*Publ. Math. Debrecen*
**46 / 1-2** (1995), 137–147

# Cocycles on cancellative semigroups

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**Abstract.** The main result proved is that every symmetric cocycle on a cancellative abelian semigroup into a divisible abelian group must be a coboundary (i.e., a Cauchy difference). This result (with the additional hypothesis that the range is uniquely divisible) was reported by M. Hosszú in 1971, but there is a gap in his proof.

## 1. Introduction

Let $M$ denote an abelian monoid, with 0 as identity element, and let $G$ be an abelian group. A function $F : M^2 \to G$ is called a *cocycle* (on $M$ to $G$) if for all $x, y, z \in M$

$$F(x,y) + F(x+y,z) = F(x,y+z) + F(y,z). \tag{1}$$

A cocycle is *symmetric* if, in addition,

$$F(x,y) = F(y,x) \tag{2}$$

for all $x, y \in M$.

For each function $f$ from $M$ to $G$ we define $\hat{f} : M^2 \to G$ by

$$\hat{f}(x,y) := f(x) + f(y) - f(x+y). \tag{3}$$

Then it is easy to verify that $\hat{f}$ is a symmetric cocycle on $M$ to $G$. We say a symmetric cocycle $F$ on $M$ to $G$ is a *coboundary* if there is a function $f$ from $M$ to $G$ such that $F = \hat{f}$ on $M^2$. It is clear that $\hat{f} = \hat{g}$ if and only if there is a homomorphism (i.e., an additive function) $A : M \to G$ such that $f = g + A$ on $M$.

We recall that an abelian group $G$ is *divisible* if, given $n \in N$ ($= \{1, 2, 3, \dots\}$) and $\gamma \in G$ there is an $\alpha \in G$ such that $n\alpha = \gamma$. We note

*Mathematics Subject Classification*: primary 39B52; secondary 39B62, 20M14.

that $\alpha$ need not be unique; for example $\mathbf{Q}/\mathbf{Z}$ is divisible. (Here $\mathbf{Q}$ and $\mathbf{Z}$ are the additive groups of rationals and integers, respectively.)

The cocycle equation (1) has a long and rich history, with connections to factor systems and group extensions (see e.g., Baer [2]), cohomology theory (see e.g., Eilenberg and MacLane [6] and MacLane [13]), information theory (see e.g., Ng [14] and Ebanks, Kannappan and Ng [5]), formal groups (see Fröhlich [8]), the algebra of polyhedra (see Jessen [11]), and Jordan derivations (see Davison [3]), among others. Among the many results about solutions of the cocycle equation, we mention here only a few. J. Erdös [7] proved that every symmetric cocycle on an abelian group into a divisible abelian group is a coboundary. This result was proved in a different way (along with other related results) by Jessen, Karpf and Thorup [12]. In Ebanks [4] it was shown that the same is true of symmetric cocycles on certain classes of abelian monoids.

The focus of our attention in the present paper is the following claim of M. Hosszú [10]. Let $M$ be a cancellative abelian monoid and $G$ a *uniquely* divisible abelian group; that is $n\alpha = \gamma$ has unique solution $\alpha$, given $\gamma \in G$ and $n \in N$. (So $G$ is in fact a vector space over the rationals.) Then every symmetric cocycle on $M$ into $G$ is a coboundary. Unfortunately, the proof given in [10] contains a gap (on page 214) where it is claimed that "by the cancellation law $f(x+y)$ may be defined uniquely by $(7')\ldots$" In fact, if $x+y = x'+y'$ with neither $x$ nor $y$ equal to either $x'$ or $y'$, then $(7')$ does not show that $f(x+y)$ is well-defined.

Our purpose in this paper is to give a correct proof of (a stronger version of) Hosszú's statement. We strengthen it by dropping the hypothesis of uniqueness of divisibility in $G$.

The main result proved here is the following.

**Theorem.** *If $M$ is a cancellative abelian monoid and $G$ is a divisible abelian group, then every symmetric cocycle on $M$ to $G$ is a coboundary.*

Note that this theorem extends trivially to the case $M$ is a cancellative abelian semigroup. In case $M$ comes without an identity, we adjoin an identity 0 to $M$ by defining $0+x = x+0 = x$ for all $x \in M$, and $0+0 = 0$. Then any symmetric cocycle $F$ on $M$ to $G$ extends to a symmetric cocycle $\bar{F}$ on $M \cup \{0\}$ to $G$ by defining $\bar{F}(x,0) = \bar{F}(0,x) = \bar{F}(0,0) =$ an arbitrary element of $G$ for all $x \in M$. By the theorem, $\bar{F}$ is a coboundary, hence so is $F$.

In Section 2 we show how, if a certain extension property holds, Zorn's Lemma (transfinite induction) may be used to carry out the proof. In Section 3 we prove the result when $M$ is cyclic, and in Section 4 we show that the extension property does indeed hold true. Finally, in Section 5, we discuss the situation for (1) without (2).

NOTATION. For $x \in M$, $\langle x\rangle = N_0 x$ is the submonoid generated by $x$. Of course $N_0 = \{0, 1, 2, 3, \dots\}$ is the basic cyclic monoid of the natural numbers under addition.

## 2. Generalities and Zorn

Most of the results proved here use only the fact $F$ is a symmetric cocycle; the structures of the domain and codomain are irrelevant as long as they are abelian.

**Lemma 1.** *Suppose $F$ is a symmetric cocycle on $M$ to $G$. Then*

$$F(x, 0) = F(0, z) = F(0, 0),$$

*for all $x, z \in M$. Moreover, if $F$ is a coboundary, say $F = \hat{f}$, then*

$$F(x, 0) = F(0, y) = f(0)$$

*for all $x, y \in M$.*

PROOF. The first statement follows from (1) immediately by putting $y = 0$. The second part follows from (3) by putting $y = 0$, respectively $x = 0$.

**Lemma 2.** *Suppose $F$ is a symmetric cocycle on $M$ to $G$. Then for all $x, y, u, v$ in $M$*

$$F(x+y, u+v) = F(x+u, y+v) + F(x, u) + F(y, v) - F(x, y) - F(u, v).$$

PROOF. Using (1) we have

$$F(u, v)+F(x, y)+F(x+y, u+v) = F(u, v)+F(x, y+u+v)+F(y, u+v).$$

Then using (2), and (1) again, we have

$$\begin{aligned} F(u, v)+F(x, y)+F(x+y, u+v) &= F(u, v)+F(u+v, y)+F(x, y+u+v) \\ &= F(u, y+v)+F(v, y)+F(x, y+u+v) \\ &= F(y, v)+F(x, u+y+v)+F(u, y+v) \\ &= F(y, v)+F(x, u)+F(x+u, y+v). \end{aligned}$$

**Lemma 3.** *Let $F$ be a symmetric cocycle on $M$ to $G$. Suppose $X, Y$ are submonoids of $M$ and $h : X + Y \to G$ is a function such that $F$ and $\hat{h}$ agree when restricted to $(X \cup Y)^2$. Then $F$ and $\hat{h}$ agree on $(X + Y)^2$.*

Proof. Let $x, x' \in X$ and $y, y' \in Y$. Then $(x+y, x'+y') \in (X+Y)^2$. By Lemma 2,

$$F(x+y, x'+y') = F(x+x', y+y') + F(x, x') + F(y, y') - F(x, y) - F(x', y').$$

Now each of $x + x', y + y', x, x', y, y'$ belongs to $X \cup Y$ so using the fact that $F = \hat{h}$ on $(X \cup Y)^2$ we deduce that

$$\begin{aligned} F(x + y, x' + y') = &[h(x + x') + h(y + y') - h(x + x' + y + y')] \\ &+ [h(x) + h(x') - h(x + x')] + [h(y) + h(y') - h(y + y')] \\ &- [h(x) + h(y) - h(x + y)] - [h(x') + h(y') - h(x' + y')] \\ = &\, h(x + y) + h(x' + y') - h(x + x' + y + y') = \hat{h}(x+y, x'+y'). \end{aligned}$$

So $F = \hat{h}$ on $(X + Y)^2$ as claimed.

We now introduce the extension property. Suppose $F$ is a symmetric cocycle on $M$ to $G$. We say that the pair $(S, f)$ is a *coboundary pair* for $F$ if $S$ is a submonoid of $M$, $f$ is a function from $S$ to $G$ and $F = \hat{f}$ on $S^2$. If $(S, f)$ and $(T, g)$ are coboundary pairs for $F$, then we define $\leq$ as follows: $(S, f) \leq (T, g)$ if $S \subset T$ and $g$ agrees with $f$ on $S$. Finally, we say $F$ has the *extension property* if whenever $(S, f)$ is a coboundary pair for $F$ and $x \in M \backslash S$ there is a function $h : \langle x \rangle + S \to G$ such that $(S, f) \leq (\langle x \rangle + S, h)$.

**Lemma 4.** *Suppose $F$ is a symmetric cocycle on $M$ to $G$. Suppose $\Lambda$ is a linearly ordered set and that $\{(S_\lambda, f_\lambda) : \lambda \in \Lambda\}$ is a chain of coboundary pairs for $F$. Put $S = \cup S_\lambda$ and $f = \cup f_\lambda$, then $(S, f)$ is a coboundary pair for $F$, and $(S_\lambda, f_\lambda) \leq (S, f)$ $\forall \lambda \in \Lambda$.*

Proof. Let $x, y \in S$. Then there is a $\lambda \in \Lambda$ such that $x, y \in S_\lambda$. Thus

$$\begin{aligned} F(x, y) = \hat{f}_\lambda(x, y) &= f_\lambda(x) + f_\lambda(y) - f_\lambda(x + y) \\ &= f(x) + f(y) - f(x + y) = \hat{f}(x, y), \end{aligned}$$

and so $(S, f)$ is a coboundary pair.

**Proposition 1.** *Suppose the symmetric cocycle $F$ has the extension property. Then $F$ is a coboundary.*

Proof. Lemma 4 shows that each chain of coboundary pairs for $F$ has a least upper bound, so by Zorn's Lemma the collection of all coboundary pairs has a maximal element, say $(S, f)$. If $S = M$ we are done, for then $F = \hat{f}$ on $M^2$ so $F$ is a coboundary. If $S \neq M$, let $x \in M \setminus S$. Then by the extension property $(S, f) \leq (\langle x \rangle + S, h)$. But this contradicts the maximality of $(S, f)$, so in fact $M \setminus S = \emptyset$.

The final result of this section does not mention cocycles but plays an important role.

**Lemma 5.** *Suppose $S$ is a nonempty submonoid of the abelian monoid $M$. Let $x \in M$ and set $J_x = \{n \in N_0 : (nx + S) \cap S \neq \emptyset\}$. Then there is a $p \in N_0$ such that $J_x = \langle p \rangle$.*

Proof. First we show that $J = J_x$ is a submonoid of $N_0$. Clearly $0 \in J$. Suppose $n, n' \in J$. Then there are $s, s' \in S$ such that $nx + s \in S$ and $n'x + s' \in S$. Hence $nx + s + n'x + s' = (n + n')x + (s + s') \in S$ and so $n + n' \in J$.

Now if $J \cap N = \emptyset$ then $J = \langle 0 \rangle$. If $J \cap N \neq \emptyset$ let $p \in J \cap N$ be the smallest element. We claim that $J = \langle p \rangle$. For suppose that $n \in J$. Then $n = qp + r$ for $q, r \in N_0$ with $0 \leq r < p$. Also there is an $s \in S$ such that $nx + s \in S$, and similarly $px + s_0 \in S$ for some $s_0 \in S$. Thus $nx + s + qs_0 \in S$. But $nx + s + qs_0 = q(px + s_0) + s + rx$, hence $r \in J$. But $r < p$, so $r = 0$ else $p$ is not least. Therefore $n = qp \in \langle p \rangle$.

## 3. The cyclic case

**Proposition 2.** *Let $M$ be a cyclic cancellative monoid and $G$ a divisible abelian group. Then every cocycle on $M$ to $G$ is a coboundary.*

Proof. Let $F$ be a cocycle on $M$ to $G$. We will define a function $f$ such that $F = \hat{f}$ (incidentally proving that $F$ must be symmetric).

Case 1. Suppose $M = \langle a \rangle$ is finite. Then $M$ is a finite cyclic group of order $m \geq 1$, say. Since $G$ is divisible we can choose $\gamma \in G$ such that

$$m\gamma = \sum_{x \in \langle a \rangle} F(a, x). \tag{4}$$

We define $f(a) := \gamma$, and for $n \in N$,

$$f(na) := nf(a) - \sum_{j=1}^{n-1} F(a, ja). \tag{5}$$

This gives a rule for $f$ but doesn't show that $f$ is a function. For this we need to prove that if $na = ka$ then $f(na) = f(ka)$. Now $na = ka$ (with

$n \neq k$) if and only if $n, k$ differ by a multiple of $m$, so it suffices to prove that for every $n$

$$f((n+m)a) = f(na).$$

Now, according to (5),

$$\begin{aligned} f((n+m)a) &= (n+m)f(a) - \sum_{j=1}^{n+m-1} F(a, ja) \\ &= nf(a) + mf(a) - \sum_{j=1}^{n-1} F(a, ja) - \sum_{j=n}^{n+m-1} F(a, ja) \\ &= f(na) + mf(a) - \sum_{x \in \langle a \rangle} F(a, x) = f(na) \end{aligned}$$

by our choice of $f(a)$. So $f$, given by (4) and (5), is a function. Observe that $f(0) = f(ma) = F(a, ma) = F(a, 0)$, hence $f(0) = F(0, 0)$ by Lemma 1.

Case 2. Suppose $M = \langle a \rangle$ is infinite. Define $f(0) := F(0, 0)$, $f(a) = 0$ and $f(na)$ by (5). Note that $na = ka$ implies $n = k$ and thus $f(na) = f(ka)$, so $f$ is a function.

We now show that in either case the function $f$ is such that $F = \hat{f}$. First, by Lemma 1 we have for all $x, z \in \langle a \rangle$

$$\begin{aligned} F(x, 0) = F(0, z) = F(0, 0) &= f(0) = f(x) + f(0) - f(x) \\ &= f(0) + f(z) - f(z) = \hat{f}(x, 0) = \hat{f}(0, z) = \hat{f}(0, 0). \end{aligned}$$

Next, for every $n \in N$

$$F(a, na) = \hat{f}(a, na)$$

follows immediately from (5).

Assume now that for some $k \in N$

$$F(ka, na) = \hat{f}(ka, na)$$

for all $n \in N_0$. Then from (1)

$$\begin{aligned} F((k+1)a, na) &= F(a + ka, na) = F(a, ka + na) + F(ka, na) - F(a, ka) \\ &= \hat{f}(a, (k+n)a) + \hat{f}(ka, na) - \hat{f}(a, ka) = \hat{f}((k+1)a, na) \end{aligned}$$

for every $n \in N_0$. Thus by induction on $k$ the result is true for all $k, n \in N_0$; that is $F = \hat{f}$.

Results very similar to Proposition 2 are known already. For example, the following shows that the cancellativity hypothesis on a cyclic $M$ can be dropped if we assume that our cocycle is symmetric.

**Proposition 2′.** *Let $M$ be a cyclic monoid and $G$ a divisible abelian group. Then every symmetric cocycle on $M$ to $G$ is a coboundary.*

Proof. Let $F : M^2 \to G$ be a symmetric cocycle. If $M$ is finite, then it contains a minimal ideal. By Theorem 1 in [4], $F$ is a coboundary.

If $M$ is infinite, then it is isomorphic to $N_0$. By Theorem 3 in [12], $F$ is a coboundary.

For other related remarks, see Section 5.

## 4. The general case

In view of Proposition 1 it suffices to prove the following result.

**Proposition 3.** *Suppose $F$ is a symmetric cocycle on the cancellative abelian monoid $M$. Then $F$ has the extension property.*

Proof. Let $(S, f)$ be a coboundary pair for $F$ and suppose $x \in M \setminus S$. By Proposition 2 there is a function $g : \langle x\rangle \to G$ such that $F = \hat{g}$ on $\langle x\rangle^2$. We use $f$ and $g$ to define a rule $h : \langle x\rangle + S \to G$ by

$$h(mx + s) := m\alpha + g(mx) + f(s) - F(mx, s). \tag{6}$$

What we have to do first is show there is an $\alpha \in G$ such that $h$ is a function. Then we show that $(\langle x\rangle + S, h)$ is a coboundary pair for $F$ such that $(S, f) \leq (\langle x\rangle + S, h)$; that is, $(\langle x\rangle + S, h)$ is an extension of $(S, f)$.

Part I. We show that $h$ is a function, for appropriate choice of $\alpha \in G$. Suppose that $mx + s = m'x + s'$ for $m \geq m' \in N_0$ and $s, s' \in S$. We need to show that $h(mx + s) = h(m'x + s')$. By (6), that means we have to prove

$$m\alpha + g(mx) + f(s) - F(mx, s) = m'\alpha + g(m'x) + f(s') - F(m'x, s'). \tag{7}$$

Now $m = m' + n$ for some $n \in N_0$, so by cancellation $nx + s = s'$. By Lemma 5, $n \in J_x = \langle p\rangle$ for some $p \in N_0$. Thus $n = qp$ for some $q \in N_0$, and $px + s_0 = t_0$ for some $s_0, t_0 \in S$.

If $p = 0$, then $n = 0$, $m = m'$, $s = s'$, and (7) is satisfied for arbitrary $\alpha \in G$.

Now suppose $p > 0$. Since $G$ is divisible, we can choose $\alpha \in G$ such that

$$p\alpha = F(px, s_0) - g(px) - f(s_0) + f(t_0). \tag{8}$$

With this $\alpha$, we show that (7) holds. Using $m = m' + n$ and $s' = nx + s$, we can rewrite (7) as

$$\begin{aligned} &n\alpha + g((m'+n)x) + f(s) - F(m'x + nx, s) \\ &= g(m'x) + f(nx+s) - F(m'x, nx+s). \end{aligned}$$

Since $F$ is a cocycle, this is equivalent to

$$\begin{aligned} &n\alpha + g((m'+n)x) + f(s) - F(nx, s) \\ &= g(m'x) + f(nx+s) - F(m'x, nx). \end{aligned}$$

And since $F = \hat{g}$ on $\langle x\rangle^2$, this reduces further to

$$n\alpha + f(s) - F(nx, s) = f(nx+s) - g(nx), \tag{9}$$

for all $n \in J_x = \langle p\rangle$.

Next, recalling that $n = qp$, that $px + s_0 = t_0$, that $nx + s \in S$, that $F$ is a cocycle, and that $F = \hat{f}$ on $S^2$, we calculate that

$$\begin{aligned} F(nx, s) &= F(qs_0, nx) + F(qs_0 + nx, s) - F(qs_0, nx+s) \\ &= F(qs_0, qpx) + F(qt_0, s) - F(qs_0, nx+s) \\ &= F(qs_0, qpx) + f(qt_0) + f(s) - f(qt_0 + s) \\ &\quad - f(qs_0) - f(nx+s) + f(qt_0+s) \\ &= F(qs_0, qpx) + f(qt_0) - f(qs_0) + f(s) - f(nx+s). \end{aligned}$$

Substituting this into (9) and using the symmetry of $F$, we find that it suffices to prove

$$F(qpx, qs_0) = qp\alpha + g(qpx) + f(qs_0) - f(qt_0) \tag{10}$$

for all $q \in N_0$.

We complete Part I by establishing (10) by induction. For $q = 0$, it is obviously true (cf. Lemma 1). For $q = 1$, it is true by (8), our choice of $\alpha$. Now assume (10) is valid for $q = k \in N$. By Lemma 2, we find that

$$\begin{aligned} &F((k+1)px, (k+1)s_0) = F(kpx + px, ks_0 + s_0) \\ &= F(kpx + ks_0, px + s_0) + F((kpx, ks_0) + F(px, s_0) - F(kpx, px) \\ &-F(ks_0, s_0) = [f(kt_0) + f(t_0) - f(kt_0 + t_0)] + [kp\alpha + g(kpx) \\ &+f(ks_0) - f(kt_0)] + [p\alpha + g(px) + f(s_0) - f(t_0)] \\ &-[g(kpx) + g(px) - g(kpx + px)] - [f(ks_0) + f(s_0) - f(ks_0 + s_0)] \\ &= (k+1)p\alpha + g((k+1)px) + f((k+1)s_0) - f(k+1)t_0), \end{aligned}$$

which is (10) for $q = k+1$. Therefore (10) is proved, and this finishes Part I.

Part II. We show that $(\langle x\rangle + S, h)$ is an extension of $(S, f)$. First, put $m = 0$ in (6) to get

$$h(s) = g(0) + f(s) - F(0, s)$$

for all $s \in S$. In view of Lemma 1, this means $h$ agrees with $f$ on $S$. Also, $s = 0$ in (6) yields

$$h(mx) = m\alpha + g(mx) + f(0) - F(mx, 0) = m\alpha + g(mx),$$

so $\hat{h} = \hat{g} = F$ on $\langle x\rangle^2$. Moreover, (6) can be written now as

$$F(mx, s) = h(mx) + h(s) - h(mx + s),$$

showing (since $F$ is symmetric) that $F$ agrees with $\hat{h}$ on $(\langle x\rangle \cup S)^2$.

Finally, by Lemma 3, we see that $F$ and $\hat{h}$ agree on $(\langle x\rangle + S)^2$. In other words, $(\langle x\rangle + S, h)$ is a coboundary pair for $F$, and $(S, f) \leq (\langle x\rangle + S, h)$. This completes the proof of Proposition 3.

Combining Propositions 1 and 3, we have proved our Theorem.

## 5. Cocycles without symmetry

From our result about symmetric cocycles, we can derive a result about arbitrary cocycles if we place an additional hypothesis on the codomain. This observation was first made by Hosszú [9] and Aczél [1]. Recall that an abelian group $G$ is *uniquely 2-divisible* if, given $\gamma \in G$, there is a unique $\alpha \in G$ such that $2\alpha = \gamma$. This $\alpha$ can be denoted $\frac{1}{2}\gamma$.

**Corollary.** *Let $M$ be a cancellative abelian monoid, and let $G$ be a divisible abelian group which is uniquely 2-divisible. Then every cocycle $F$ on $M$ to $G$ is of the form*

$$F = \hat{f} + A \tag{11}$$

*for some $f : M \to G$ and skew-symmetric biadditive $A : M^2 \to G$.*

Proof. We use the unique 2-divisibility of $G$ to split $F$ into symmetric and skew-symmetric parts. Define $H, A : M^2 \to G$ by

$$H(x, y) := \frac{1}{2}[F(x, y) + F(y, x)], \quad A(x, y) := \frac{1}{2}[F(x, y) - F(y, x)].$$

Clearly, $H$ is symmetric, $A$ is skew-symmetric, and $F = H + A$. Furthermore, since $M$ is abelian, it is easy to check that both $H$ and $A$ are cocycles. Hence $H$ is a coboundary.

It only remains to be shown that $A$ is biadditive. Since $A$ is skew-symmetric, it suffices to prove that $A$ is additive in one variable. This is achieved by the following calculation, which uses (1) three times.

$$\begin{gathered}2A(x+y,z) = F(x+y,z) - F(z,x+y)\\ = [F(x,y+z)+F(y,z)-F(x,y)] - [F(z+x,y)+F(z,x)-F(x,y)]\\ = F(x,z+y) - F(x+z,y) + F(y,z) - F(z,x)\\ = [F(x,z)-F(z,y)] + F(y,z) - F(z,x) = 2A(x,z) + 2A(y,z).\end{gathered}$$

Conversely, any function $F$ given by (11), with $A$ biadditive, is a cocycle. This concludes the proof of the corollary.

*Remark.* By the splitting used in the proof above, we can also prove the following consequence of Proposition $2'$: Every cocycle on a cyclic monoid into a divisible abelian group uniquely divisible by 2, is a coboundary.

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*(Received March 9, 1994)*