Publ. Math. Debrecen
91/3-4 (2017), 349–367
DOI: 10.5486/PMD.2017.7643

On the Gauss map of minimal Lorentzian surfaces in 4-dimensional semi-Riemannian space forms with index 2

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Abstract. In this paper, we study minimal Lorentzian surfaces with finite type Gauss map in 4-dimensional semi-Riemannian space forms with index of 2. First, we give the complete classification of Lorentzian surfaces in the semi-Euclidean space $\mathbb{E}^4_2$ with pointwise 1-type Gauss map. Then, we study all Lorentzian minimal surfaces in $\mathbb{S}^4_2(1)$ regarding their Gauss map. In particular, we proved that a Lorentzian minimal surface in $\mathbb{S}^4_2(1)$ has 2-type Gauss map if and only if it has constant Gaussian curvature and non-zero constant normal curvature.

1. Introduction

The notion of finite type maps has been studied by many geometers since it was first introduced by B.-Y. Chen in the middle of the 1980's during his program of understanding the finite type submanifolds in semi-Euclidean spaces, [9], [11], [12]. In particular, after the problem "*To what extent does the type of the Gauss map of a submanifold of $\mathbb{E}^m_r$ determine the submanifold?*" was presented by B.-Y. Chen and P. Piccinni in [12], submanifolds with finite type Gauss map have been worked in many articles, see [5], [6], [10], [23], [26].

Let $\mathbb{E}^m_s$ denote the semi-Euclidean space with the canonical semi-Euclidean metric tensor of index r given by

$$g = -\sum_{i=1}^{s} dx_i^2 + \sum_{j=s+1}^{m} dx_j^2,$$

Mathematics Subject Classification: Primary: 53B25; Secondary: 53C50.
Key words and phrases: Minkowski space-time, semi-Riemannian space forms, finite type map, pointwise 1-type Gauss map, Lorentzian surface, finite type submanifolds, minimal surfaces.

where $(x_1, x_2, \dots, x_n)$ is a rectangular coordinate system in E_s^m, and let M be a (semi-)Riemannian submanifold of a semi-Euclidean space $\mathbb{E}_r^m$. A map ϕ defined on M into another semi-Euclidean space $\mathbb{E}_S^N$ is said to be of k-type if it can be expressed as

$$\phi = \phi_0 + \phi_1 + \cdots + \phi_k,$$

for some eigenvectors $\phi_1, \phi_2, \dots, \phi_k$ corresponding from k distinct eigenvalues of Δ [11], [12], where ϕ_0 is a constant vector.

From the above definition one can see that a submanifold M has (global) 1-type Gauss map ν if and only if the equation

$$\Delta\nu = \lambda(\nu + C) \tag{1.1}$$

is satisfied for a constant vector C and $\lambda \in \mathbb{R}$. Similarly, a submanifold M is said to have pointwise 1-type Gauss map if the Laplacian of its Gauss map takes the form

$$\Delta\nu = f(\nu + C) \tag{1.2}$$

for a smooth function f and a constant vector C. More precisely, a pointwise 1-type Gauss map is called *of the first kind* if (1.2) is satisfied for $C = 0$, and *of the second kind* if $C \neq 0$. Moreover, if (1.2) is satisfied for a non-constant function f, then M is said to have *proper* pointwise 1-type Gauss map [9].

Nowadays, the study of submanifolds with pointwise 1-type Gauss map is a very active research subject (cf. [1], [2], [3], [4], [14], [17], [18], [19], [24], [25]). For example, the second-named author studied marginally trapped surfaces in the Minkowski space-time $\mathbb{E}_1^4$ in terms of type of their Gauss map in [24]. Most recently, Dursun and Bektaş have studied flat Lorentzian rotational surfaces in the Minkowski space $\mathbb{E}_1^4$ with pointwise 1-type Gauss map [19]. Further, in [2], the authors obtained some classifications of general rotational surfaces in the semi-Euclidean space $\mathbb{E}_2^4$ in terms of type of their Gauss map.

On the other hand, a submanifold M is said to have 2-type Gauss map if and only if

$$\nu = \nu_0 + \nu_1 + \nu_2, \quad \Delta\nu_i = \lambda_i\nu_i, \quad i = 1, 2 \tag{1.3}$$

is satisfied for some eigenvalues λ_1, λ_2 of Δ and a constant vector ν_0. Submanifolds with 2-type Gauss map are studied in several papers [5], [10], [11], [12]. Before we proceed, we would like to note that if M has a 2-type Gauss map, then the weakly elliptic, semi-linear, forth degree partial differential equation

$$\Delta^2\nu + \xi\Delta\nu + \eta\nu = C \tag{1.4}$$

is necessarily satisfied for a constant vector C and some constants ξ, η. However, satisfying (1.4) is not a sufficient condition for ν in order to be of 2-type (see, for example, [11], [12], [21]).

In this work, we study Lorentzian minimal surfaces in the semi-Euclidean spaces in terms of type of their Gauss map. In Section 2, after describing our notations, we give a summary of the basic facts and formulas that we will use. In Section 3, we obtain the complete classification of minimal Lorentzian surfaces with pointwise 1-type Gauss map. In particular, we proved that there are non-planar minimal surfaces in $\mathbb{E}^4_2$ proper pointwise 1-type Gauss map of the second kind. We would like to note that the non-existence of such surfaces in $\mathbb{E}^4_1$ was obtained by the second-named author in [25]. Finally, in Section 4, we obtain the complete classification of Lorentzian minimal surfaces in the pseudo-Euclidean space form $\mathbb{S}^4_2(1)$.

2. Preliminaries

2.1. Basic notations, formulas and definitions. Let $\mathbb{E}^m_s$ denote the semi-Euclidean space with the canonical semi-Euclidean metric tensor g of index s. We put

$$\mathbb{S}^{m-1}_s(r^2) = \{x \in \mathbb{E}^m_s : \langle x, x\rangle = r^{-2}\},$$
$$\mathbb{H}^{m-1}_{s-1}(-r^2) = \{x \in \mathbb{E}^m_s : \langle x, x\rangle = -r^{-2}\},$$

where $\langle\ ,\ \rangle$ is the indefinite inner product of $\mathbb{E}^m_s$. We will also use the following notation:

$$R^m_s(c) = \begin{cases} \mathbb{S}^m_s(c) & \text{if } c > 0, \\ \mathbb{E}^m_s & \text{if } c > 0, \\ \mathbb{H}^m_s(c) & \text{if } c < 0. \end{cases}$$

Let v be a non-zero vector in $\mathbb{E}^m_r$. v is called space-like, time-like or light-like if $\langle v, v\rangle$ is positive, negative or zero, respectively.

We want to state the following well-known lemmas that we will use later.

Lemma 2.1 ([22]). *Let U be a real vector space with a non-degenerated inner product $\langle\ ,\ \rangle$ with index 1. Then, two light-like vectors v_1, v_2 are linearly dependent if and only if $\langle v_1, v_2\rangle = 0$.*

Lemma 2.2 ([22]). *Let V be a subspace of a real vector space U, and $\langle\ ,\ \rangle$ a non-degenerated inner product defined in U. Then, $\langle\ ,\ \rangle|_V$ is non-degenerated if and only if $V \cap V^\perp = \{0\}$.*

Let M be an n-dimensional semi-Riemannian submanifold of the semi-Euclidean space $\mathbb{E}^m_r$. We denote the Levi–Civita connections of $\mathbb{E}^m_r$ and M by $\widetilde{\nabla}$ and ∇, respectively. The Gauss and Weingarten formulas are given, respectively, by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$

$$\widetilde{\nabla}_X \xi = -A_\xi(X) + D_X \xi, \tag{2.2}$$

for any tangent vector field X, Y and normal vector field ξ on M, where h, D and A are the second fundamental form, the normal connection and the shape operator of M, respectively. On the other hand, the shape operator A and the second fundamental form h of M are related by

$$\langle A_\xi X, Y\rangle = \langle h(X,Y), \xi\rangle. \tag{2.3}$$

The Gauss, Codazzi and Ricci equations are given, respectively, by

$$R(X,Y,Z,W) = \langle h(Y,Z), h(X,W)\rangle - \langle h(X,Z), h(Y,W)\rangle, \tag{2.4a}$$

$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z), \tag{2.4b}$$

$$\langle R^D(X,Y)\xi, \eta\rangle = \langle [A_\xi, A_\eta]X, Y\rangle, \tag{2.4c}$$

where R, R^D are the curvature tensors associated with connections ∇ and D, respectively, and

$$(\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

2.2. Lorentzian surfaces in $\mathbb{E}^4_2$. Now, we consider a Lorentzian surface M in the semi-Euclidean space $\mathbb{E}^4_2$. Let $\{f_1, f_2; f_3, f_4\}$ be a local pseudo-orthogonal frame field on M consisting of light-like vector fields such that $\langle f_1, f_2\rangle = \langle f_3, f_4\rangle = -1$.

Remark 2.3. We will say M is properly contained in the semi-Euclidean space $\mathbb{E}^4_2$ if it has no open part that lies on a non-degenerate hyperplane of $\mathbb{E}^4_2$.

The mean curvature vector H and Gaussian curvature K of M are defined by

$$H = -h(f_1, f_2), \tag{2.5a}$$

$$K = R(f_1, f_2, f_2, f_1). \tag{2.5b}$$

M is said to be minimal if $H \equiv 0$. On the other hand, for a smooth map ϕ, the Laplace operator of M is given by

$$\Delta\phi = f_1 f_2(\phi) + f_2 f_1(\phi) - (\nabla_{f_1} f_2)(\phi) - (\nabla_{f_2} f_1)(\phi). \tag{2.6}$$

The relative null space at p of M is defined as

$$\mathcal{N}_p(M) = \{X \in T_pM | h(X,Y) = 0, \quad \text{for all} \quad Y \in T_pM\}.$$

We say M has degenerated relative null bundle if $(\mathcal{N}_p(M), \langle\ ,\ \rangle)$ is a degenerated inner product space for all $p \in M$.

2.3. The Gauss map. Let M be a minimal Lorentzian surface in $\mathbb{E}^4_2$, and $\{f_1, f_2; f_3, f_4\}$ a pseudo-orthogonal frame field on M. Then, the smooth map ν defined by

$$\begin{aligned} \nu : M &\to \mathbb{H}^5_3(-1) \subset \mathbb{E}^6_4 \\ p &\mapsto \nu(p) = (f_1 \wedge f_2)(p) \end{aligned} \tag{2.7}$$

is called the (tangent) Gauss map of M. For a geometric interpretation of the Gauss map of M, see [12], [13], [19].

In the next lemma, we provide the Laplacian of the Gauss map of a minimal surface in $\mathbb{E}^4_2$ (see [13, Lemma 3.2]).

Lemma 2.4. *Let M be a Lorentzian surface in $\mathbb{E}^4_2$, and $\{f_1, f_2\}$ be a pseudo-orthogonal base field of the tangent bundle of M. Then, the Gauss map $\nu = f_1 \wedge f_2$ of M satisfies*

$$\Delta\nu = 2K\nu + 2h(f_1, f_1) \wedge h(f_2, f_2), \tag{2.8}$$

where K is the Gaussian and h is the second fundamental form of M, respectively.

3. Minimal surfaces in $\mathbb{E}^4_2$ and their Gauss maps

In this section, we consider minimal Lorentzian surfaces in $\mathbb{E}^4_2$ in terms of type of their Gauss maps.

We state the following lemma and theorem that we will use later.

Lemma 3.1 ([7]). *Let M be a Lorentzian surface in a semi Euclidean space $\mathbb{E}^m_s$. Then there exists a local coordinate system (s, t) such that the induced metric is of the form*

$$g = -m^2(ds \otimes dt + ds \otimes dt), \quad s \in I_1,\ t \in I_2, \tag{3.1}$$

for a non-vanishing function $m = m(s,t)$, where I_1 and I_2 are some open intervals. Moreover, the Levi–Civita connection of M is given by

$$\nabla_{\partial_s}\partial_s = \frac{2m_s}{m}\partial_s, \quad \nabla_{\partial_s}\partial_t = 0, \quad \nabla_{\partial_t}\partial_t = \frac{2m_t}{m}\partial_t, \tag{3.2}$$

and the Gaussian curvature of M becomes

$$K = \frac{2(mm_{st} - m_s m_t)}{m^4}. \tag{3.3}$$

Remark 3.2. If $m(s,t) = m_1(s)m_2(t)$ for some smooth non-vanishing functions m_1, m_2 or, equivalently, $K = 0$, then by a suitable change of coordinates we can assume $m(s,t) = 1$.

Theorem 3.3 ([8], [15])**.** *Every minimal Lorentzian surface in $\mathbb{E}^m_s$ is locally congruent to a translation surface defined by*

$$x(s,t) = \alpha(s) + \beta(t), \tag{3.4}$$

where $\alpha(s)$ and $\beta(t)$ are two null curves defined on open intervals I_1 and I_2, respectively, in the semi-Euclidean space $\mathbb{E}^m_s$ and satisfy $\langle \alpha', \beta' \rangle = -m^2(s,t) \neq 0$.

Remark 3.4. Let M be a minimal Lorentizan surface given by (3.4). Then, from the Codazzi equation (2.4b) one can obtain

$$D_{\partial_t} h(\partial_s, \partial_s) = 0, \tag{3.5a}$$

$$D_{\partial_s} h(\partial_t, \partial_t) = 0, \tag{3.5b}$$

where s, t are the local coordinates given in Lemma 3.1.

3.1. Pointwise 1-type Gauss map of the first kind. In this subsection, we focus on minimal Lorentzian surfaces whose Gauss map satisfying (1.2) for $C = 0$. We want to note that from Lemma 2.4, one can see that having such a Gauss map is equivalent to having flat normal bundle for a Lorentzian minimal surface in $\mathbb{E}^4_2$.

First, we obtain the following lemma.

Lemma 3.5. *Let M be a Lorentzian surface in $\mathbb{E}^m_s$. Then M has degenerated relative null bundle if and only if it is congruent to the surface given by*

$$x(s,t) = s\eta_0 + \beta(t), \quad \langle \eta_0, \beta(t) \rangle \neq 0, \tag{3.6}$$

where η_0 is a constant light-like vector and β is a null curve in $\mathbb{E}^m_s$ which contains no open part of a line.

Proof. Let $\mathcal{N}_p(M)$ be degenerated for all $p \in M$. Then, Lemma 2.2 implies that $\mathcal{N}_p(M) = \text{span}\{f_1\}$ for a light-like vector field f_1 tangent to M. Let f_2 be the light-like tangent vector field such that $\langle f_1, f_2\rangle = -1$. Then, we have $h(f_1, f_1) = h(f_1, f_2) = 0$. Thus, the Gauss equation (2.4a) implies $K = 0$. Therefore, we may assume $m(s,t) = 1$.

On the other hand, from $h(f_1, f_2) = 0$, we have that M is minimal. Thus, the position vector x of M is (3.4) for some light-like curves α, β in $\mathbb{E}^m_s$ satisfying $\langle \alpha', \beta'\rangle = -1$. Moreover, we can assume $f_1 = \partial_s$, $f_2 = \partial_t$. By combining $h(f_1, f_1) = 0$, (2.1) and (3.2), we get

$$\alpha'' = \widetilde{\nabla}_{\partial_s}\partial_s = 0,$$

i.e, α is a light-like line. Hence M is congruent to the surface given by (3.6). Moreover, we have $\langle \eta_0, \eta_0\rangle = 0$ and $\langle \eta_0, \beta'(t)\rangle \neq 0$. Note that if β has an open part $\mathcal{M}$ of a line, then we have $\mathcal{N}_p(M) = T_pM$ for all $p \in \mathcal{M}$, which is a contradiction.

The converse of the Lemma follows from a direct computation. Thus, the proof is completed. □

Next, we prove the following proposition, which is the classification of minimal surfaces whose Gauss map satisfies (1.2) for $C = 0$.

Proposition 3.6. *There exist four families of minimal Lorentzian surfaces in the semi-Euclidean space $\mathbb{E}^4_2$ with pointwise 1-type Gauss map of the first kind:*

(i) *a minimal Lorentzian surface lying in a hyperplane $\mathbb{E}^3_2$ of $\mathbb{E}^4_2$;*

(ii) *a minimal Lorentzian surface lying in a hyperplane $\mathbb{E}^3_1$ of $\mathbb{E}^4_2$;*

(iii) *a surface with degenerated relative null space given by* (3.6)*;*

(iv) *a surface lying on a degenerated hyperplane given by*

$$x(s,t) = \left(\phi_1(s) + \phi_2(t), \frac{\sqrt{2}}{2}(s+t), \frac{\sqrt{2}}{2}(s-t), \phi_1(s) + \phi_2(t)\right), \tag{3.7}$$

where $\phi_i : I_i \to \mathbb{R}$ are some smooth, non-vanishing functions, and I_i are some open intervals for $i = 1, 2$.

Conversely, every minimal Lorentzian surface with pointwise 1-type Gauss map of the first kind in the semi-Euclidean space $\mathbb{E}^4_2$ is congruent to an open portion of a surface obtained from these types of surfaces.

Proof. If M is a minimal surface lying in 3-dimensional Minkowski space $\mathbb{E}^3_1$, then [17, Lemma 3.2] implies that M has pointwise 1-type Gauss map (see also [16]). Therefore, the surfaces given in case (i) and case (ii) have pointwise

1-type Gauss map of the first kind. On the other hand, a direct calculation yields that the surfaces given in case (iii) and case (iv) of the theorem have harmonic Gauss map. Now, we want to prove the converse of this theorem.

Let M be a minimal Lorentzian surface in $\mathbb{E}^4_2$. We consider a local coordinate system (s,t) satisfying (3.1) for a non-vanishing function m and the pseudo-orthonormal base field $\{f_1, f_2\}$ of tangent bundle of M, given by $f_1 = m^{-1}\partial_s$ and $f_2 = m^{-1}\partial_t$. Since M is minimal, we have $h(f_1, f_2) = 0$. Let x be the position vector of M given by (3.4).

Now, we assume that M has pointwise 1-type Gauss map of the first kind, i.e., (1.2) is satisfied for $C = 0$. Then, (1.2) and (2.8) imply $h(f_1, f_1) \wedge h(f_2, f_2) = 0$, from which we see that $h(f_1, f_1)$ and $h(f_2, f_2)$ are linearly dependent.

If $h(f_1, f_1) = h(f_2, f_2) = 0$, then we have $h = 0$, which implies M is a Lorentzian plane. Thus, case (i) or (ii) of the theorem is satisfied. On the other hand, if $h(f_1, f_1) = 0$, $h(f_2, f_2) \neq 0$, then M has degenerated relative null bundle. Lemma 3.5 implies case (iii) of the theorem. Therefore, we assume that $h(f_1, f_1)$ and $h(f_2, f_2)$ are non-vanishing on M, and we have 3 cases subject to their causality.

Case 1. $h(f_1, f_1)$ and $h(f_2, f_2)$ are space-like. In this case, we consider the local orthogonal base field $\{e_3, e_4\}$ of the tangent bundle of M such that

$$e_3 = \frac{h(\partial_s, \partial_s)}{\langle h(\partial_s, \partial_s), h(\partial_s, \partial_s)\rangle^{1/2}}.$$

Then, Remark 3.4 implies

$$h(\partial_s, \partial_s) = A^2(s)e_3, \quad h(\partial_t, \partial_t) = \varepsilon B^2(t)e_3, \tag{3.8}$$

for some non-vanishing smooth functions A, B, where $\varepsilon = \pm 1$. Next, we define new coordinates S, T such that $S = \int\limits_{s_0}^{s} A(\xi)d\xi$ and $T = \int\limits_{t_0}^{t} B(\xi)d\xi$. Then (3.8) becomes

$$h(\partial_S, \partial_S) = \varepsilon h(\partial_T, \partial_T) = e_3, \quad \varepsilon = \pm 1. \tag{3.9}$$

Moreover, because of Remark 3.4, we have $D_{\partial_S} h(\partial_T, \partial_T) = D_{\partial_T} h(\partial_S, \partial_S) = 0$, from which and (3.9) we have $De_3 = 0$, that is, e_3 is parallel. As M has codimension of 2, e_4 is also parallel. Moreover, by combining (2.3) and (3.9), we obtain $A_4 = 0$. Thus, we have $\widetilde{\nabla} e_4 = 0$, i.e., e_4 is constant. Therefore, M lies on a hyperplane Π whose normal is e_4. Since e_4 is time-like, we have case (i) of the theorem.

Case 2. $h(f_1, f_1)$ and $h(f_2, f_2)$ are time-like. In a similar way to the previous case, we obtain case (ii) of the theorem.

Case 3. $h(f_1, f_1)$ and $h(f_2, f_2)$ are light-like. In this case, since these normal vector fields are linearly dependent, Lemma 2.1 implies $\langle h(f_1, f_1), (f_2, f_2)\rangle = 0$. Therefore, the Gauss equation (2.4a) implies $K = 0$. Because of Remark 3.2, we can assume $m(s, t) = 1$, from which and (3.2) we get $\nabla_{\partial_s}\partial_s = \nabla_{\partial_s}\partial_t = \nabla_{\partial_t}\partial_t = 0$.

Let f_3 be a normal light-like vector field given by

$$f_3 = h(\partial_s, \partial_s) = \widetilde{\nabla}_{\partial_s}\partial_s. \tag{3.10}$$

Since $h(\partial_s, \partial_s)$ and $h(\partial_t, \partial_t)$ are linearly dependent, there exists a non-vanishing function a such that $h(\partial_t, \partial_t) = af_3$. This equation, $h(\partial_s, \partial_t) = 0$ and (3.10) give $\langle h(X, Y), f_3\rangle = 0$ for all vector fields X, Y tangent to M. Thus, (2.3) implies $A_3 = 0$. On the other hand, from (3.5) we get $D_{\partial_t} f_3 = 0$ and $D_{\partial_s} f_3 = -(a_s/a)f_3$. Therefore, we have $\widetilde{\nabla}_{\partial_t} f_3 = 0$ and $\widetilde{\nabla}_{\partial_s} f_3 = -(a_s/a)f_3$. Hence, we obtain

$$f_3 = f_3(s) = b_1(s)c_0$$

for a constant light-like normal vector c_0, where $b_1 = 1/a$. Thus, (3.10) implies

$$\widetilde{\nabla}_{\partial_s}\partial_s = b_1(s)c_0. \tag{3.11}$$

In a similar way, we get

$$\widetilde{\nabla}_{\partial_t}\partial_t = b_2(t)c_0 \tag{3.12}$$

for a smooth non-vanishing function b_2.

By combining (3.4) with (3.11) and (3.12), we have

$$\alpha''(s) = b_1(s)c_0, \tag{3.13a}$$

$$\beta''(t) = b_2(t)c_0. \tag{3.13b}$$

By integrating these equations, we see that M is congruent to the surface given by

$$x(s, t) = (\phi_1(s) + \phi_2(t))\, c_0 + tc_1 + sc_2, \tag{3.14}$$

for some smooth non-constant functions ϕ_1, ϕ_2 and some constant vectors c_1, c_2. Since the induced metric of M is $g = -(ds \otimes dt + ds \otimes dt)$, we have $\langle x_s, x_s\rangle = \langle x_t, x_t\rangle = 0$ and $\langle x_s, x_t\rangle = -1$. Thus, we have c_1, c_2 are light-like vectors such that $\langle c_1, c_2\rangle = -1$ and $\langle c_0, c_1\rangle = \langle c_0, c_2\rangle = 0$. Up to the isometries of $\mathbb{E}^4_2$, we may choose $c_0 = (1, 0, 0, 1)$, $c_1 = (0, 1/\sqrt{2}, 1/\sqrt{2}, 0)$ and $c_2 = (0, 1/\sqrt{2}, -1/\sqrt{2}, 0)$. By using these equations on (3.14), we get (3.7). □

Next, we state two direct consequences of Proposition 3.6.

Corollary 3.7. *A minimal Lorentzian surface in* $\mathbb{E}^4_2$ *with proper pointwise 1-type Gauss map of the first kind lies on a hyperplane of* $\mathbb{E}^4_2$.

Theorem 3.8. *Let* M *be a minimal Lorentzian surface properly contained in the semi-Euclidean space* $\mathbb{E}^4_2$. *Then, the following statements are equivalent:*

(i) M *has pointwise 1-type Gauss map of the first kind;*

(ii) M *has harmonic Gauss map;*

(iii) M *is congruent to either the surface given by* (3.6) *or the surface given by* (3.7).

3.2. Pointwise 1-type Gauss map of the second kind. In this subsection, we obtain a family of minimal Lorentzian surfaces in $\mathbb{E}^4_2$ with pointwise 1-type Gauss map of the second kind.

First, we obtain the following characterization; then, we get the following classification theorem of minimal Lorentzian surfaces in $\mathbb{E}^4_2$ with pointwise 1-type Gauss map of the second kind.

Lemma 3.9. *Let* M *be a minimal surface in the semi-Euclidean space* $\mathbb{E}^4_2$ *with non-harmonic Gauss map, and* $\{f_1, f_2\}$ *a pseudo-orthonormal base field of the tangent bundle of* M. *Then,* M *has pointwise 1-type Gauss map of the second kind if and only if the normal vector fields* $h(f_1, f_1)$ *and* $h(f_2, f_2)$ *are light-like and linearly independent. In this case,* (1.2) *is satisfied for the smooth function* f *and the constant vector* C *given by*

$$f = 4K, \tag{3.15}$$

$$C = -\frac{1}{2}\left(\nu + f_3 \wedge f_4\right), \tag{3.16}$$

where K *is the Gaussian curvature of* M.

Proof. For the proof of the necessary part of the theorem, we assume that M has pointwise 1-type Gauss map of the second kind. Then the equation (1.2) is satisfied for a smooth function f and a constant vector $C \neq 0$. From (1.2) and (2.8), we have

$$C = C_1 f_1 \wedge f_2 + C_2 f_3 \wedge f_4 \tag{3.17}$$

for some smooth functions C_1, C_2, where $\{f_3, f_4\}$ is a pseudo-orthonormal base field of the normal bundle of M. By applying f_1 and f_2 to (3.17), we obtain

$$\begin{aligned} f_1(C) &= f_1(C_1)f_1\wedge f_2 + C_1 f_1(f_1\wedge f_2) + f_1(C_2)f_3\wedge f_4 + C_2 f_1(f_3\wedge f_4)\\ &= f_1(C_1)f_1\wedge f_2 + C_1 h(f_1,f_1)\wedge f_2 + f_1(C_2)f_3\wedge f_4\\ &\quad - C_2A_3(f_1)\wedge f_4 + C_2A_4(f_1)\wedge f_3, \end{aligned}$$

from which we get

$$\begin{aligned} f_1(C) = f_1(C_1)f_1\wedge f_2 + f_1(C_2)f_3\wedge f_4 + (h^4_{11}C_1 - h^4_{11}C_2)f_2\wedge f_3\\ + (h^3_{11}C_1 + h^3_{11}C_2)f_2\wedge f_4, \end{aligned} \tag{3.18}$$

where $h^a_{ij} = \langle h(f_i, f_j), f_a\rangle,\ i,j = 1,2, a = 3,4$. Similarly, we have

$$\begin{aligned} f_2(C) = f_2(C_1)f_1\wedge f_2 + f_2(C_2)f_3\wedge f_4 + (-h^4_{22}C_1 - h^4_{22}C_2)f_1\wedge f_3\\ + (-h^3_{22}C_1 + h^3_{22}C_2)f_1\wedge f_4. \end{aligned} \tag{3.19}$$

Since C is constant, (3.18) and (3.19) imply that there are some constants C_1, C_2 satisfying

$$h^4_{11}(C_1 - C_2) = h^3_{11}(C_1 + C_2) = 0, \tag{3.20a}$$

$$h^3_{22}(-C_1 + C_2) = h^4_{22}(-C_1 - C_2) = 0. \tag{3.20b}$$

Note that if $h(f_1, f_1) = 0$ or $h(f_2, f_2) = 0$, then (2.8) implies $\Delta\nu = 0$, which gives us a contradiction. Since C is non-zero, without loss of generality, we may assume $h^3_{11} = h^4_{22} = 0$ and $C_1 = C_2$. Therefore, we obtain $h(f_1, f_1)$ and $h(f_2, f_2)$ are proportional to f_3 and f_4, respectively. Hence, they are light-like and linearly independent.

Conversely, assume that $h(f_1, f_1)$ and $h(f_2, f_2)$ are linearly independent. Then, we can choose the pseudo-orthogonal base field $\{f_3, f_4\}$ of the normal bundle of M such that

$$h(f_1, f_1) = f_3 \text{ and } h(f_2, f_2) = -Kf_4.$$

Then, (2.8) becomes

$$\Delta\nu = 2K(\nu - f_3\wedge f_4). \tag{3.21}$$

By a simple calculation, one can see that (1.2) is satisfied for f and C given by (3.15) and (3.16), and C is constant. Thus, in this case M has pointwise 1-type Gauss map of the second kind. $\square$

Theorem 3.10 (The Classification Theorem)**.** *Let M be a minimal Lorentzian surface properly contained in the semi-Euclidean space $\mathbb{E}^4_2$ with non-harmonic Gauss map. Then M has pointwise 1-type Gauss map of the second kind if and only if it is locally congruent to the surface given by*

$$x(s,t) = (\phi_1(s) + \phi_2(t), s + t, s + \cos c\ t + \sin c\ \phi_2(t), \\ \phi_1(s) - \sin c\ t + \cos c\ \phi_2(t)) \tag{3.22}$$

for some smooth non-linear functions ϕ_1, ϕ_2 and a constant $c \in (0, 2\pi)$, where $\varepsilon = \pm 1$. In this case, (1.2) is satisfied for $f = 4K$.

Proof. Let M be the minimal Lorentzian surface given by (3.4) for some light-like curves $\alpha(s), \beta(t)$.

If we assume that M has pointwise 1-type Gauss map of the second kind, then, Proposition 3.9 implies that $h(f_1, f_1)$ and $h(f_2, f_2)$ are light-like and linearly independent, where $f_1 = \frac{1}{m}\partial_s$, $f_2 = \frac{1}{m}\partial_t$ and $m = (-\langle \alpha'(s), \beta'(t)\rangle)^{1/2}$. By combining Gauss formula (2.1) with (3.2), we obtain

$$h(\partial_s, \partial_s) = \alpha'' - \frac{2m_s}{m}\alpha', \tag{3.23a}$$

$$h(\partial_t, \partial_t) = \beta'' - \frac{2m_t}{m}\beta'. \tag{3.23b}$$

On the other hand, since $\langle \alpha', \alpha' \rangle = 0$, we may assume

$$\alpha'(s) = R(s)(\cos\theta(s), \sin\theta(s), \sin\gamma(s), \cos\gamma(s)) \tag{3.24}$$

for some smooth functions R, θ, γ. By using $\langle h(\partial_s, \partial_s), h(\partial_s, \partial_s)\rangle = 0$, (3.23a) and (3.24), we obtain $\gamma'^2 = \theta'^2$, which implies $\gamma(s) = \varepsilon_1\theta(s) + c_1$ for a constant c_1, where $\varepsilon_1 = \pm 1$. Thus, (3.24) implies

$$\alpha'(s) = (\alpha_1(s), \alpha_2(s), \varepsilon_1 \sin c_1\ \alpha_1(s) + \varepsilon_1 \cos c_1\ \alpha_2(s), \\ \cos c_1\ \alpha_1(s) - \sin c_1\ \alpha_2(s)) \tag{3.25}$$

for some smooth functions α_1, α_2.

By integrating this equation and using the inverse function theorem, we see that we can assume

$$\alpha(s) = (\phi_1(s), s, \varepsilon_1 \sin c_1\ \phi_1(s) + \varepsilon_1 \cos c_1\ s, \cos c_1\ \phi_1(s) - \sin c_1\ s) \tag{3.26}$$

for a smooth function ϕ_1. Note that if ϕ_1 is linear, i.e., $\phi_1'' \equiv 0$, then α becomes an open part of light-like line. In this case, M is congruent to the surface given by (3.6), and Theorem 3.8 implies M has harmonic Gauss map, which yields a contradiction. Thus, ϕ_1 must be non-linear.

In a similar way, we have

$$\beta(t) = (\phi_2(t), t, \varepsilon_2 \sin c_2\ \phi_2(t) + \varepsilon_2 \cos c_2\ t, \cos c_2\ \phi_2(t) - \sin c_2\ t) \tag{3.27}$$

for another constant c_2 and a smooth non-linear function ϕ_2, where $\varepsilon_2 = \pm 1$. By combining (3.4) with (3.26) and (3.27), we get

$$\begin{aligned} x(s,t) = (&\phi_1(s) + \phi_2(t), s+t, \varepsilon_1 \sin c_1\ \phi_1(s) + \varepsilon_2 \sin c_2\ \phi_2(t) + \varepsilon_1 \cos c_1\ s \\ &+ \varepsilon_2 \cos c_2\ t, \cos c_1\ \phi_1(s) + \cos c_2\ \phi_2(t) - \sin c_1\ s - \sin c_2\ t), \end{aligned}$$

which is congruent to the surface given by

$$\begin{aligned} x(s,t) = (&\phi_1(s) + \phi_2(t), s+t, s + \varepsilon \cos c\ t + \varepsilon \sin c\ \phi_2(t), \\ &\phi_1(s) - \sin c\ t + \cos c\ \phi_2(t)), \end{aligned}$$

for $\varepsilon = \varepsilon_1\varepsilon_2$ and $c = c_2 - \varepsilon c_1$. In addition, a direct computation yields $c = 0$ or $c = 2\pi$ implies $\Delta\nu = 0$. Hence, we obtain a contradiction. Therefore, we can assume $c \in (0, 2\pi)$.

On the other hand, if $\varepsilon = -1$, then we have

$$\begin{aligned} m^2(s,t) = -\langle \alpha'(s), \beta'(t)\rangle = &\left(\phi_1'(s)\sqrt{1-\cos c} + \frac{|\sin c|\sqrt{1+\cos c}}{\sin c}\right) \\ &\left(\phi_2'(t)\sqrt{1-\cos c} + \frac{|\sin c|\sqrt{1+\cos c}}{\sin c}\right). \end{aligned}$$

By combining this equation and (3.3), we obtain $K = 0$, which yields a contradiction. Hence, M is congruent to the surface given by (3.22).

The converse of the theorem follows from a direct calculation. □

In the remaining of this section, we want to obtain the complete classification of minimal Lorentzian surfaces whose Gauss map satisfies (1.1) for a constant λ. First, we obtain the the following lemma.

Lemma 3.11. *Let M be the minimal Lorentzian surface given in Theorem 3.10. If M has constant Gaussian curvature, then it must be flat.*

Proof. Let M be the surface given by (3.22). Then, α and β are light-like curves given by (3.26) and (3.27). By a direct computation, we get

$$m^2(s,t) = -\langle \alpha'(s), \beta'(t)\rangle = (1-\cos c)(\phi_1'(s)\phi_2'(t) + 1) + \sin c(\phi_1'(s) - \phi_2'(t)).$$

By combining this equation with (3.3), we obtain

$$m^6 K = 2(1 - \cos c)\phi_1''(s)\phi_2''(t). \tag{3.28}$$

Now, suppose that K is a non-zero constant. Then, from (3.28) we have $m(s,t) = m_1(s)m_2(t)$ for some smooth functions m_1, m_2. However, Remark (3.2) implies that $K = 0$, which yields us a contradiction. □

By combining Theorem 3.10 and Lemma 3.11, we obtain:

Proposition 3.12. *There exists no minimal Lorentzian surface in the semi-Euclidean space* $\mathbb{E}^4_2$ *with the Gauss map satisfying* (1.1) *for* $\lambda \neq 0$ *and* $C \neq 0$.

From the Theorem 3.8 and Proposition 3.12 we have:

Theorem 3.13. *There exists no minimal Lorentzian surface properly contained in the semi-Euclidean space* $\mathbb{E}^4_2$ *with non-harmonic 1-type Gauss map.*

4. Minimal surfaces in $\mathbb{S}^4_2(1)$ with 2-type Gauss map

In this section, we consider minimal surfaces in $\mathbb{S}^4_2(1)$ with finite type Gauss map.

Before we proceed, we would like to state the following results obtained by the second-named author and Dursun in [20].

Theorem 4.1 ([20])**.** *Let* M *be a Lorentzian surface lying fully in* $\mathbb{S}^4_2(1) \subset \mathbb{E}^5_2$. *Then,* M *is minimal in* $\mathbb{S}^4_2(1)$ *with the constant Gaussian curvature* K *and non-zero constant normal curvature* K^D *if and only if it is the surface given by*

$$x(s,t) = \left(\frac{1}{2}s^2 + \frac{27}{40}\langle \alpha'''(t), \alpha'''(t)\rangle\right)\alpha(t) + \frac{3}{2}s\alpha'(t) + \frac{3}{2}\alpha''(t), \tag{4.1}$$

where α *is a null curve in the light cone* $\mathcal{LC}$ *of* $\mathbb{E}^5_2$ *satisfying*

$$\langle \alpha''(t), \alpha''(t)\rangle = \frac{4}{9}. \tag{4.2}$$

Corollary 4.2 ([20])**.** *Let* M *be an oriented minimal Lorentzian surface in* $\mathbb{S}^4_2(1) \subset \mathbb{E}^5_2$ *with the Gaussian curvature* K *and normal curvature* K^D. *If* K *and* $K^D \neq 0$ *are constant, then* $K = \frac{1}{3}$ *and* $|K^D| = \frac{2}{3}$.

In the following, we consider a Lorentzian surface M^2_1 in the semi Riemannian space $S^4_2(1)$ with 2-type Gauss map ν. In this case, ν satisfies (1.4) for some constants ξ, η.

Let (s,t) be the local coordinate system on M given in Lemma 3.1, and consider local pseudo-orthonormal frame field $\{f_1, f_2\}$ of the tangent bundle of M given by $f_1 = m^{-1}\partial_s$ and $f_2 = m^{-1}\partial_t$. The Levi–Civita connection of M takes the form

$$\nabla_{f_i} f_1 = \varphi_i f_1, \quad \nabla_{f_i} f_2 = -\varphi_i f_2, \tag{4.3}$$

for smooth functions φ_1 and φ_2 given by

$$\varphi_1 = \frac{m_s}{m^2}, \quad \varphi_2 = -\frac{m_t}{m^2}. \tag{4.4}$$

On the other hand, since the normal space of M in $\mathbb{S}^4_2(1)$ has dimension 2 and index 1, there are two null vector fields f_3, f_4 tangent to $\mathbb{S}^4_2(1)$ and normal to M satisfying $\langle f_3, f_4\rangle = -1$. Now, assume that M is minimal in $S^4_2(1)$. Then, the second fundamental form h of M in $\mathbb{E}^5_2$ takes the form

$$h(f_1, f_1) = af_3 + bf_4, \tag{4.5a}$$

$$h(f_1, f_2) = x, \tag{4.5b}$$

$$h(f_2, f_2) = cf_3 + df_4 \tag{4.5c}$$

for some smooth functions a, b, c, d. By considering (2.3), we also get

$$A_3(f_1) = bf_2, \qquad A_3(f_2) = df_1, \tag{4.6a}$$

$$A_4(f_1) = af_2, \qquad A_4(f_2) = cf_1. \tag{4.6b}$$

Thus, the Gaussian curvature K of M becomes

$$K = ad + bc + 1, \tag{4.7}$$

and the normal curvature K^D of M is

$$K^D = ad - bc. \tag{4.8}$$

Next, we obtain the following lemma.

Lemma 4.3. *Let M be a Lorentzian minimal surface in $\mathbb{S}^4_2(1)$, and f_1, f_2, f_3, f_4 the null vectors described above. Then, the tangent Gauss map $\nu = f_1 \wedge f_2$ and $\mu = f_3 \wedge f_4$ satisfy*

$$\Delta\nu = (4 - 2K)\nu + 2K^D\mu, \tag{4.9}$$

and

$$\begin{aligned}\Delta^2\nu = &\Big(-2\Delta K + (4-2K)^2 + 4{K^D}^2\Big)\nu + \Big(2\Delta K^D + 2K^D(4-2K+2-2K)\Big)\mu\\ &-4cf_1(K-K^D)f_1\wedge f_3 + 4af_2(K+K^D)f_2\wedge f_3 - 4df_1(K+K^D)f_1\wedge f_4\\ &+4bf_2(K-K^D)f_2\wedge f_4 + 4f_1(K)(f_2\wedge x) - 4f_2(K)(f_1\wedge x) \qquad (4.10)\end{aligned}$$

Proof. By a direct long computation using (4.3)–(4.6) and considering (4.7), (4.8), we get (4.9) and

$$\Delta\mu = (4-2K)\mu + 2K^D\nu. \qquad (4.11)$$

By using (4.9), we obtain

$$\begin{aligned}\Delta^2\nu = &\Delta(4-2K)\nu + (4-2K)\Delta\nu + \Delta(2K^D)\mu + 2K^D\Delta\mu\\ &+2f_1(4-2K)f_2(\nu) + 2f_2(4-2K)f_1(\nu)\\ &+2f_1(2K^D)f_2(\mu) + 2f_2(2K^D)f_1(\mu)\end{aligned}$$

Next, we combine this equation with (4.9) and (4.11) to get (4.10). □

Theorem 4.4. *Let M be a connected minimal Lorentzian surface in $\mathbb{S}^4_2(1)$. Then, M has a 2-type Gauss map if and only if it has constant Gaussian curvature and non-zero constant normal curvature.*

Proof. Let M be a minimal Lorentzian surface in $\mathbb{S}^4_2(1)$ with 2-type Gauss map. In order to prove the necessary part of the theorem, we, on the contrary, assume that K and K^D are not constant on M. Then its Gauss map ν satisfies (1.4) for a constant vector C and some constants λ_1, λ_2.

By considering (4.9), (4.10), we obtain

$$\langle C, f_3\wedge x\rangle = 0, \qquad (4.12a)$$
$$\langle C, f_4\wedge x\rangle = 0, \qquad (4.12b)$$
$$\langle C, f_1\wedge x\rangle = -4f_1(K), \qquad (4.12c)$$
$$\langle C, f_2\wedge x\rangle = 4f_2(K), \qquad (4.12d)$$
$$\langle C, f_1\wedge f_3\rangle = 4bf_2(K-K^D), \qquad (4.12e)$$
$$\langle C, f_2\wedge f_3\rangle = -4df_1(K+K^D), \qquad (4.12f)$$
$$\langle C, f_1\wedge f_4\rangle = 4af_2(K+K^D), \qquad (4.12g)$$
$$\langle C, f_2\wedge f_4\rangle = -4cf_1(K-K^D). \qquad (4.12h)$$

By considering (4.12a), (4.12b) and using (4.6), (4.6), we get

$$\langle C, bf_2 \wedge x + f_1 \wedge f_3\rangle = 0, \tag{4.13a}$$
$$\langle C, df_1 \wedge x + f_2 \wedge f_3\rangle = 0, \tag{4.13b}$$
$$\langle C, af_2 \wedge x + f_1 \wedge f_4\rangle = 0, \tag{4.13c}$$
$$\langle C, cf_1 \wedge x + f_2 \wedge f_4\rangle = 0. \tag{4.13d}$$

Finally, we combine (4.13) with (4.12) to get

$$b(4f_2(K)) + 4bf_2(K - K^D) = 0, \tag{4.14a}$$
$$d(-4f_1(K)) - 4df_1(K + K^D) = 0, \tag{4.14b}$$
$$a(4f_2(K)) + 4af_2(K + K^D) = 0, \tag{4.14c}$$
$$c(-4f_1(K)) - 4cf_1(K - K^D) = 0, \tag{4.14d}$$

which gives

$$bf_2(2K - K^D) = 0, \tag{4.15a}$$
$$df_1(2K + K^D) = 0, \tag{4.15b}$$
$$af_2(2K + K^D) = 0, \tag{4.15c}$$
$$cf_1(2K - K^D) = 0. \tag{4.15d}$$

Now, consider an open, connected subset $\mathcal{M}$ of M on which ∇K and ∇K^D do not vanish. Note that if $ad \neq 0$ and $bc \neq 0$ at $p \in \mathcal{M}$, then (4.15) implies that K and K^D are constant on a neighborhood of p in $\mathcal{M}$, which yields a contradiction. Therefore, we have either $ad = 0$ or $bc = 0$ at p. Moreover, if $ad = bc = 0$ on an open subset $\mathcal{O}$ of $\mathcal{M}$, then we have $K^D = 0$ and $K = 1$ on $\mathcal{O}$ because of (4.7) and (4.8). Hence, we have either $ad = 0, bc \neq 0$ or $ad \neq 0, bc = 0$ on $\mathcal{O}$.

We assume that $ad \neq 0$, $bc = 0$ on $\mathcal{O}$. Then, we have $K = ad + 1$, $K^D = ad$ from (4.7) and (4.8). In addition, (4.15b), (4.15c) yield that $2K + 2K^D = \text{const}$ over $\mathcal{O}$. By combining these equations, we see that K and K^D are constants on $\mathcal{O}$, which yields a contradiction. Therefore, we have $K^D = \text{const} \neq 0$. In a similar way, we obtain the same result for the case $bc \neq 0, ad = 0$. Hence, we proved the necessary part.

In order to prove the sufficiency part, we assume that M has constant Gaussian curvature K and non-zero constant normal curvature K^D. Then, up to re-orientation, we may assume $K = 1/3$ and $K^D = -2/3$ because of Corollary 4.2. Thus, (4.9) and (4.11) imply

$$\Delta\nu = \frac{10}{3}\nu - \frac{4}{3}\mu \quad \text{and} \quad \Delta(\mu) = -\frac{4}{3}\nu + \frac{4}{3}\mu.$$

By a direct computation, one can check that $\nu_1 = \frac{4}{5}\nu - \frac{2}{5}\mu$ and $\nu_2 = \frac{1}{5}\nu + \frac{2}{5}\mu$ satisfy $\nu = \nu_1 + \nu_2$, $\Delta\nu_1 = 4\nu_1$ and $\Delta\nu_2 = \frac{2}{3}\nu_2$. Hence, M is 2-type. □

By combining Theorems 4.1 and 4.4, we obtain the following classification result.

Theorem 4.5. *Let M be a connected minimal Lorentzian surface in $\mathbb{S}^4_2(1)$. Then, M has a 2-type Gauss map if and only if it is congruent to the surface given by* (4.1) *for a null curve α in the light cone $\mathcal{LC}$ of $\mathbb{E}^5_2$ satisfying* (4.2)*. In this case,* (1.3) *is satisfied for $\nu_1 = \frac{4}{5}\nu - \frac{2}{5}\mu$, $\nu_2 = \frac{1}{5}\nu + \frac{2}{5}\mu$, $\lambda_1 = 4$ and $\lambda_2 = \frac{2}{3}$.*

Corollary 4.6. *There are no minimal Lorentzian surface in $\mathbb{S}^4_2(1)$ with null 2-type Gauss map.*

Acknowledgements. This work is obtained during the TÜBİTAK 1001 project 'Y_EUCL2TIP' (Project Number: 114F199).

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(Received March 16, 2016; revised October 24, 2016)