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Warped product pseudo-slant immersions in Sasakian manifolds

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Dedicated to Professor Lajos Tamássy on the occasion of his 94th birthday

Abstract. Recently, we have proved that there do not exist warped product semi-slant submanifolds of Sasakian manifolds other than contact CR-warped products which have been studied by Hasegawa and Mihai. In this paper, we introduce another class of submanifolds, called warped product pseudo-slant submanifolds. A characterization theorem for such immersions is obtained. Also, we establish an inequality for the squared norm of the second fundamental form in terms of the warping function and the slant angle. Furthermore, the equality case in the statement of the inequality is investigated, and we give two examples of pseudo-slant and warped product pseudo-slant submanifolds.

1. Introduction

The geometry of slant submanifolds has intensely been studied since B.-Y. Chen defined and studied slant immersions in complex geometry as a natural generalization of both holomorphic and totally real immersions [7], [8]. Later, A. Lotta extended this study to almost contact metric manifolds [20]. After that, Cabrerizo *et al.* [5] studied and characterized these submanifolds in case of K-contact and Sasakian manifolds. To generalize these submanifolds, N. Papaghiuc [21] introduced the notion of another class of submanifolds, called semi-slant submanifolds, and then this idea was further extended by Cabrerizo *et al.*

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for semi-slant submanifolds in contact metric manifolds [4]. Later, A. Carriazo [6] introduced another class of submanifolds, called anti-slant submanifolds.

On the other hand, the concept of warped products was introduced by Bishop and O'Neill [2] to construct examples of Riemannian manifolds with negative curvature. They defined these manifolds as follows:

Let M_1 and M_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and a positive differentiable function f on M_1. Consider the product manifold $M_1 \times M_2$ with its projections $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$. Then their warped product manifold $M = M_1 \times_f M_2$ is the Riemannian manifold $M_1 \times M_2 = (M_1 \times M_2, g)$ equipped with the Riemannian structure such that

$$g(X, Y) = g_1(\pi_{1*}X, \pi_{1*}Y) + (f \circ \pi_1)^2 g_2(\pi_{2*}X, \pi_{2*}Y),$$

for any vector field X, Y tangent to M, where $*$ is the symbol for the tangent maps. A warped product manifold $M = M_1 \times_f M_2$ is said to be *trivial*, or simply a *Riemannian product manifold*, if the warping function f is constant. Let X be a vector field on M_1, and Z be a vector field on M_2, then from [2, Lemma 7.3], we have

$$\nabla_X Z = \nabla_Z X = (X \ln f) Z, \tag{1.1}$$

where ∇ is the Levi–Civita connection on M. If $M = M_1 \times_f M_2$ is a warped product manifold, then M_1 is a totally geodesic submanifold, and M_2 is a totally umbilical submanifold of M [2]. We note that warped product manifolds have their application to general relativity. Many spacetime models such as Robertson–Walker spacetime, asymptotically flat spacetime, Schwarzschild spacetime and Reissner–Nordström spacetime are examples of warped product manifolds, for details see [17].

In the beginning of this century, B.-Y. Chen initiated the study of warped product CR-submanifolds of Kaehler manifolds [9], [10]. He established several fundamental results on the existence of such warped product submanifolds, including optimal inequalities and characterisations [11], [12]. Later, Hasegawa and Mihai extended this study by investigating contact CR-warped product submanifolds in Sasakian manifolds [16]. For the detailed survey on warped product manifolds and warped product submanifolds, we refer to [13], [15], [14].

Recently, warped product pseudo-slant submanifolds of Kaehler manifolds were studied by B. Sahin under the name of Hemi-slant warped products [24]. He obtained many interesting results, including a characterization and a sharp relation for the squared norm of the second fundamental form by using the mixed totally geodesic condition. In the context of contact metric manifold, we have seen that there is no warped product semi-slant submanifold in Sasakian manifolds [1], [25].

In this paper, we study the warped product submanifolds where one of the factor is slant and another is anti-invariant, and we call such submanifolds *warped product pseudo-slant submanifolds* (warped product hemi-slant submanifolds in the same sense of Sahin [24]) of Sasakian manifolds.

The paper is organized as follows. In Section 2, we review some basic formulas and definitions for almost contact metric manifolds and their submanifolds. In Section 3, we recall the definitions of slant and pseudo-slant submanifolds, as well as provide an example and some basic results which are useful for the next section. In Section 4, we study warped product pseudo-slant submanifolds. At the beginning of this section, we construct an example of such warped product immersions and then obtain a characterization result. In the same section, we also establish an inequality for the squared norm of the second fundamental form in terms of warping function and the slant angle. The equality case is also considered.

2. Preliminaries

An *almost contact manifold* is a $(2n+1)$ odd-dimensional manifold $\widetilde{M}$ which carries a tensor field φ of the tangent space, a vector field ξ, called *characteristic* or *Reeb vector field*, and a 1-form η satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

where $I : T\widetilde{M} \to T\widetilde{M}$ is the identity mapping. From the definition, it follows that $\varphi\xi = 0$, $\eta \circ \varphi = 0$, and the $(1,1)$-tensor field φ has constant rank $2n$ (cf. [3]). An almost contact manifold $(\widetilde{M}, \varphi, \eta, \xi)$ is said to be *normal* when the tensor field $N_\varphi = [\varphi, \varphi] + 2d\eta \otimes \xi$ vanishes identically, where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. It is known that any almost contact manifold $(\widetilde{M}, \varphi, \eta, \xi)$ admits a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for any $X, Y \in \Gamma(T\widetilde{M})$, the Lie algebra of vector fields on $\widetilde{M}$. This metric g is called a *compatible metric*, and the manifold $\widetilde{M}$ together with the structure

(φ, ξ, η, g) is called an *almost contact metric manifold*. As an immediate consequence of (2.2), one has $\eta(X) = g(X, \xi)$ and $g(\varphi X, Y) = -g(X, \varphi Y)$. Hence, the fundamental 2-form Φ of $\widetilde{M}$ is defined $\Phi(X, Y) = g(X, \varphi Y)$ and the manifold is said to be a *contact metric manifold* if $\Phi = d\eta$. If ξ is a Killing vector field with respect to g, the contact metric structure is called a *K-contact structure*. A normal contact metric manifold is said to be a *Sasakian manifold*. In terms of the covariant derivatives of φ, the Sasakian condition can be expressed by

$$(\widetilde{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X \tag{2.3}$$

for all $X, Y \in \Gamma(T\widetilde{M})$, where $\widetilde{\nabla}$ is the Levi–Civita connection of g. From the formula (2.3), it follows that $\widetilde{\nabla}_X \xi = -\varphi X$. The covariant derivative of φ is defined by

$$(\widetilde{\nabla}_X \varphi)Y = \widetilde{\nabla}_X \varphi Y - \varphi \widetilde{\nabla}_X Y$$

for all $X, Y \in \Gamma(T\widetilde{M})$.

Now, let M be an isometrically immersed submanifold in $\widetilde{M}$ with induced metric g. Let $\Gamma(TM)$ the Lie algebra of vector fields on M and $\Gamma(T^\perp M)$, the set of all vector fields normal to M. If we denote the Levi–Civita connection of M by ∇, then the Gauss and Weingarten formulas are respectively given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \quad \widetilde{\nabla}_X V = -A_V X + \nabla^\perp_X V, \tag{2.4}$$

for any vector field $X,\ Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, where $\nabla^\perp$ is the normal connection in the normal bundle, σ is the second fundamental form, and A_V is the shape operator (corresponding to the normal vector field V) for the immersion of M into $\widetilde{M}$. They are related by $g(\sigma(X, Y), V) = g(A_V X, Y)$.

For any $X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, the tangential and normal components of φX and φV are respectively decomposed as

$$(a)\ \varphi X = TX + FX, \qquad (b)\ \varphi V = tV + fV. \tag{2.5}$$

Let $p \in M$ and $\{e_1, \ldots, e_m, \ldots, e_{2n+1}\}$ be an orthonormal basis of the tangent space $T_p\widetilde{M}$ such that $e_1, \ldots, e_m$ are tangent to M at p. We denote by H the mean curvature vector, that is $H(p) = \frac{1}{m}\sum_{i=1}^m \sigma(e_i, e_i)$. Also, we set

$$\sigma^r_{ij} = g(\sigma(e_i, e_j), e_r), \quad i, j \in \{1, \ldots, m\}, \quad r \in \{n+1, \ldots, 2n+1\}, \tag{2.6}$$

and

$$\|\sigma\|^2 = \sum_{i,j=1}^{m} g(\sigma(e_i, e_j), \sigma(e_i, e_j)). \tag{2.7}$$

The gradient $\vec{\nabla} f$ of a smooth function f on a manifold M is defined as $g(\vec{\nabla} f, X) = Xf$, for any $X \in \Gamma(TM)$. As a consequence, we have

$$\|\vec{\nabla} f\|^2 = \sum_{i=1}^{n} (e_i(f))^2 \tag{2.8}$$

for an orthonormal frame $\{e_1, \ldots, e_n\}$ on M.

A submanifold M normal to ξ in a Sasakian manifold is said to be a *C-totally real* submanifold. In this case, φ maps any tangent space of M into the normal space, that is, $\varphi(T_pM) \subset T_p^{\perp}M$, for every $p \in M$.

For submanifolds tangent to the structure vector field ξ, there are different classes of submanifolds. We mention the following:

(i) A submanifold M tangent to ξ is called an *invariant submanifold* if φ preserves the tangent space of M, that is, $\varphi(T_pM) \subset T_pM$, for every $p \in M$.

(ii) A submanifold M tangent to ξ is called an *anti-invariant submanifold* if φ maps any tangent space of M into the normal space, that is, $\varphi(T_pM) \subset T_p^{\perp}M$, for every $p \in M$.

(iii) A submanifold M tangent to ξ is called a *contact CR-submanifold* if there exists a pair of orthogonal distributions $\mathcal{D} : p \to \mathcal{D}_p$ and $\mathcal{D}^{\perp} : p \to \mathcal{D}_p^{\perp}$, $\forall\, p \in M$ such that $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \langle\xi\rangle$, where $\langle\xi\rangle$ is the 1-dimensional distribution spanned by the structure vector field ξ, $\mathcal{D}$ is invariant, i.e., $\varphi\mathcal{D} = \mathcal{D}$ and $\mathcal{D}^{\perp}$ is anti-invariant, i.e., $\varphi\mathcal{D}^{\perp} \subseteq T^{\perp}M$.

There are other classes of submanifolds tangent to ξ, which we discuss in the next section.

3. Slant and pseudo-slant immersions

Slant immersions in complex geometry were defined and studied by B.-Y. Chen [7], [8]. Later on, A. Lotta introduced the notion of slant immersions of a Riemannian manifold into an almost contact metric manifold, and he discussed some properties of such immersions [20]. In [5], Cabrerizo *et al.* studied slant submanifolds of Sasakain manifolds.

A submanifold M tangent to ξ is said to be *slant* if, for any $p \in M$ and any $X \in T_pM$, linearly independent to ξ, the angle between φX and T_pM is a constant $\theta \in [0, \pi/2]$, which is called the *slant angle* of M in $\widetilde{M}$. Invariant and anti-invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A slant immersion which is not invariant nor anti-invariant is called a *proper slant immersion*.

We recall the following result for slant submanifolds.

Theorem 3.1 ([5]). *Let M be a submanifold of an almost contact metric manifold $\widetilde{M}$, such that $\xi \in \Gamma(TM)$. Then M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$T^2 = \lambda(-I + \eta \otimes \xi). \tag{3.1}$$

Furthermore, if θ is slant angle, then $\lambda = \cos^2\theta$.

The following relations are straightforward consequence of (3.1):

$$g(TX, TY) = \cos^2\theta[g(X, Y) - \eta(X)\eta(Y)], \tag{3.2}$$

$$g(FX, FY) = \sin^2\theta[g(X, Y) - \eta(X)\eta(Y)], \tag{3.3}$$

for any $X, Y \in \Gamma(TM)$.

Now, for a slant submanifold M, we prove the following result for later use.

Theorem 3.2. *Let M be a proper slant submanifold of an almost contact metric manifold $\widetilde{M}$. Then*

$$(a)\ tFX = \sin^2\theta(-X + \eta(X)\xi), \qquad (b)\ fFX = -FTX, \tag{3.4}$$

for any $X \in \Gamma(TM)$.

Proof. From (2.5) (a), we have $\varphi^2 X = \varphi TX + \varphi FX$, for any $X \in \Gamma(TM)$. Using (2.1) and again by (2.5), we derive

$$-X + \eta(X)\xi = T^2X + FTX + tFX + fFX.$$

Then, using Theorem 3.1 and equating the tangential and normal components, we get the desired result. □

Pseudo-slant submanifolds were defined by Carriazo in [6] under the name of *anti-slant submanifolds* as a particular class of bi-slant submanifolds. However, the term "anti-slant" makes it seem as though there is no slant part, which is not the case. Later, he called these classes of submanifolds as *pseudo-slant submanifolds*. He defined these submanifolds as follows:

A submanifold M of an almost contact metric manifold $\widetilde{M}$ is said to be a *pseudo-slant submanifold* if there exists a pair of orthogonal distributions $\mathcal{D}^{\perp}$ and $\mathcal{D}^{\theta}$ on M such that $TM = \mathcal{D}^{\perp} \oplus \mathcal{D}^{\theta} \oplus \langle\xi\rangle$, $\mathcal{D}^{\perp}$ is anti-invariant, that is, $\varphi(\mathcal{D}^{\perp}) \subset T^{\perp}M$ and $\mathcal{D}^{\theta}$ is slant with angle $\theta \neq 0$.

In this case, we call the angle θ the slant angle of the submanifold M. Denote the dimensions of $\mathcal{D}^{\perp}$ and $\mathcal{D}^{\theta}$ by n_1 and n_2, respectively. Then invariant (resp. anti-invariant) and proper slant submanifolds are the special cases of pseudo-slant submanifolds with $n_1 = 0$, $\theta = 0$ (resp. $n_2 = 0$) and $n_1 = 0$, respectively. Similarly, contact CR-submanifolds are also the special cases of pseudo-slant submanifolds with slant angle $\theta = 0$. A pseudo-slant submanifold is said to be proper if neither any $n_i = 0,\ i = 1, 2$, nor $\theta = 0$ or $\pi/2$.

A pseudo-slant submanifold M is said to be *mixed totally geodesic* if $\sigma(X, Z) = 0$, for any $X \in \Gamma(\mathcal{D}^{\theta} \oplus \langle\xi\rangle)$ and $Z \in \Gamma(\mathcal{D}^{\perp})$.

Let M be a pseudo-slant submanifold of an almost contact metric manifold $\widetilde{M}$. Then the normal bundle $T^{\perp}M$ can be decomposed as

$$T^{\perp}M = \varphi\mathcal{D}^{\perp} \oplus F\mathcal{D}^{\theta} \oplus \nu, \quad \varphi\mathcal{D}^{\perp} \perp F\mathcal{D}^{\theta},$$

where ν is the normal invariant subbundle under φ.

For the examples of slant submanifolds of Sasakian manifolds, we refer to [5]. In the following, we construct an example of a pseudo-slant submanifold in a Sasakian manifold.

Example 3.1. Consider $\mathbf{R}^9$ with its usual Sasakian structure (φ, ξ, η, g), given by

$$\varphi\left\{\sum_{i=1}^{4}\left(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i}\right) + Z\frac{\partial}{\partial z}\right\} = \sum_{i=1}^{4}\left(Y_i\frac{\partial}{\partial x^i} - X_i\frac{\partial}{\partial y^i}\right) + \sum_{i=1}^{4} Y_i y^i\frac{\partial}{\partial z},$$

$$\xi = 2\frac{\partial}{\partial z}, \quad \eta = \frac{1}{2}\left(dz - \sum_{i=1}^{4} y^i dx^i\right),$$

$$g = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{4}(dx^i \otimes dx^i + dy^i \otimes dy^i),$$

where $(x^1, \ldots, x^4, y^1, \ldots, y^4, z)$ are Cartesian coordinates. Now, for any $\theta \in [0, \frac{\pi}{2})$,

$$x(u, v, w, s, z) = 2(u, 0, w, 0, 0, v, s\cos\theta, s\sin\theta, z)$$

defines a 5-dimensional proper pseudo-slant submanifold M. Then the tangent space of M spanned by the orthogonal tangent vectors

$$e_1 = 2\left(\frac{\partial}{\partial x^1} + y^1\frac{\partial}{\partial z}\right),\quad e_2 = 2\frac{\partial}{\partial y^2},\quad e_3 = 2\left(\frac{\partial}{\partial x^3} + y^3\frac{\partial}{\partial z}\right),$$
$$e_4 = \cos\theta\left(2\frac{\partial}{\partial y^3}\right) + \sin\theta\left(2\frac{\partial}{\partial y^4}\right),\quad e_5 = 2\frac{\partial}{\partial z} = \xi.$$

Thus the tangent space $TM = \mathcal{D}^\perp \oplus \mathcal{D}^\theta \oplus \langle\xi\rangle$, where $\mathcal{D}^\perp = \mathrm{Span}\{e_1, e_2\}$ and $\mathcal{D}^\theta = \mathrm{Span}\{e_3, e_4\}$.

Now, we have the following useful results for pseudo-slant submanifolds.

Proposition 3.1 ([19]). *Let M be a pseudo-slant submanifold of a Sasakian manifold $\widetilde{M}$. Then, the anti-invariant distribution $\mathcal{D}^\perp$ is always integrable.*

Lemma 3.1. *On a pseudo-slant submanifold M of a Sasakian manifold $\widetilde{M}$, we have*

$$g(\nabla_X Z, Y) = \sec^2\theta\{g(\sigma(X,Z), FTY) - g(\sigma(X,TY), \varphi Z)\} \tag{3.5}$$

for any $X, Y \in \Gamma(\mathcal{D}^\theta \oplus \langle\xi\rangle)$ and $Z \in \Gamma(\mathcal{D}^\perp)$.

Proof. From (2.4) and (2.2), we have

$$g(\nabla_X Y, Z) = g(\widetilde{\nabla}_X Y, Z) = g(\varphi\widetilde{\nabla}_X Y, \varphi Z) = g(\widetilde{\nabla}_X \varphi Y, \varphi Z) - g((\widetilde{\nabla}_X \varphi)Y, \varphi Z)$$

for any $X, Y \in \Gamma(\mathcal{D}^\theta \oplus \langle\xi\rangle)$ and $Z \in \Gamma(\mathcal{D}^\perp)$. Using (2.5) (a) and (2.3), we derive

$$g(\nabla_X Y, Z) = g(\widetilde{\nabla}_X TY, \varphi Z) + g(\widetilde{\nabla}_X FY, \varphi Z).$$

Then, from (2.4) and the covariant derivative property of φ, we find

$$g(\nabla_X Y, Z) = g(\sigma(X,TY), \varphi Z) - g(FY, \varphi\widetilde{\nabla}_X Z) - g(FY, (\widetilde{\nabla}_X\varphi)Z).$$

Using (2.3) and (2.2), we obtain

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\sigma(X,TY), \varphi Z) + g(\varphi FY, \widetilde{\nabla}_X Z) \\ &= g(\sigma(X,TY), \varphi Z) + g(tFY, \widetilde{\nabla}_X Z) + g(fFY, \widetilde{\nabla}_X Z). \end{aligned}$$

Thus by Theorem 3.2, we get

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\sigma(X,TY), \varphi Z) - \sin^2\theta g(Y, \widetilde{\nabla}_X Z) \\ &\quad - \eta(Y)\sin^2\theta g(\widetilde{\nabla}_X \xi, Z) - g(FTY, \widetilde{\nabla}_X Z). \end{aligned}$$

Using (2.3), (2.4) and the orthogonality of two distributions, we derive

$$g(\nabla_X Y, Z) = g(\sigma(X,TY), \varphi Z) + \sin^2\theta g(\widetilde{\nabla}_X Y, Z) - g(\sigma(X,Z), FTY).$$

Hence, the assertion follows from the last relation. □

Proposition 3.2. *Let M be a pseudo-slant submanifold of a Sasakian manifold $\widetilde{M}$. Then*

$$\begin{aligned}\cos^2\theta g([X,Y],Z) &= g(\sigma(X,TY),\varphi Z) - g(\sigma(Y,TX),\varphi Z)\\ &\quad + g(\sigma(Y,Z),FTX) - g(\sigma(X,Z),FTY),\end{aligned} \tag{3.6}$$

for any $X,Y\in\Gamma(\mathcal{D}^\theta\oplus\langle\xi\rangle)$ and $Z\in\Gamma(\mathcal{D}^\perp)$.

Proof. By Lemma 3.1, we have

$$g(\nabla_X Y,Z) = \sec^2\theta\{g(\sigma(X,TY),\varphi Z) - g(\sigma(X,Z),FTY)\} \tag{3.7}$$

for any $X,Y\in\Gamma(\mathcal{D}^\theta\oplus\langle\xi\rangle)$ and $Z\in\Gamma(\mathcal{D}^\perp)$. By interchanging X by Y in (3.7), we find

$$g(\nabla_Y X,Z) = \sec^2\theta\{g(\sigma(Y,TX),\varphi Z) - g(\sigma(Y,Z),FTX)\}. \tag{3.8}$$

Thus the result follows from (3.7) and (3.8). □

4. Warped product pseudo-slant submanifolds

Recently, Sahin proved the non-existence of warped product semi-slant submanifolds in Kaehler manifolds [23]. Then, he considered warped product pseudo-slant (hemi-slant) submanifolds of Kaehler manifolds [24]. He proved many interesting results, including characterization and inequality for such submanifolds, by using the mixed totally geodesic condition. In the context of almost contact metric manifolds, we have seen that there do not exist warped product semi-slant submanifolds other than contact CR-warped products in Sasakian manifolds [1], [25]. We have also proved the non-existence of warped product pseudo-slant submanifolds of the form $M_\perp\times_f M_\theta$ of a Sasakian manifold $\widetilde{M}$ in [25], where $M_\perp$ and M_θ are anti-invariant and proper slant submanifolds of a Sasakian manifold $\widetilde{M}$, respectively. In this paper, we consider the warped products of the form $M_\theta\times_f M_\perp$.

First, we give the following example of a warped product pseudo-slant submanifold of an almost contact metric manifold.

Example 4.1. Consider a submanifold M of $\mathbf{R}^7$ with the Cartesian coordinates $(x_1,y_1,x_2,y_2,x_3,y_3,z)$ and the almost contact structure

$$\varphi\left(\frac{\partial}{\partial x_i}\right)=\frac{\partial}{\partial y_i},\quad \varphi\left(\frac{\partial}{\partial y_j}\right)=-\frac{\partial}{\partial x_j},\quad \varphi\left(\frac{\partial}{\partial z}\right)=0,\quad 1\le i,j\le 3.$$

For any vector field $X = \lambda_i \frac{\partial}{\partial x_i} + \mu_j \frac{\partial}{\partial y_j} + \nu \frac{\partial}{\partial z} \in \Gamma(T\mathbf{R}^7)$, then we have

$$g(X, X) = \lambda_i^2 + \mu_j^2 + \nu^2, \quad g(\varphi X, \varphi X) = \lambda_i^2 + \mu_j^2$$

and

$$\varphi^2(X) = -\lambda_i \frac{\partial}{\partial x_i} - \mu_j \frac{\partial}{\partial y_j} = -X + \eta(X)\xi$$

for any $i, j = 1, 2, 3$. It is clear that $g(\varphi X, \varphi X) = g(X, X) - \eta^2(X)$. Thus, (φ, ξ, η, g) is an almost contact metric structure on $\mathbf{R}^7$. Let us consider the immersion χ of M into $\mathbb{R}^7$ as

$$\chi(u, v, w, z) = (v, u, u\cos w, \sqrt{3}\, v\cos w, u\sin w, \sqrt{3}\, v\sin w, z).$$

Then the tangent bundle TM of M is spanned by the following orthogonal vector fields:

$$e_1 = \frac{\partial}{\partial y_1} + \cos w\, \frac{\partial}{\partial x_2} + \sin w\, \frac{\partial}{\partial x_3}, \quad e_2 = \frac{\partial}{\partial x_1} + \sqrt{3}\, \cos w\, \frac{\partial}{\partial y_2} + \sqrt{3}\, \sin w\, \frac{\partial}{\partial y_3},$$
$$e_3 = -u\sin w\, \frac{\partial}{\partial x_2} - \sqrt{3}\, v\sin w\, \frac{\partial}{\partial y_2} + u\cos w\, \frac{\partial}{\partial x_3} + \sqrt{3}\, v\cos w\, \frac{\partial}{\partial y_3}; \quad e_4 = \frac{\partial}{\partial z}.$$

Then, with respect to the given almost contact structure, we obtain

$$\begin{aligned}
\varphi e_1 &= -\frac{\partial}{\partial x_1} + \cos w\, \frac{\partial}{\partial y_2} + \sin w\, \frac{\partial}{\partial y_3}, \\
\varphi e_2 &= \frac{\partial}{\partial y_1} - \sqrt{3}\, \cos w\, \frac{\partial}{\partial x_2} - \sqrt{3}\, \sin w\, \frac{\partial}{\partial x_3}, \quad \varphi e_4 = 0, \\
\varphi e_3 &= -u\sin w\, \frac{\partial}{\partial y_2} + \sqrt{3}\, v\sin w\, \frac{\partial}{\partial x_2} + u\cos w\, \frac{\partial}{\partial y_3} - \sqrt{3}\, v\cos w\, \frac{\partial}{\partial x_3}.
\end{aligned}$$

Since φe_3 is orthogonal to TM, $\mathcal{D}^\perp = \mathrm{Span}\{e_3\}$ is an anti-invariant distribution, and $\mathcal{D}^\theta = \mathrm{Span}\{e_1, e_2\}$ is a proper slant distribution with slant angle $\theta = \frac{5\pi}{12}$ such that $\xi = e_4$ tangent to $\mathcal{D}^\theta$. Hence M is a proper pseudo-slant submanifold of $\mathbb{R}^7$. It is easy to show that the slant distribution $\mathcal{D}^\theta \oplus <\xi>$ is integrable. We denote the integral manifolds of $\mathcal{D}^\perp$ and $\mathcal{D}^\theta \oplus <\xi>$ by $M_\perp$ and M_θ, respectively. Then the metric tensor g of the product manifold M is given by

$$g = dz^2 + 2du^2 + 4dv^2 + (u^2 + 3v^2)dw^2 = g_1 + \left(\sqrt{u^2 + 3v^2}\right)^2 g_2,$$

where $g_1 = dz^2 + 2du^2 + 4dv^2$ is the metric tensor of M_θ and g_2 is the metric tensor of $M_\perp$. Thus M is a warped product pseudo-slant submanifold of the form $M_\theta \times_f M_\perp$ with warping function $f = \sqrt{u^2 + 3v^2}$.

Now, we have the following results for later use.

Lemma 4.1. *Let $M = M_\theta \times {}_f M_\perp$ be a warped product submanifold of a Sasakian manifold $\widetilde{M}$ such that $\xi \in \Gamma(TM_\theta)$, where $M_\perp$ and M_θ are anti-invariant and proper slant submanifolds of $\widetilde{M}$, respectively. Then*

(i) $g(\sigma(X,Y),\varphi Z) = g(\sigma(X,Z),FY)$;

(ii) $g(\sigma(Z,W),FTX) = g(\sigma(Z,TX),\varphi W) - \cos^2\theta(X\ln f)g(Z,W)$;

(iii) $g(\sigma(Z,W),FX) = g(\sigma(Z,X),\varphi W) + \{(TX\ln f) + \eta(X)\}g(Z,W)$;

or any $X,Y \in \Gamma(TM_\theta)$ and $Z,W \in \Gamma(TM_\perp)$.

Proof. From (2.4), we have

$$g(\sigma(X,Y),\varphi Z) = g(\widetilde{\nabla}_X Y,\varphi Z)$$

for any $X,Y \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$. Then, using (2.2) and the covariant derivative property of φ, we obtain

$$g(\sigma(X,Y),\varphi Z) = g((\widetilde{\nabla}_X\varphi)Y,Z) - g(\widetilde{\nabla}_X\varphi Y,Z).$$

Using (2.3) and the fact that ξ is tangent to M_θ, we derive

$$g(\sigma(X,Y),\varphi Z) = g(TY,\widetilde{\nabla}_X Z) + g(FY,\widetilde{\nabla}_X Z). \tag{4.1}$$

Hence, the first part of the lemma follows from (4.2) by using (2.4) and (1.1). For the other parts, for any $Z,W \in \Gamma(TM_\perp)$ and $X \in \Gamma(TM_\theta)$, we have

$$g(\sigma(Z,W),FTX) = g(\sigma(Z,W),\varphi TX) = g(\widetilde{\nabla}_Z W,\varphi TX) - g(\nabla_Z W,T^2X).$$

Using (2.2), (3.1) and the covariant derivative property of φ, we derive

$$\begin{aligned} g(\sigma(Z,W),FTX) = &- g(\widetilde{\nabla}_Z\varphi W,TX) + g((\widetilde{\nabla}_Z\varphi)W,TX) \\ &+ \cos^2\theta g(\nabla_Z W,X) + \eta(X)\cos^2\theta g(\nabla_Z\xi,W). \end{aligned}$$

Thus from (2.3), (2.4) and the fact that ξ is tangent to M_θ, we obtain

$$g(\sigma(Z,W),FTX) = g(A_{\varphi W}Z,TX) - \cos^2\theta g(W,\nabla_Z X). \tag{4.2}$$

Hence, the second part follows from (4.3) and (1.1). If we replace X by TX in (ii), then we have

$$g(\sigma(Z,W),FX) = g(\sigma(Z,X),\varphi W) - \eta(X)g(\sigma(Z,\xi),\varphi W) + (TX\ln f)g(Z,W).$$

Part (iii) of the lemma follows from the above relation by using the fact that for a Sasakian manifold, we have $\sigma(Z,\xi) = -\varphi Z$. Thus, the proof is complete. □

Now, we give the following characterization for pseudo-slant submanifolds.

Theorem 4.1. *Let M be a proper pseudo-slant submanifold of a Sasakian manifold $\widetilde{M}$ such that ξ is tangent to the slant distribution $\mathcal{D}^\theta$. Then M is a locally warped product manifold of the form $M_\theta \times_\mu M_\perp$ such that $M\theta$ is a proper slant submanifold and $M_\perp$ is an anti-invariant submanifold of $\widetilde{M}$ if and only if*

$$A_{\varphi Z}TX - A_{FTX}Z = \cos^2\theta\, X(\mu)Z, \tag{4.3}$$

for any $X \in \Gamma(\mathcal{D}^\theta \oplus \langle\xi\rangle)$ and $Z \in \Gamma(\mathcal{D}^\perp)$, where μ is a function on M such that $W(\mu) = 0$, for any $W \in \Gamma(\mathcal{D}^\perp)$.

Proof. Let $M = M_\theta \times_f M_\perp$ be a warped product submanifold of a Sasakian manifold $\widetilde{M}$. Then, for any $X, Y \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$, we have

$$g(A_{\varphi Z}TX, Y) = g(h(TX, Y), \varphi Z) = g(\widetilde{\nabla}_Y TX, \varphi Z).$$

Then, from (2.2) and the covariant derivative of φ, we have

$$g(A_{\varphi Z}TX, Y) = g((\widetilde{\nabla}_Y \varphi)TX, Z) - g(\widetilde{\nabla}_Y \varphi TX, Z).$$

Using (2.5) (a) and (2.3), we get

$$g(A_{\varphi Z}TX, Y) = -g(\widetilde{\nabla}_Y T^2X, Z) - g(\widetilde{\nabla}_Y FTX, Z).$$

By Theorem 3.1, the above equation takes the form

$$g(A_{\varphi Z}TX, Y) = \cos^2\theta\, g(\widetilde{\nabla}_Y X, Z) - \cos^2\theta\, \eta(X) g(\widetilde{\nabla}_Y \xi, Z) + g(A_{FTX}Y, Z).$$

From the symmetry of the shape operator A and the characteristic equation of the Sasakian structure, we obtain

$$g(A_{\varphi Z}TX - A_{FTX}Z, Y) = -\cos^2\theta\, g(X, \nabla_Y Z).$$

Then, by using (1.1) and the orthogonality of vector fields, we get

$$g(A_{\varphi Z}TX - A_{FTX}Z, Y) = 0.$$

Thus, we conclude that $A_{\varphi Z}TX - A_{FTX}Z$ lies in $TM_\perp$. This fact can also be obtained from Lemma 4.1 (i) by interchanging Y by TY. Using this fact with Lemma 4.1 (ii), we get (4.3).

Conversely, if M is a pseudo-slant submanifold with the slant distribution $\mathcal{D}^\theta$ tangent to ξ and the anti-invariant distribution $\mathcal{D}^\perp$ such that (4.3) holds, then, by Proposition 3.2, we have

$$\cos^2\theta\, g([X,Y],Z) = g(A_{\varphi Z}TY - A_{FTY}Z, X) - g(A_{\varphi Z}TX - A_{FTX}Z, Y)$$

for any $X, Y \in \Gamma(\mathcal{D}^\theta \oplus \langle\xi\rangle)$ and $Z \in \Gamma(\mathcal{D}^\perp)$. Using (4.3) and the orthogonality of vector fields, we get either $\cos\theta = 0$ or $g([X,Y],Z) = 0$. Since M is a proper pseudo-slant submanifold, we get that $\theta \neq \frac{\pi}{2}$, thus we conclude that $\mathcal{D}^\theta \oplus \langle\xi\rangle$ is integrable. Also, from Lemma 3.1, we have

$$\cos^2\theta\, g(\nabla_X Y, Z) = g(A_{\varphi Z}TY - A_{FTY}Z, X).$$

Thus, using the given conditions of (4.3), we get either $\cos^2\theta = 0$ or $g(\nabla_X Y, Z) = 0$, but, since M is a proper pseudo-slant submanifold, we get that $\theta \neq \frac{\pi}{2}$, which means that the leaves of the distribution $\mathcal{D}^\theta \oplus \langle\xi\rangle$ are totally geodesic in M. Also, we know that $\mathcal{D}^\perp$ is integrable (Proposition 3.1), and if we consider $\sigma^\perp$ be the second fundamental form of a leaf $M_\perp$ of $\mathcal{D}^\perp$ in M, then, for any $X \in \Gamma(\mathcal{D}^\theta \oplus \langle\xi\rangle)$ and $Z, W \in \Gamma(\mathcal{D}^\perp)$, we have

$$g(\sigma^\perp(Z,W), X) = g(\nabla_Z W, X) = g(\widetilde{\nabla}_Z W, X).$$

Then, from (2.2), we get

$$g(\sigma^\perp(Z,W), X) = g(\varphi\widetilde{\nabla}_Z W, \varphi X) - \eta(X) g(\widetilde{\nabla}_Z \xi, W).$$

Using (2.3) and (2.5) (a), we derive

$$g(\sigma^\perp(Z,W), X) = g(\widetilde{\nabla}_Z \varphi W, TX) + g(\widetilde{\nabla}_Z \varphi W, FX) - g((\widetilde{\nabla}_Z \varphi)W, FX).$$

Again by using (2.3), (2.4) and the fact that ξ is tangent to M_θ, we obtain

$$\begin{aligned} g(\sigma^\perp(Z,W), X) &= -g(\varphi W, \widetilde{\nabla}_Z TX) - g(\varphi W, \widetilde{\nabla}_Z FX) \\ &= -g(\varphi W, \sigma(Z, TX)) + g(W, \widetilde{\nabla}_Z \varphi FX) - g(W, (\widetilde{\nabla}_Z \varphi)FX). \end{aligned}$$

Then, from (2.5) (b) and (2.3), we arrive at

$$g(\sigma^\perp(Z,W), X) = -g(A_{\varphi W}TX, Z) + g(W, \widetilde{\nabla}_Z tFX) + g(W, \widetilde{\nabla}_Z fFX).$$

Thus, by Theorem 3.2, we get

$$\begin{aligned} g(\sigma^\perp(Z,W), X) = &- g(A_{\varphi W}TX, Z) + \sin^2\theta g(\widetilde{\nabla}_Z W, X) \\ &- \eta(X)\sin^2\theta g(W, \widetilde{\nabla}_Z \xi) - g(W, \widetilde{\nabla}_Z FTX). \end{aligned}$$

Using (2.3), (2.4) and the symmetry of the shape operator, we obtain

$$\cos^2\theta\, g(\sigma^{\perp}(Z,W),X) = -g(A_{\varphi W}TX - A_{FTX}W, Z).$$

Then, from (4.3), we derive

$$g(\sigma^{\perp}(Z,W),X) = -(X\mu)g(Z,W),$$

which means that $\sigma^{\perp}(Z,W) = -g(Z,W)\vec{\nabla}\mu$, where $\vec{\nabla}\mu$ is the gradient of the function μ. Thus $M_{\perp}$ is a totally umbilical submanifold of M with mean curvature vector $H^{\perp} = -\vec{\nabla}\mu$. We can show that $H^{\perp}$ is parallel with the normal connection D^N of $M_{\perp}$ in M (see [26]). Thus the leaves of $\mathcal{D}^{\perp}$ are totally umbilical with parallel mean curvature $H^{\perp}$ in M, that is, $M_{\perp}$ is an extrinsic sphere in M. Hence, by a result of Hiepko [18], M is a locally warped product manifold of the form $M_{\theta}\times_{\mu} M_{\perp}$. This completes the proof of the theorem. □

Now, we construct the following frame fields on warped product pseudo-slant submanifolds. Let $M = M_{\theta}\times_f M_{\perp}$ be an m-dimensional warped product pseudo-slant submanifold of a $(2n+1)$-dimensional Sasakian manifold $\widetilde{M}$ such that $M_{\perp}$ is an n_1-dimensional anti-invariant submanifold of $\widetilde{M}$, and M_{θ} is a proper slant submanifold of $\widetilde{M}$ with the dimension $n_2 = 2p+1$ such that ξ is tangent to M_{θ}. Let us consider the tangent spaces of $M_{\perp}$ and M_{θ} by $\mathcal{D}^{\perp}$ and $\mathcal{D}^{\theta}\oplus\langle\xi\rangle = \Xi$ instead of $TM_{\perp}$ and TM_{θ}, respectively. We set the orthonormal frame fields of $\mathcal{D}^{\perp}$ and $\mathcal{D}^{\theta}\oplus\langle\xi\rangle = \Xi$, respectively, as $\{e_1, e_2, \dots, e_{n_1}\}$ and $\{e_{n_1+1} = e_1^*, \dots, e_{n_1+p} = e_p^*, e_{n_1+p+1} = e_{p+1}^* = \sec\theta T e_1^*, \dots, e_{n_1+2p} = e_{2p}^* = \sec\theta T e_p^*, e_m = e_{2p+1}^* = \xi\}$, where θ is the slant angle of the immersion. Then the orthonormal frame fields of the normal subbundles of $\varphi\mathcal{D}^{\perp}$, $F\mathcal{D}^{\theta}$ and ν, respectively, are $\{e_{m+1} = \tilde{e}_1 = \varphi e_1, e_{m+2} = \tilde{e}_2 = \varphi e_2, \dots, e_{m+n_1} = \tilde{e}_{n_1} = \varphi e_{n_1}\}$, $\{e_{m+n_1+1} = \tilde{e}_{n_1+1} = \csc\theta F e_1^*, e_{m+n_1+2} = \tilde{e}_{n_1+2} = \csc\theta F e_2^*, \dots, e_{m+n_1+p} = \tilde{e}_{n_1+p} = \csc\theta F e_p^*, e_{m+n_1+p+1} = \tilde{e}_{n_1+p+1} = \csc\theta\sec\theta FT e_1^*, \dots, e_{2m-1} = \tilde{e}_{m-1} = \csc\theta\sec\theta FT e_p^*\}$ and $\{e_{2m} = \tilde{e}_m, \dots, e_{2n+1} = \tilde{e}_{2(n-m+1)}\}$.

Theorem 4.2. *Let $M = M_{\theta}\times_f M_{\perp}$ be a mixed totally geodesic warped product submanifold of a Sasakian manifold $\widetilde{M}$ such that $\xi\in\Gamma(TM_{\theta})$, where M_{θ} is a proper slant submanifold, and $M_{\perp}$ is an n_1-dimensional anti-invariant submanifold of $\widetilde{M}$. Then we have the following:*

(i) *The squared norm of the second fundamental form of M satisfies*

$$\|\sigma\|^2 \geq n_1\cot^2\theta\,\|\vec{\nabla}\ln f\|^2, \tag{4.4}$$

where $\vec{\nabla}\ln f$ is the gradient of $\ln f$ along M_{θ}.

(ii) *If the equality sign in (4.5) holds identically, then M_θ is totally geodesic in $\widetilde{M}$, and $M_\perp$ is a totally umbilical submanifold of $\widetilde{M}$.*

Proof. From the definition of σ, we have

$$\|\sigma\|^2=\sum_{i,j=1}^{m}g(\sigma(e_i,e_j),\sigma(e_i,e_j))=\sum_{r=m+1}^{2n+1}\sum_{i,j=1}^{m}g(\sigma(e_i,e_j),e_r).$$

Using the frame fields of $\mathcal{D}^\perp$ and $\mathcal{D}^\theta\oplus\langle\xi\rangle$, we find

$$\|\sigma\|^2=\sum_{r=m+1}^{2n+1}\sum_{l,k=1}^{n_1}g(\sigma(e_l,e_k),e_r)^2+\sum_{r=m+1}^{2n+1}\sum_{i,j=1}^{2p+1}g(\sigma(e_i^*,e_j^*),e_r)^2.$$

The above relation can be separated for the $\phi\mathcal{D}^\perp$, $F\mathcal{D}^\theta$ and μ components as follows:

$$\begin{aligned}\|\sigma\|^2=&\sum_{r=1}^{n_1}\sum_{l,k=1}^{n_1}g(\sigma(e_l,e_k),\tilde{e}_r)^2+\sum_{r=n_1+1}^{2p+n_1}\sum_{l,k=1}^{n_1}g(\sigma(e_l,e_k),\tilde{e}_r)^2\\&+\sum_{r=m}^{2(n-m+1)}\sum_{l,k=1}^{n_1}g(\sigma(e_l,e_k),\tilde{e}_r)^2+\sum_{r=1}^{n_1}\sum_{i,j=1}^{2p+1}g(\sigma(e_i^*,e_j^*),\tilde{e}_r)^2\\&+\sum_{r=n_1+1}^{2p+n_1}\sum_{i,j=1}^{2p+1}g(\sigma(e_i^*,e_j^*),\tilde{e}_r)^2+\sum_{r=m}^{2(n-m+1)}\sum_{i,j=1}^{2p+1}g(\sigma(e_i^*,e_j^*),\tilde{e}_r)^2.\end{aligned}\tag{4.5}$$

We shall leave all other positive terms except the second term to be evaluated, then we obtain

$$\begin{aligned}\|\sigma\|^2\geq&\sum_{r=n_1+1}^{2p+n_1}\sum_{l,k=1}^{n_1}g(\sigma(e_l,e_k),\tilde{e}_r)^2\\=&\sum_{r=n_1+1}^{p+n_1}\sum_{l,k=1}^{n_1}g(\sigma(e_l,e_k),\tilde{e}_r)^2+\sum_{r=n_1+p+1}^{2p+n_1}\sum_{l,k=1}^{n_1}g(\sigma(e_l,e_k),\tilde{e}_r)^2.\end{aligned}$$

Then, from the adopted frame of $F\mathcal{D}^\theta$, we obtain

$$\|\sigma\|^2\geq\sum_{i=1}^{p}\sum_{l,k=1}^{n_1}g(\sigma(e_l,e_k),\csc\theta Fe_i^*)^2+\sum_{i=1}^{p}\sum_{l,k=1}^{n_1}g(\sigma(e_l,e_k),\csc\theta\sec\theta FTe_i^*)^2.$$

Hence, by Lemma 4.1 (ii)–(iii), we derive

$$\begin{aligned}\|\sigma\|^2 \geq \csc^2\theta \sum_{i=1}^{p}\sum_{l,k=1}^{n_1}(Te_i^*\ln f + \eta(e_i^*))^2 g(e_l,e_k)^2 \\ + \cot^2\theta \sum_{i=1}^{p}\sum_{l,k=1}^{n_1}(e_i^*\ln f)^2 g(e_l,e_k)^2 \\ = \csc^2\theta \sum_{i=1}^{2p+1}\sum_{l,k=1}^{n_1}(Te_i^*\ln f)^2 g(e_l,e_k)^2 + n_1\csc^2\theta \\ + \cot^2\theta \sum_{i=1}^{p}\sum_{l,k=1}^{n_1}(e_i^*\ln f)^2 g(e_l,e_k)^2 \\ - \csc^2\theta \sum_{i=p+1}^{2p+1}\sum_{l,k=1}^{n_1}(Te_i^*\ln f + \eta(e_i^*))^2 g(e_l,e_k)^2.\end{aligned}$$

Using the considered frame fields, the above expression can be written as

$$\begin{aligned}\|\sigma\|^2 \geq n_1\csc^2\theta \sum_{i=1}^{2p+1} g(e_i^*, T\vec{\nabla}\ln f)^2 + n_1\csc^2\theta + n_1\cot^2\theta \sum_{i=1}^{p}(e_i^*\ln f)^2 \\ - n_1\csc^2\theta \sum_{i=1}^{p} g(e_{p+i}^*, T\vec{\nabla}\ln f)^2 - n_1\csc^2\theta.\end{aligned}$$

To satisfy (2.8), the above inequality can be simplified as

$$\begin{aligned}\|\sigma\|^2 \geq n_1\csc^2\theta\|T\vec{\nabla}\ln f\|^2 + n_1\cot^2\theta \sum_{i=1}^{p}(e_i^*\ln f)^2 \\ - n_1\csc^2\theta\sec^2\theta \sum_{i=1}^{p} g(Te_i^*, T\vec{\nabla}\ln f)^2.\end{aligned}$$

Using (3.2) and the fact that for a warped product submanifold of a Sasakian manifold, $\xi\ln f = 0$, we arrive at

$$\|\sigma\|^2 \geq n_1\cot^2\theta\|\vec{\nabla}\ln f\|^2 + n_1\cot^2\theta \sum_{i=1}^{p}(e_i^*\ln f)^2 - n_1\cot^2\theta \sum_{i=1}^{p}(e_i^*\ln f)^2,$$

which is inequality (i). If the equality holds in (i), then, from the leaving terms of (4.5), we conclude that

$$\sigma(\Xi,\Xi) \perp F\mathcal{D}^\theta, \quad \sigma(\Xi,\Xi) \perp \nu \;\Rightarrow\; \sigma(\Xi,\Xi) \in \varphi\mathcal{D}^\perp, \tag{4.6}$$

and

$$\sigma(\mathcal{D}^\perp,\mathcal{D}^\perp) \perp \varphi\mathcal{D}^\perp, \quad \sigma(\mathcal{D}^\perp,\mathcal{D}^\perp) \perp \nu \;\Rightarrow\; \sigma(\mathcal{D}^\perp,\mathcal{D}^\perp) \in F\mathcal{D}^\theta. \tag{4.7}$$

Also, from the fourth term of (4.6) and Lemma 4.1 (i), we get $\sigma(\Xi,\Xi)\perp\varphi\mathcal{D}^{\perp}$. Thus, by using (4.7) and the fact that $\sigma(\Xi,\Xi)\perp\varphi\mathcal{D}^{\perp}$, we get $\sigma(\Xi,\Xi)=0$, which implies that M_θ is totally geodesic in $\widetilde{M}$ due to M_θ being totally geodesic in M [2], [9]. Furthermore, since M is mixed geodesic, from Lemma 4.1 (ii) and (4.7), we have

$$g(\sigma(Z,W),FTX)=-\cos^2\theta(X\ln f)g(Z,W) \tag{4.8}$$

for any $Z,W\in\Gamma(\mathcal{D}^{\perp})$ and $X\in\Gamma(\Xi)$. Hence, since $M_\perp$ is totally umbilical in M [2], [9], it follows that $M_\perp$ is totally umbilical in $\widetilde{M}$. Thus, the proof is complete. □

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