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On the Diophantine equations $(x-1)^3+x^5+(x+1)^3=y^n$ and $(x-1)^5+x^3+(x+1)^5=y^n$

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Abstract. In this paper, we prove that the Diophantine equations $(x-1)^3+x^5+(x+1)^3=y^n$ and $(x-1)^5+x^3+(x+1)^5=y^n$ have no integer solutions with $x \neq 0$ and $n > 1$, unless $(x, y, n) = (1, \pm 3, 2)$ for the first equation.

1. Introduction

The Diophantine equation

$$1^k + 2^k + \cdots + x^k = y^n, k, n \geq 2$$

was considered by a number of authors (see, e.g., [2], [7], [13], [14], [15], [16], [17], [18], [23], [24]). A generalization is to consider the equation

$$(x+1)^k + (x+2)^k + \cdots + (x+m)^k = y^n, k, n \geq 2.$$

Meng Bai and the author [25] solved this equation for $k = 2$, $m = x$, and Bennett, Patel and Siksek [4] for $k = 3$ and $2 \leq m \leq 50$. When $m = 3$, we usually redefine variables and consider the equation

$$(x-1)^k + x^k + (x+1)^k = y^n. \tag{1}$$

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Cassels [10] proved that $x = 0, 1, 2, 24$ are the only integer solutions to this equation for $k = 3$, $n = 2$. For general $n > 1$, the author [26] provides all the integer solutions for $k = 2, 3, 4$, and Bennett, Patel and Siksek [3] for $k = 5, 6$.

In this paper, we consider a variation of equation (1), that is,

$$(x-1)^k + x^m + (x+1)^k = y^n,$$

and obtain the following results.

Theorem 1.1. *The equation*

$$(x-1)^3 + x^5 + (x+1)^3 = y^n \tag{2}$$

has only the integer solutions $(x, y, n) = (1, \pm 3, 2)$ *with* $x \neq 0$ *and* $n > 1$.

Theorem 1.2. *The equation*

$$(x-1)^5 + x^3 + (x+1)^5 = y^n \tag{3}$$

has no integer solutions (x, y, n) *with* $x \neq 0$ *and* $n > 1$.

2. Some preliminary results

In this section, we present some lemmas which will help us to prove Theorem 1.1 and Theorem 1.2. The first lemma is due to Nagell [8].

Lemma 2.1. *If* $n > 1$, *then the equation*

$$x^2 + 5 = y^n$$

has only the integer solutions $(x, y, n) = (\pm 2, \pm 3, 2)$.

Lemma 2.2. *If* $n > 1$, *then the equation*

$$x^2 + 5 = 2y^n$$

has no integer solutions.

Proof. Obviously, $\gcd(x, y) = 1$. Then, by [1, Theorem 2], it has no integer solutions for $n \geq 3$. If $n = 2$, one has $x^2 + 5 = 2y^2$, modulo 8 yields no integer solutions. $\square$

A special case of Theorem 1 in [9], which we need in this paper, is the following result.

Lemma 2.3. *Let* u, $r \geq 0$, $n \geq 3$ *be integers, then the equation*

$$19^u x^n - 2^r y^n = \pm 1$$

has no integer solutions with $x, y > 0$, *unless* $u = 1$, $r = 0$, $n = 3$ *and* $(x, y) = (3, 8)$.

3. The modular approach

We introduce some basic concepts and notation for the modular approach we used in this paper.

Let E be an elliptic curve over $\mathbb{Q}$ of conductor N. For a prime of good reduction l, we write $\#E(\mathbb{F}_l)$ for the number of points on E over the finite field $\mathbb{F}_l$, and let $a_l(E) = l+1-\#E(\mathbb{F}_l)$. By a *newform* f, we will always mean a cuspidal newform of weight 2 with respect to $\Gamma_0(N_0)$ for some positive integer N_0, and N_0 will be called the *level* of f. Write $f = q + \sum_{i\geq 2} c_i q^i$ the q-expansion of f, then c_n will be called the *Fourier coefficients* of f. Let $\mathbb{K} = \mathbb{Q}(c_2, c_3, \dots)$ be the field obtained by adjoining to $\mathbb{Q}$ the Fourier coefficients of f, then $\mathbb{K}$ is a finite and totally real extension of $\mathbb{Q}$ (see, e.g., [12, Chapter 15]).

We shall say that the curve E arises modulo p from the newform f (and write $E \sim_p f$) if there is a prime ideal $\mathfrak{p}$ of $\mathbb{K}$ above p such that for all but finitely many primes l we have $a_l(E) \equiv c_l \pmod{\mathfrak{p}}$(see [12, Definition 15.2.1]).

We have the following result, which is just [9, Lemma 2.1].

Proposition 3.1. *Assume that $E \sim_p f$. There exists a prime ideal $\mathfrak{p}$ of $\mathbb{K}$ above p such that, for all primes l,*

(i) *if $l \nmid pNN_0$, then $a_l(E) \equiv c_l \pmod{\mathfrak{p}}$,*

(ii) *if $l || N$ but $l \nmid pN_0$, then $\pm(l+1) \equiv c_l \pmod{\mathfrak{p}}$.*

Moreover, if f is rational, then the above can be relaxed slightly as follows, for all primes l,

(i) *if $l \nmid NN_0$, then $a_l(E) \equiv c_l \pmod{p}$,*

(ii) *if $l || N$ but $l \nmid N_0$, then $\pm(l+1) \equiv c_l \pmod{p}$.*

4. Proofs of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1. Expanding the left hand side of equation (2), one has

$$x(x^4 + 2x^2 + 6) = y^n. \tag{4}$$

Since $\gcd(x, x^4 + 2x^2 + 6) = \gcd(x, 6) \in \{1, 2, 3, 6\}$, equation (4) implies one of the following cases:

$$\begin{array}{llll}
\text{(i)} & x = z^n, & x^4 + 2x^2 + 6 = w^n, & y = zw; \\
\text{(ii)} & x = 2^{n-1}z^n, & x^4 + 2x^2 + 6 = 2w^n, & y = 2zw;
\end{array}$$

$$\text{(iii)} \quad x = 3^{n-1}z^n, \quad x^4 + 2x^2 + 6 = 3w^n, \quad y = 3zw;$$
$$\text{(iv)} \quad x = 6^{n-1}z^n, \quad x^4 + 2x^2 + 6 = 6w^n, \quad y = 6zw.$$

In case (i), we obtain $(x^2+1)^2+5 = w^n$, and by Lemma 2.1, one has $(x, y, n) = (1, \pm 3, 2)$. In case (ii), we get $(x^2+1)^2+5 = 2w^n$, and this equation has no integer solutions by Lemma 2.2.

In cases (iii) and (iv), without loss of generality, we assume $n = p$ and p is a prime. We proceed to prove that equation (2) has no integer solutions for $p \geq 11$ in case (iii), and for $p \geq 7$ in case (iv). The remaining cases will be treated at the end of the proof. Assume $x \neq 0$ in the following discussion.

In case (iii), we apply Proposition 3.1 and the multi-Frey approach [9] to bound p. Let $x = 3u$, one has

$$(9u^2+1)^2+5 = 3w^p$$

and

$$2w^p = (2+3u^2)^2 + 45u^4 = (2+3u^2)^2 + 5 \times 3^{p-6}(3^3z^4)^p.$$

It is obvious that $\gcd(9u^2+1, 5) = 1$ in the first equation. If $5|2+3u^2$ in the second equation, then $\mathrm{ord}_5\,(45u^4)$=1, a contradiction. Therefore, one has $\gcd(2w^p, 45u^4) = 1$. To a possible solution (u, w) with $u \neq 0$, we associate the Frey curves [5]

$$E_{1,u} : Y^2 = X^3 + 2(9u^2+1)X^2 - 5X$$

for the first equation, and

$$E_{2,u} : Y^2 = X^3 + 2(3u^2+2)X^2 + 2(27u^4+6u^2+2)X$$

for the second equation, with conductors $N = 2^5 \,\mathrm{rad}(15w) = 2^5 \times 3 \times 5 \,\mathrm{rad}_{\{3,5\}}(w)$ and $N = 2^6 \,\mathrm{rad}(30uw) = 2^7 \times 3 \times 5 \,\mathrm{rad}_{\{2,3,5\}}(w)$, respectively, where for a finite set S of primes, we denote

$$\mathrm{rad}_S(a) = \prod_{p|a, p \neq q, q \in S} p.$$

Then, by [5, Lemma 3.3], there are newforms f, g of levels $N(E_{1,u})_p = 2^5 \times 3 \times 5 = 480$ and $N(E_{2,u})_p = 2^7 \times 3 \times 5 = 1920$ such that $E_{1,u} \sim_p f$ and $E_{2,u} \sim_p g$.

There are 8 rational newforms at level 480, and 28 newforms at level 1920, with 4 non-rational, numbered in Stein's Table [19] by $f_1, f_2, \ldots, f_8$ and g_1, $g_2, \ldots, g_{28}$, respectively. We choose $l = 7, 11, 13$ to get the bound $p \leq 7$ for the newforms at level 1920 by Proposition 3.1, except g_1, g_4. Then, we use the multi-Frey approach to consider the pair (f_i, g_1) and (f_i, g_4) with $1 \le i \le 8$. We get $p \le 7$ while choosing the primes $7 \le l \le 19$ for the 15 pairs, and $l = 31$ for the left pair (f_7, g_4). In fact, for the pair (f_7, g_4), one has $c_{31}(f_7) = a_{31}(E_{1,10}) = -4$, but $c_{31}(g_4) = 10 \neq -10 = a_{31}(E_{2,10})$.

For the case (iv), let $x = 6v$, then we have

$$w^p = 1 + 12v^2 + 216v^4 = (1 + 6v^2)^2 + 180v^4 = (1 + 6v^2)^2 + 5 \times 6^{p-6}(6^3 z^4)^p.$$

If $5 | 1 + 6v^2$, then $\mathrm{ord}_5\,(180v^4)$=1 is a contradiction. Therefore, we have $\gcd(w^p, 180v^4) = 1$. To a possible solution (v, w) with $v \neq 0$, we associate the Frey curve [5]

$$E_v : Y^2 + XY = X^3 + 6\left(\frac{v}{2}\right)^2 X^2 - 45\left(\frac{v}{2}\right)^4 X,$$

with conductor $N = \mathrm{rad}(30vw) = 30\,\mathrm{rad}_{\{2,3,5\}}(vw)$. Then, by [5, Lemma 3.3], there is a newform of level $N(E_v)_p = 30$ such that $E_v \sim_p f$. There is only one rational newform, and choosing $l = 7$ leads to $p \le 5$.

We proceed to treat the small primes p for the cases (iii) and (iv). Write $d = 3, 6$, then we have

$$x^4 + 2x^2 + 6 = dw^p. \tag{5}$$

If $p = 2$, we write $X = dx^2$, $Y = d^2xw$. From (5), it follows that (X, Y) is an integral point on the elliptic curve

$$E_d : Y^2 = X^3 + 2dX^2 + 6d^2X.$$

Appealing to Magma [6], we get that the integral points on these curves are $(0, 0)$ for $d = 3$ and $(0, 0), (96, 1008)$ for $d = 6$, which yields no integer solutions with $x \neq 0$ for the equation (2). If $p = 3$, write $X = dw$, $Y = d(x^2 + 1)$. From (5), one has the elliptic curve

$$E'_d : Y^2 = X^3 - 5d^2.$$

According to Magma, we get the integral points $(21, \pm\, 96)$ for $d = 3$ and $(6, \pm\, 6)$, $(69, \pm\, 573)$ for $d = 6$, and we also obtain no integer solutions with $x \neq 0$ for the equation (2).

It remains to deal with the prime $p = 5, 7$ for the case (iii), and only $p = 5$ for the case (iv), since we have $p \le 5$ for this case. Let $t = x^2 + 1$, and rewrite (5) as

$$t^2 + 5 = dw^p. \tag{6}$$

For $p = 5$, we have two genus 2 hyperelliptic curves $t^2 = 3w^5 - 5$ and $t^2 = 6w^5 - 5$ from (6). The rank of the Jacobians of these curves is 1, so classical

Chabauty [11] applies, which is also implemented in Magma (see, e.g., [20], [21], [22]). It is not difficult to determine a point having infinite order on these Jacobians, and after that Chabauty's method combined with the Mordell–Weil sieve provides the points. In the first case, only the point at infinity is a solution, while in the second case the point at infinity and the points $(1, \pm 1)$ are, which corresponds to $x = 0$.

For $p = 7$, we need to treat the equation

$$t^2 + 5 = 3w^7. \tag{7}$$

Let $K = \mathbb{Q}(\sqrt{-5})$. This field has class number 2 and ring of integers $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$. Since $3|x$, we obtain

$$(t + \sqrt{-5})\mathcal{O}_K = (3, 1 + \sqrt{-5})\mathfrak{a}^7$$

from $t = x^2 + 1 \equiv 1 \pmod 3$. Observe that

$$(1 + \sqrt{-5})\mathcal{O}_K = (3, 1 + \sqrt{-5})(2, 1 + \sqrt{-5})$$

and

$$(2, 1 + \sqrt{-5})^2 = 2\mathcal{O}_K.$$

Then we write

$$(t + \sqrt{-5})\mathcal{O}_K = (3, 1 + \sqrt{-5})(2, 1 + \sqrt{-5})^7((2, 1 + \sqrt{-5})^{-1}\mathfrak{a})^7,$$

that is,

$$(t + \sqrt{-5})\mathcal{O}_K = (1 + \sqrt{-5})(8)(2^{-1}(2, 1 + \sqrt{-5})\mathfrak{a})^7.$$

Therefore, $(2, 1 + \sqrt{-5})\mathfrak{a}$ must be principle, and since it is an integral ideal, we obtain

$$t + \sqrt{-5} = 8(1 + \sqrt{-5})(2^{-1}(a + b\sqrt{-5}))^7,$$

with a, b being some integers. Expanding the right hand side corresponds to the Thue equation

$$a^7 + 7a^6b - 105a^5b^2 - 175a^4b^3 + 875a^3b^4 + 525a^2b^5 - 875ab^6 - 125b^7 = 16.$$

It has no integer solutions modulo 29. From the discussion above, this completes the proof of Theorem 1.1. $\square$

Proof of Theorem 1.2. Expanding the left hand side of equation (3), one has

$$x(x^2+10)(2x^2+1)=y^n. \tag{8}$$

Since $\gcd(2x^2+1,x)=1$, $\gcd(2x^2+1,x^2+10)=\gcd(2x^2+1,2x^2+20)=\gcd(2x^2+1,19)=1$ or 19, equation (8) implies

$$2x^2+1=19^\alpha z^n,$$

with $\alpha=0$, 1 or $n-1$. By Lemma 2.3, it has no integer solutions with $n\geq 3$ and $x\neq 0$.

We are left to treat $n=2$. By the discussion above, we only need to solve the equations $z^2=x(x^2+10)$ and $z^2=19x(x^2+10)$. Let $v=19z$, $u=19x$, then the last equation can be written as $v^2=u^3+3610u$. Appealing to Magma, the only integral points on these two curves are $(0,0)$, and hence $x=0$. $\square$

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