_f F_2)$ be a warped product Finsler manifold, where F_1 is conformal to a Euclidean metric with smooth function φ on $\mathbb{R}^n$. If $\check{G}^\gamma_{\mu\nu}$ is independent of y_2, then the warped product Finsler manifold (M, F) is a gradient Ricci soliton with potential function h if and only if the functions f, φ and h satisfy*

$$\frac{1}{2}\frac{\partial^2 \ln\varphi^2}{\partial(x_1^i)^2} - \frac{1}{4}\left(\frac{\partial \ln\varphi^2}{\partial x_1^j}\right)^2 + \frac{1}{4}\frac{1}{f^2}\frac{\partial \ln\varphi^2}{\partial x_1^j}\frac{\partial f^2}{\partial x_1^j} - \frac{1}{2}\frac{\partial^2 f^2}{\partial(x_1^k)^2} + \frac{\partial^2 h}{\partial(x_1^k)^2} - \frac{1}{2}\frac{\partial \ln\varphi^2}{\partial x_1^j}\frac{\partial h}{\partial x_1^j} = K\frac{1}{\varphi^2}, \quad (16)$$

$$\frac{1}{2}\frac{\partial f^2}{\partial x_1^i}\frac{\partial h}{\partial x_1^i}\varphi^2 - \frac{\partial\varphi^2}{\partial x_1^i}\frac{\partial f^2}{\partial x_1^i} - \varphi^2\frac{\partial^2 f^2}{\partial(x_1^i)^2} + \frac{1}{2}\frac{1}{f^2}\varphi^2\left(\frac{\partial f^2}{\partial x_1^i}\right)^2 + \frac{1}{2}\varphi^2\frac{\partial \ln\varphi^2}{\partial x_1^i}\frac{\partial f^2}{\partial x_1^i} = Kf^2 - K_2, \quad (17)$$

$$\frac{\partial^2 h}{\partial x_1^k \partial x_1^l} = \frac{1}{2}\frac{\partial^2 f^2}{\partial x_1^k \partial x_1^l}. \quad (18)$$

Proof. Consider $(M_1, F_1) = (\mathbb{R}^n, [\frac{1}{\varphi^2}(\delta^i_j y_1^i y_1^j)]^{1/2})$, so we have

$$\hat{G}^i = \frac{1}{4}\hat{g}^{ik}\left(\frac{\partial^2 F_1^2}{\partial y_1^k \partial x_1^h}y_1^h - \frac{\partial F_1^2}{\partial x_1^k}\right) = \frac{1}{4}\frac{\partial \ln\varphi^2}{\partial x_1^i}(y_1^h)^2 - \frac{1}{2}\frac{\partial \ln\varphi^2}{\partial x_1^h}y_1^i y_1^h,$$

$$\hat{F}^j_{kl} = \frac{1}{2}\hat{g}^{jh}\left(\frac{\delta \hat{g}_{hk}}{\delta x_1^l} + \frac{\delta \hat{g}_{hl}}{\delta x_1^k} - \frac{\delta \hat{g}_{kl}}{\delta x_1^h}\right) = \frac{1}{2}\frac{\partial \ln\varphi^2}{\partial x_1^j}\hat{g}_{kl}. \quad (19)$$

Applying (6) and (7), we get

$$\hat{G}(\hat{\nabla}_h f^2, \hat{\nabla}_h f^2) = 2\varphi^2 \left(\frac{\partial f^2}{\partial x_1^i}\right)^2,$$

$$\hat{\Delta}_h f^2 = 2\frac{\partial \varphi^2}{\partial x_1^i}\frac{\partial f^2}{\partial x_1^i} + 2\varphi^2 \frac{\partial^2 f^2}{\partial (x_1^i)^2} - n\varphi^2 \frac{\partial \ln \varphi^2}{\partial x_1^i}\frac{\partial f^2}{\partial x_1^i}. \tag{20}$$

Also, we obtain the Ricci tensor of the base space as $\hat{\mathrm{Ric}}_{kl} = \frac{1}{2}\frac{\partial^2 \hat{K}_i^i}{\partial y_1^k \partial y_1^l}$, where the Berwald's formula is given by

$$\begin{aligned}
\hat{K}_i^i &= 2\frac{\partial \hat{G}^i}{\partial x_1^i} - \hat{G}_j^i \hat{G}_i^j - y_1^j \frac{\partial \hat{G}_i^i}{\partial x_1^j} + 2\hat{G}^j \hat{G}_{ji}^i \\
&= \frac{1}{2}\frac{\partial^2 \ln \varphi^2}{\partial (x_1^i)^2}(y_1^h)^2 - \frac{\partial^2 \ln \varphi^2}{\partial x_1^i \partial x_1^h} y_1^i y_1^h - \frac{1}{2}\frac{\partial \ln \varphi^2}{\partial x_1^j}\frac{\partial \ln \varphi^2}{\partial x_1^i} y_1^i y_1^j \\
&\quad + \frac{1}{2}\frac{\partial^2 \ln \varphi^2}{\partial x_1^j \partial x_1^h} y_1^j y_1^h + \frac{1}{2}\frac{\partial \ln \varphi^2}{\partial x_1^j}\frac{\partial \ln \varphi^2}{\partial x_1^h} y_1^j y_1^h \\
&\quad + \frac{1}{4}\left(\frac{\partial \ln \varphi^2}{\partial x_1^j} y_1^i\right)^2 + \frac{1}{4}\left(\frac{\partial \ln \varphi^2}{\partial x_1^i} y_1^j\right)^2 - \frac{1}{4}\left(\frac{\partial \ln \varphi^2}{\partial x_1^j} y_1^h\right)^2.
\end{aligned} \tag{21}$$

Since (M, F) is a gradient Ricci soliton with the potential function h and constant K, equations (12), (13) and (14) are satisfied. On the other hand, by Theorem 3.2 and Corollary 3.3, the function h is independent of x_2, and the fiber space is an Einstein with constant K_2. Then the third equation of (14) reduces to

$$K_2 \breve{g}_{\mu\nu} + \breve{g}_{\mu\nu}\left[-\frac{1}{2}\hat{\Delta}_h f^2 + \frac{1}{4}\frac{1}{f^2}\hat{G}(\hat{\nabla}_h f^2, \hat{\nabla}_h f^2)\right] + \frac{1}{2}\frac{\partial f^2}{\partial x_1^i}\frac{\partial h}{\partial x_1^i}\varphi^2 \breve{g}_{\mu\nu} = K f^2 \breve{g}_{\mu\nu}.$$

By substituting (20) into the above equation, we obtain (17).

Now, we consider two possible cases. The first is the case of $k = l$, and the other is $k \neq l$. If $k = l$, then the Ricci tensor is given by

$$\hat{\mathrm{Ric}}_{kl} = \frac{1}{2}\frac{\partial^2 \ln \varphi^2}{\partial (x_1^i)^2} - \frac{1}{4}\left(\frac{\partial \ln \varphi^2}{\partial x_1^j}\right)^2. \tag{22}$$

By applying (19), we have

$$\hat{G}_{kl}^j = \frac{1}{2}\frac{\partial \ln \varphi^2}{\partial x_1^j}, \qquad \hat{F}_{kl}^j = \frac{1}{2}\frac{\partial \ln \varphi^2}{\partial x_1^j}. \tag{23}$$

Inserting (22), (23) into the first equations of (12) and (13), and by using these results for the first equations of (14), we obtain (16).

Finally, in the case of $k \neq l$, we have $\hat{\mathrm{Ric}}_{kl} = \hat{G}^j_{kl} = \hat{F}^j_{kl} = 0$. Then we gain

$$\frac{\partial^2 h}{\partial x_1^k \partial x_1^l} = \frac{1}{2} \frac{\partial^2 f^2}{\partial x_1^k \partial x_1^l}. \qquad \square$$

Now, we want to find solutions of the system (16), (17) and (18) of the form $\varphi(\omega)$, $f(\omega)$ and $h(\omega)$, where $\omega = \sum_1^n \varepsilon^i x_1^i$, $\varepsilon^i \in \mathbb{R}$. The following theorem provides the system of ordinary differential equations that must be satisfied by such solutions.

Theorem 3.5. *Let $(\mathbb{R}^n \times M_2, F_1 \times_f F_2)$ be a warped product Finsler manifold, where F_1 is conformal to a Euclidean metric with function φ. Suppose that $\check{G}^\gamma_{\mu\nu}$ is independent of y_2, and $\varphi(\omega)$, $f(\omega)$, $h(\omega)$ are smooth functions on $\mathbb{R}^n$, where $\omega = \sum_1^n \varepsilon^i x_1^i$, $\varepsilon^i \in \mathbb{R}$. Then the warped product Finsler manifold is a gradient Ricci soliton with potential function h if and only if the functions f, φ and h satisfy*

$$h'' = (f')^2 + ff'', \tag{24}$$

$$(\varepsilon^i)^2[f\varphi\varphi'' - f(\varphi')^2] - (\varepsilon^j)^2[\varphi\varphi' f' - f(\varphi')^2 - f\varphi\varphi' h'] = Kf, \tag{25}$$

$$(\varepsilon^i)^2[ff'h'\varphi^2 - \varphi\varphi' ff' - ff''\varphi^2] = Kf^2 - K_2, \tag{26}$$

whenever $\sum_i (\varepsilon^i)^2 \neq 0$, and

$$h'' = (f')^2 + ff'', \qquad K = K_2 = 0, \tag{27}$$

whenever $\sum_i (\varepsilon^i)^2 = 0$.

Proof. Let $(\mathbb{R}^n \times M_2, F_1 \times_f F_2)$ be a warped Finslerian gradient Ricci soliton, $F_1^2 = \frac{1}{\varphi^2}(\delta^i_j y_1^i y_1^j)$ and $\check{G}^\gamma_{\mu\nu\gamma} = 0$. By Corollary 3.3, the Finsler metric F_2 is an Einstein with constant K_2. We also assume that $\varphi(\omega)$, $f(\omega)$, $h(\omega)$ are smooth functions on $\mathbb{R}^n$, where $\omega = \sum_1^n \varepsilon^i x_1^i$, $\varepsilon^i \in \mathbb{R}$. Then we have

$$\frac{\partial h}{\partial x_1^i} = \varepsilon^i h' \qquad \frac{\partial^2 h}{\partial x_1^i \partial x_2^j} = \varepsilon^i \varepsilon^j h'', \qquad \frac{\partial f^2}{\partial x_1^i} = 2\varepsilon^i ff'$$

$$\frac{\partial^2 f^2}{\partial x_1^i \partial x_2^j} = 2\varepsilon^i\varepsilon^j (f')^2 + 2\varepsilon^i\varepsilon^j ff'', \qquad \frac{\partial \varphi^2}{\partial x_1^i} = 2\varepsilon^i \varphi\varphi'$$

$$\frac{\partial\, ln\, \varphi^2}{\partial x_1^i} = 2\varepsilon^i \frac{\varphi'}{\varphi}, \qquad \frac{\partial^2 ln\, \varphi^2}{\partial x_1^i \partial x_1^j} = 2\varepsilon^i\varepsilon^j \frac{\varphi''}{\varphi} - 2\varepsilon^i\varepsilon^j \left(\frac{\varphi'}{\varphi}\right)^2. \tag{28}$$

Applying these expressions into (18), we get

$$\varepsilon^k\varepsilon^l h'' = \varepsilon^k\varepsilon^l[(f')^2 + ff''].$$

If there exist $k \neq l$ such that $\varepsilon^k\varepsilon^l \neq 0$, then we obtain (24).

Similarly, inserting (28) into equation (16), we have

$$(\varepsilon^i)^2\varphi\varphi'' - (\varepsilon^i)^2(\varphi')^2 - (\varepsilon^j)^2(\varphi')^2 + (\varepsilon^j)^2\varphi\varphi'\frac{f'}{f} - (\varepsilon^k)^2\varphi^2(f')^2 - (\varepsilon^k)^2\varphi^2 ff'' + (\varepsilon^k)^2\varphi^2 h'' - (\varepsilon^j)^2\varphi\varphi' h' = K. \quad (29)$$

Using the relation between h'', f' and f'', we get (25).

Analogously, we apply (28) into the (17) and obtain (26). Hence, if $\omega = \sum_i(\varepsilon^i)^2 \neq 0$, we get the desired result. If $\omega = \sum_i(\varepsilon^i)^2 = 0$, equation (24) holds, and equations (25) and (26) imply that $K = K_2 = 0$.

Now, consider the manner in which for all $k \neq l$, $\varepsilon^k\varepsilon^l = 0$. Then $\omega = x_1^{k_0}$, and equation (24) is trivially satisfied. For $k \neq k_0$, by applying (29), we get (25). In the case of $k = k_0$, equation (29) implies (24). If $k = k_0$ or $k \neq k_0$, we get (26) from (28) and (17). This completes the proof of the theorem. □

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