Publ. Math. Debrecen
91/3-4 (2017), 391–402
DOI: 10.5486/PMD.2017.7718

Summability of general Fourier series

By LARRY GOGOLADZE (Tbilisi) and VAKHTANG TSAGAREISHVILI (Tbilisi)

Abstract. In the paper, the sufficient conditions are stated which should be satisfied by functions of orthonormal systems (ONS) $\{\varphi_n(x)\}$ such that the Fourier series of every function with bounded variation is summable by the method (C,α), $\alpha > 0$, a.e. on $[0,1]$. It is also shown that the obtained results are best possible in a certain sense.

1. Definitions and auxiliary theorems

By $V(0,1)$ we denote a class of all functions with bounded variation on $[0,1]$. $\overset{1}{\underset{0}{\vee}}(f)$ denotes the variation of function $f(x)$ on $[0,1]$.

Let $\{\varphi_n(x)\}$ be a real valued orthonormal system on $[0,1]$, and

$$\widehat{\varphi}_n(f) = \int_0^1 f(x)\varphi_n(x)\,dx, \qquad n = 1,2,\dots,$$

are the Fourier coefficients of function $f(x) \in L(0,1)$.

Introduce the notation

$$B_n(a) = \max_{1\le i\le n} \left| \int_0^{\frac{i}{n}} \Phi_n(a,x)\,dx \right|, \tag{1.1}$$

where

$$\Phi_n(a,x) = \sum_{k=1}^{n} a_k\lambda_k\varphi_k(x),$$

(a_k) and (λ_k) are some sequences of numbers.

Mathematics Subject Classification: 42C10.
Key words and phrases: Fourier coefficients, summability, trigonometric system, Walsh system, Haar system, finite variation, Lebesgue function.
The work was supported by the Shota Rustaveli National Science Foundation. Grant No. FR/102/5-100/14.

Besides, let
$$L_n^{(1)}(x)=\int_0^1\left|\frac{1}{n}\sum_{k=1}^{n}\sum_{m=1}^{k}\varphi_m(x)\varphi_m(t)\right|dt \tag{1.2}$$
be a Lebesgue function.

We have the following:

Theorem 1.1 (Menshov–Rademacher, see [1, p. 87])**.** *If for some sequence* (c_n) *the condition*
$$\sum_{n=1}^{\infty}c_n^2\log^2 n<+\infty \tag{1.3}$$
holds, then a.e. on $[0,1]$ *the series*
$$\sum_{n=1}^{\infty}c_n\varphi_n(x) \tag{1.4}$$
converges, where $\{\varphi_n(x)\}$ *is an arbitrary ONS.*

Theorem 1.2 (Kachmarz, see [1, pp. 207–209])**.** *If* (λ_n) *is a non-decreasing convex sequence of positive numbers which should satisfy the conditions*

(a) $\lim\limits_{n\to\infty}\lambda_n=+\infty$,

(b) $L_n^{(1)}(x)=O(\lambda_n)$ *a.e. on* $[0,1]$,

then if
$$\sum_{n=1}^{\infty}c_n^2\lambda_n<+\infty, \tag{1.5}$$
then series (1.4) *is* (C,α) $(\alpha>0)$ *summable a.e. on* $[0,1]$.

Remark. Without loss of generality, it may be supposed that $\lambda_n\le\log^2 n$. From the convexity of the sequence (λ_n), we easily get the following: there exists the natural N such that $\lambda_n\ge\log^2 n$ or $\lambda_n<\log^2 n$, when $n>N$ $(n=1,2,\dots)$. In the first case, from (1.5) it follows that (1.3) is true, and hence, according to Theorem 1.1 the series (1.4) will converge and indeed (C,α), $\alpha>0$, is summable a.e. on $[0,1]$. There remains the second case. Therefore, without loss of generality, in Theorem 1.2 we will suppose that $\lambda_n<\log^2 n$.

In [4], it is proved. Let $f(x)$, $\Phi(x)\in L_2(0,1)$, and let $f(x)$ be a function everywhere finite on $[0,1]$. Then
$$\int_0^1 f(x)\Phi(x)\,dx=\sum_{i=1}^{n-1}\left(f\left(\frac{i}{n}\right)-f\left(\frac{i+1}{n}\right)\right)\int_0^{\frac{i}{n}}\Phi(x)\,dx$$
$$+\sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\left(f(x)-f\left(\frac{i}{n}\right)\right)\Phi(x)dx+f(1)\int_0^1\Phi(x)dx. \tag{1.6}$$

2. Main problem

From S. Banach's theorem [2], it follows that if $f(x) \in L_2(0,1)$ ($f \not\sim 0$), then there exists ONS $\{\varphi_n(x)\}$ such that the Fourier series of this function is not summable by the Cesaro method (C,α), $\alpha > 0$, a.e. on $[0,1]$ with respect to this system.

A. M. Olevskii [3] has proved that if $f(x) \in L_2(0,1)$ ($f \not\sim 0$) is an arbitrary function and $(a_n) \in \ell_2$ is any sequence of numbers, then there exists ONS such that

$$\widehat{\varphi}_n(f) = c \cdot a_n, \qquad n = 1, 2, \dots,$$

and c is some number.

Thus it is clear that the Fourier coefficients with bounded variation in the general case do not satisfy condition (1.5).

On the bases of the above reasoning, if it is necessary that ONS $\{\varphi_n(x)\}$ guaranteed the summability of the Fourier series by the method (C,α), $\alpha > 0$, a.e. on $[0,1]$ of every function from $V(0,1)$, the functions of the system $\{\varphi_n(x)\}$ should satisfy some conditions.

In the present paper, the above conditions are studied so that the Fourier coefficients of every function from $V(0,1)$ satisfied condition (1.5).

3. Main results

Theorem 3.1. *Let $\{\varphi_n(x)\}$ be ONS on $[0,1]$, and let (λ_n) be a convex sequence of numbers, $\lim_{n\to\infty} \lambda_n = +\infty$, and*

$$L_n^{(1)}(x) = O(\lambda_n) \quad (\lambda_n = O(\log^2 n)).$$

Then, if for any sequence $(a_n) \in \ell_2$

$$B_n(a) = O(1) \left(\sum_{k=1}^{n} a_k^2 \lambda_k \right)^{\frac{1}{2}}, \tag{3.1}$$

then the Fourier series of every function from $V(0,1)$ is summable by the method $(C, \alpha > 0)$ a.e. on $[0,1]$.

Proof. Assume in equality (1.6) that $f(x) \in V(0,1)$ and $\Phi(x) = \Phi_n(\widehat{\varphi}, x)$, where $\Phi_n(\widehat{\varphi}, x) = \Phi_n(a, x)$ if $(a) = (\widehat{\varphi})$.

We get

$$\begin{aligned}\int_0^1 f(x)\Phi_n(\widehat{\varphi},x)\,dx &= \sum_{i=1}^{n-1}\left(f\left(\frac{i}{n}\right)-f\left(\frac{i+1}{n}\right)\right)\int_0^{\frac{i}{n}}\Phi_n(\widehat{\varphi},x)\,dx \\ &\quad+\sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\left(f(x)-f\left(\frac{i}{n}\right)\right)\Phi_n(\widehat{\varphi},x)\,dx \\ &\quad+f(1)\int_0^1\Phi_n(\widehat{\varphi},x)\,dx = M_1+M_2+M_3. \qquad (3.2)\end{aligned}$$

Putting $f(x)\in V(0,1)$, according to Theorem 3.1, we obtain (see (1.1))

$$\begin{aligned}|M_1| &\le \sup_{1\le i<n}\left|\int_0^{\frac{i}{n}}\Phi_n(\widehat{\varphi},x)\,dx\right|\sum_{i=1}^{n-1}\left|f\left(\frac{i}{n}\right)-f\left(\frac{i+1}{n}\right)\right| \\ &\le \overset{1}{\underset{0}{\vee}}(f)\cdot B_n(\widehat{\varphi}) = O(1)\left(\sum_{k=1}^{n}a_k^2\lambda_k\right)^{\frac{1}{2}}, \qquad (3.3)\\ |M_2| &\le \sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\left|f(x)-f\left(\frac{i}{n}\right)\right|\,|\Phi_n(\widehat{\varphi},x)|\,dx \\ &\le \sum_{i=1}^{n}\sup_{x\in[\frac{i-1}{n},\frac{i}{n}]}\left|f(x)-f\left(\frac{i}{n}\right)\right|\int_{\frac{i-1}{n}}^{\frac{i}{n}}\Phi_n(\widehat{\varphi},x)|\,dx \\ &\le \overset{1}{\underset{0}{\vee}}(f)\sup_{1\le i\le n}\int_{\frac{i-1}{n}}^{\frac{i}{n}}|\Phi_n(\widehat{\varphi},x)|\,dx \\ &= O(1)\,\frac{1}{\sqrt{n}}\left(\int_0^1\Phi_n^2(\widehat{\varphi},x)\,dx\right)^{\frac{1}{2}} = O(1)\,\frac{1}{\sqrt{n}}\left(\sum_{k=1}^{n}\widehat{\varphi}_k^2(f)\,\lambda_k^2\right)^{\frac{1}{2}} \\ &= O(1)\,\frac{\sqrt{\lambda_n}}{\sqrt{n}}\left(\sum_{k=1}^{n}\widehat{\varphi}_k^2(x)\,\lambda_k\right)^{\frac{1}{2}} = O(1)\left(\sum_{k=1}^{n}\widehat{\varphi}_k^2(f)\,\lambda_k\right)^{\frac{1}{2}}. \qquad (3.4)\end{aligned}$$

Besides,

$$\begin{aligned}|M_3| &= |f(1)|\left|\int_0^1\Phi_n(\widehat{\varphi},x)\,dx\right| = O(1)\max_{1\le i\le n}\left|\int_0^{\frac{i}{n}}\Phi_n(\widehat{\varphi},x)\,dx\right| \qquad (3.5)\\ &= O(1)B_n(\widehat{\varphi}) = O(1)\left(\sum_{k=1}^{n}\widehat{\varphi}_k^2(f)\lambda_k\right)^{\frac{1}{2}}.\end{aligned}$$

Taking in account (3.3), (3.4) and (3.5), from (3.2) we get

$$\int_0^1 f(x)\Phi_n(\widehat{\varphi},x)\,dx = O(1)\left(\sum_{k=1}^n \widehat{\varphi}_k^2(f)\lambda_k\right)^{\frac{1}{2}}. \tag{3.6}$$

On the other hand,

$$\sum_{k=1}^n \widehat{\varphi}_k^2(f)\lambda_k = \int_0^1 f(x)\sum_{k=1}^n \widehat{\varphi}_k(f)\lambda_k\varphi_k(x)\,dx = \int_0^1 f(x)\Phi_n(\widehat{\varphi},x)\,dx.$$

From here and (3.6) we get

$$\sum_{k=1}^n \widehat{\varphi}_k^2(f)\lambda_k = O(1), \qquad \text{i.e.} \qquad \sum_{k=1}^\infty \widehat{\varphi}_k^2(f)\lambda_k < +\infty,$$

for any function from $V(0,1)$.

Finally, from Theorem 1.2 there follows the summability of the Fourier series of any function from $V(0,1)$. □

Consider now the case when condition (3.1) is not satisfied.

Theorem 3.2. *Let $\{\varphi_n(x)\}$ be ONS on $[0,1]$, and (λ_n) a convex sequence of numbers, $\lim_{n\to\infty}\lambda_n = +\infty$ ($\lambda_n = O(\log^2 n)$), and for some sequence of numbers $(b_n)\in\ell_2$,*

$$\overline{\lim_{n\to\infty}}\frac{B_n(b)}{\left(\sum_{k=1}^n b_k^2\lambda_k\right)^{1/2}} = +\infty.$$

Then there exists a function $f_0(x)\in A$ (class of absolute continuous functions) such that

$$\sum_{n=1}^\infty \widehat{\varphi}_n^2(f_0)\lambda_n = +\infty.$$

Proof. Consider the sequence of functions

$$f_n(x) = \begin{cases} 0 \text{ for } x\in\left[0,\frac{i_n}{n}\right], \\ 1 \text{ for } x\in\left[\frac{i_{n+1}}{n},1\right], \\ \text{is continuous and linear on } \left[\frac{i_n}{n},\frac{i_{n+1}}{n}\right], \end{cases} \tag{3.7}$$

where

$$\max_{1\le i\le n}\left|\int_0^{\frac{i}{n}}\Phi_n(b,x)\,dx\right| = \left|\int_0^{\frac{i_n}{n}}\Phi_n(b,x)\,dx\right|.$$

Assume in equality (3.2) that $f(x) = f_n(x)$ and $\Phi_n(\widehat{\varphi}, x) = \Phi_n(b, x)$. Then we have

$$\begin{aligned}\int_0^1 f_n(x)\Phi_n(b,x)dx &= \sum_{i=1}^{n-1}\left(f_n\left(\frac{i}{n}\right) - f_n\left(\frac{i+1}{n}\right)\right)\int_0^{\frac{i}{n}} \Phi_n(b,x)\,dx \\ &+ \sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\left(f_n(x) - f_n\left(\frac{i}{n}\right)\right)\Phi_n(b,x)dx \\ &+ f_n(1)\int_0^1 \Phi_n(b,x)\,dx = I_1 + I_2 + I_3. \qquad (3.8)\end{aligned}$$

Taking into account (3.7), we will have

$$|I_1| = \left|\int_0^{\frac{i_n}{n}} \Phi_n(b,x)\,dx\right| = B_n(b). \qquad (3.9)$$

Then, since (see (3.7)) $|f_n(x) - f_n(\frac{i}{n})| = 1$ when $x \in \left[\frac{i_n}{n}, \frac{i_{n+1}}{n}\right]$, we have

$$\begin{aligned}|I_2| &= \left|\sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\left(f_n(x) - f_n\left(\frac{i+1}{n}\right)\right)\Phi_n(b,x)\,dx\right| \le \int_{\frac{i_n-1}{n}}^{\frac{i_n}{n}} |\Phi_n(b,x)|\,dx \\ &\le \frac{1}{\sqrt{n}}\left(\sum_{k=1}^{n} b_k^2\lambda_k^2\right)^{\frac{1}{2}} \le \frac{\sqrt{\lambda_n}}{\sqrt{n}}\left(\sum_{k=1}^{n} b_k^2\lambda_k\right)^{\frac{1}{2}} = O(1)\left(\sum_{k=1}^{n} b_k^2\lambda_k\right)^{\frac{1}{2}}. \qquad (3.10)\end{aligned}$$

Finally,

$$\begin{aligned}|I_3| &= \left|\int_0^1 \Phi_n(b,x)\,dx\right| = \left|\sum_{k=1}^{n} b_k\lambda_k\int_0^1 \varphi_k(x)\,dx\right| \\ &= \left(\sum_{k=1}^{n} b_k^2\lambda_k\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}\left(\int_0^1 \varphi_k(x)\,dx\right)^2\lambda_k\right)^{\frac{1}{2}}. \qquad (3.11)\end{aligned}$$

We can assume without loss of generality that

$$\sum_{k=1}^{\infty}\left(\int_0^1 \varphi_k(x)\,dx\right)^2\lambda_k < +\infty.$$

Otherwise, if

$$\sum_{k=1}^{\infty}\left(\int_0^1 \varphi_k(x)\,dx\right)^2\lambda_k = +\infty,$$

we get that the Fourier coefficients of the function $f_0(x) = 1$ satisfy the statement of Theorem 3.2.

Therefore, from (3.11) we get

$$|I_3| = O(1)\left(\sum_{k=1}^{n} b_k^2\lambda_k\right)^{\frac{1}{2}}. \tag{3.12}$$

From (3.9), (3.10) and (3.8), according to (3.8), we have

$$\left|\int_0^1 f_n(x)\Phi_n(b,x)\,dx\right| \geq B_n(b) - O(1)\left(\sum_{k=1}^{n} b_k^2\lambda_k\right)^{\frac{1}{2}}.$$

Hence

$$\frac{\left|\int_0^1 f_n(x)\Phi_n(b,x)\,dx\right|}{\left(\sum_{k=1}^{n} b_k^2\lambda_k\right)^{\frac{1}{2}}} \geq \frac{B_n(b)}{\left(\sum_{k=1}^{n} b_k^2\lambda_k\right)^{\frac{1}{2}}} - O(1).$$

Due to the condition of Theorem 3.2, we have

$$\overline{\lim_{n\to\infty}} \frac{\left|\int_0^1 f_n(x)\Phi_n(b,x)\,dx\right|}{\left(\sum_{k=1}^{n} b_k^2\lambda_k\right)^{\frac{1}{2}}} = \overline{\lim_{n\to\infty}} \frac{B_n(b)}{\left(\sum_{k=1}^{n} b_k^2\lambda_k\right)^{\frac{1}{2}}} = +\infty. \tag{3.13}$$

Let A be a class of all absolutely continuous functions, and let $\|f\|_A$ be norm of functions $f(x) \in A$. Then

$$\|f_n\|_A = \|f_n\|_C + \int_0^1 \left|\frac{df_n(x)}{dx}\right| dx = 2.$$

Consider on the Banach space A the sequence of bounded linear functionals

$$U_n(f) = \left(\sum_{k=1}^{n} b_k^2\lambda_k\right)^{\frac{-1}{2}} \int_0^1 f(x)\Phi_n(b,x)\,dx.$$

Since $\|f_n\|_A = 2$, taking into account (3.13), $\limsup_n |U_n(f_n)| = +\infty$, in virtue of the Banach–Steinhaus theorem, there exists the function $f_0(x) \in A$ such that

$$\overline{\lim_{n\to\infty}} \frac{1}{\left(\sum_{k=1}^{n} b_k^2\lambda_k\right)^{\frac{1}{2}}} \left|\int_0^1 f_0(x)\Phi_n(b,x)\,dx\right| = +\infty. \tag{3.14}$$

Then

$$\left|\int_0^1 f_0(x)\Phi_n(b,x)\,dx\right| = \left|\sum_{k=1}^{n} b_k\lambda_k \int_0^1 f_0(x)\varphi_k(x)\,dx\right| =$$

$$= \left| \sum_{k=1}^{n} b_k \sqrt{\lambda_k} \widehat{\varphi}_k(f_0) \sqrt{\lambda_k} \right| \le \left(\sum_{k=1}^{n} b_k^2 \lambda_k \right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} \widehat{\varphi}_k^2(f_0) \lambda_k \right)^{\frac{1}{2}}.$$

From here and (3.14), we get

$$\lim_{n\to\infty} \sum_{k=1}^{n} \widehat{\varphi}_k^2(f_0)\lambda_k = +\infty.$$

Theorem 3.2 is completely proved. □

It has been proved above that in the general case the Fourier coefficients of the function from $V(0,1)$ do not satisfy condition (3.7). It should be mentioned, however, that any ONS $\{\varphi_n(x)\}$ contains the subsystem $\{\varphi_{n_k}(x)\}$ with respect to which the Fourier coefficients of any function from $V(0,1)$ satisfy condition (1.3). We will prove this theorem below.

Theorem 3.3. *From any ONS $\{\varphi_n(x)\}$ we can single out the subsystem with respect to which the Fourier coefficients of any function from $V(0,1)$ satisfy condition* (1.5).

Proof. Let $f(x) = 1$ for $x \in [0, \frac{i}{n}]$, and $f(x) = 0$ for $x \in (\frac{i}{n}, 1]$. Then due to Bessel's inequality

$$\sum_{n=1}^{\infty} \left(\int_0^{\frac{i}{n}} \varphi_n(x)\, dx \right)^2 \le \sum_{n=1}^{\infty} \left(\int_0^{1} f(x)\varphi_n(x)\, dx \right)^2 \le \int_0^{\frac{i}{n}} f^2(x)\, dx \le 1.$$

By $p(n)$ denote the natural number such that

$$\sum_{m=p(n)}^{\infty} \left(\int_0^{\frac{i}{n}} \varphi_m(x)\, dx \right)^2 \le 2^{-2n}, \quad i = 1, 2, \dots, n.$$

Hence

$$\left| \int_0^{\frac{i}{n}} \varphi_m(x)\, dx \right| < 2^{-n}, \quad i = 1, 2, \dots, n,$$

when $m \ge p(n)$.

Assume that $p(n+1) > p(n) + 2^n$, and put

$$\psi_{2^n+1}(x) = \varphi_{p(n)+1}(x), \qquad \psi_{2^n+s}(x) = \varphi_{p(n)+s}(x),$$

where $1 \le s \le 2^n$, $n = 0, 1, \dots$.

Thus, we have obtained the sequence of functions $\{\psi_m(x)\}$ which satisfy the conditions

$$\left|\int_0^{\frac{i}{n}} \psi_m(x)\,dx\right| = \left|\int_0^{\frac{i}{n}} \varphi_{p(n)+l}(x)\,dx\right| < 2^{-n},$$

where $m = 2^n + l$ $(1 \le l < 2^n)$.

The number $j(i)$ $(1 \le i \le n)$ is chosen so that $\left|\frac{i}{n} - \frac{j(i)}{2^m}\right| < 2^{-m}$. Then using the above inequality, we get $((a_n) \in \ell_2)$

$$\begin{aligned}
&\left|\int_0^{\frac{i}{n}} \sum_{s=2^m}^{2^{m+1}-1} a_s\lambda_s\psi_s(x)\,dx\right| \\
&\quad \le \left|\int_{\frac{i}{n}}^{\frac{j(i)}{2^m}} \sum_{s=2^m}^{2^{m+1}-1} a_s\lambda_s\psi_s(x)\,dx\right| + \left|\int_0^{\frac{j(i)}{2^m}} \sum_{s=2^m}^{2^{m+1}-1} a_s\lambda_s\psi_s(x)\,dx\right| \\
&\quad \le 2^{-\frac{m}{2}} \left(\sum_{s=2^m}^{2^{m+1}-1} a_s^2\lambda_s^2\right)^{\frac{1}{2}} + \sum_{s=2^m}^{2^{m+1}-1} |a_s|\lambda_s \cdot 2^{-m}. \qquad (3.15)
\end{aligned}$$

Assume now that $n = 2^d + n_1$ $(1 \le n_1 \le 2^d)$, then according to (3.15)

$$\begin{aligned}
&\left|\int_0^{\frac{i}{n}} \sum_{m=0}^{d-1}\sum_{s=2^m}^{2^{m+1}-1} a_s\lambda_s\psi_s(x)\,dx\right| \le \sum_{m=0}^{d-1}\sum_{s=2^m}^{2^{m+1}-1} |a_s|\lambda_s \cdot 2^{-m} \\
&\quad + \sum_{m=0}^{d-1} 2^{-\frac{m}{2}} \left(\sum_{s=2^m}^{2^{m+1}-1} a_s^2\lambda_s^2\right)^{\frac{1}{2}} \le \left(\sum_{m=0}^{d-1} 2^{-m}\lambda_{2^{m+1}}\right)^{\frac{1}{2}} \left(\sum_{m=0}^{d-1}\sum_{s=2^m}^{2^{m+1}-1} a_s^2\lambda_s\right)^{\frac{1}{2}} \\
&\quad + \left(\sum_{m=0}^{d-1} 2^{-m}\lambda_{2^{m+1}}\right)^{\frac{1}{2}} \left(\sum_{m=0}^{d-1}\sum_{s=2^m}^{2^{m+1}-1} a_s^2\lambda_s\right)^{\frac{1}{2}} = O(1)\left(\sum_{m=1}^{2^d-1} a_m^2\lambda_m\right)^{\frac{1}{2}}. \qquad (3.16)
\end{aligned}$$

Besides $(n = 2^d + n_1,\ n_1 < 2^d)$

$$\begin{aligned}
&\left|\int_0^{\frac{i}{n}} \sum_{s=2^d}^{n} a_s\lambda_s\psi_s(x)\,dx\right| \le \int_{\frac{i}{n}}^{\frac{j(i)}{2^d}} \left|\sum_{s=2^d}^{n} a_s\lambda_s\psi_s(x)\,dx\right| + \sum_{s=2^d}^{n} |a_s|\lambda_s 2^{-d} \\
&\quad \le 2^{-\frac{d}{2}}\sqrt{\lambda_n}\left(\sum_{s=2^d}^{n} a_s^2\lambda_s\right)^{\frac{1}{2}} + 2^{-\frac{d}{2}}\sqrt{\lambda_n}\left(\sum_{s=2^d}^{n} a_s^2\lambda_s\right)^{\frac{1}{2}} \\
&\quad = O(1)\left(\sum_{s=2^d}^{n} a_s^2\lambda_s\right)^{\frac{1}{2}}. \qquad (3.17)
\end{aligned}$$

From (3.16) and (3.17), we can conclude that

$$B_n(a) = \max_{1\le i\le n}\left|\int_0^{\frac{i}{n}}\sum_{m=1}^{n} a_m\lambda_m\psi_m(x)\,dx\right| = O(1)\left(\sum_{m=1}^{n} a_m^2\lambda_m\right)^{\frac{1}{2}}$$

for any sequence $(a_n)\in\ell_2$. From here and Theorem 3.1, it follows that for any function from $V(0,1)$ (1.5) is satisfied. □

4. Problems of efficiency

We will call condition (1.5) efficient if it is easily verified for classical ONS (the trigonometric system, the Walsh and Haar systems, etc. …).

Theorem 4.1. *Let* $\{\varphi_n(x)\}$ *be ONS such that* (*for all* $x\in[0,1]$)

$$\int_0^x \varphi_n(t)\,dt = O\left(\frac{1}{n}\right). \tag{4.1}$$

Then condition (1.5) *is satisfied.*

Proof. We have ($(a_n)\in\ell_2$)

$$\max_{1\le i\le n}\left|\int_0^{\frac{i}{n}}\sum_{k=1}^{n} a_k\lambda_k\varphi_k(x)\,dx\right| = \max_{1\le i\le n}\left|\sum_{k=1}^{n} a_k\lambda_k\int_0^{\frac{i}{n}}\varphi_k(x)\,dx\right|$$

$$= O(1)\sum_{k=1}^{n}\frac{|a_k|\lambda_k}{k} = O(1)\left(\sum_{k=1}^{n}\frac{\lambda_k}{k^2}\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n} a_k^2\lambda_k\right)^{\frac{1}{2}} = O(1)\left(\sum_{k=1}^{n} a_k^2\lambda_k\right)^{\frac{1}{2}}.$$

The trigonometric system and the Walsh system [5] satisfy condition (1.4), and hence, according to Theorem 4.1, they satisfy (1.5). □

Theorem 4.2. *If* $\{\chi_n(x)\}$ *is a Haar system* (*see* [6, p. 57]), *then condition* (1.5) *is fulfilled.*

Proof. Using the definition of Haar's system, we get ($i=1,2,\dots,n$, $(a_n)\in\ell_2$))

$$\left|\int_0^{\frac{i}{n}}\sum_{k=2^s}^{2^{s+1}-1} a_k\lambda_k\chi_k(x)\,dx\right| \le 2^{-\frac{s}{2}}|a_{k(i)}|\lambda_{k(i)},$$

where $2^s\le k(i)\le 2^{s+1}-1$.

Consequently, $(n = 2^p + l)$,

$$\max_{1\le i\le n}\left|\int_0^{\frac{i}{n}}\sum_{k=1}^{2^p-1} a_k\lambda_k\chi_k(x)\,dx\right| = \max_{1\le i\le n}\left|\sum_{k=0}^{p-1}\int_0^{\frac{i}{n}}\sum_{m=2^k}^{2^{k+1}-1} a_m\lambda_m\chi_m(x)\,dx\right|$$

$$\le \max_{1\le i\le n}\sum_{k=0}^{p-1} 2^{-\frac{k}{2}}|a_{k(i)}|\lambda_{k(i)} \le \max_{1\le i\le n}\left(\sum_{k=0}^{p-1} a_{k(i)}^2\lambda_{k(i)}\right)^{\frac{1}{2}}\left(\sum_{k=0}^{p-1} 2^{-k}\cdot\lambda_{k(i)}\right)^{\frac{1}{2}}$$

$$\le \left(\sum_{k=0}^{p-1}\sum_{m=2^k}^{2^{k+1}-1} a_m^2\lambda_m\right)^{\frac{1}{2}}\max_{1\le i\le n}\left(\sum_{k=0}^{p-1} 2^{-k}\cdot\lambda_{k(i)}\right)^{\frac{1}{2}} = O(1)\left(\sum_{k=1}^{2^p} a_k^2\lambda_k\right)^{\frac{1}{2}}.$$

In a similar way, it is proved that

$$\max_{1\le i\le n}\left|\int_0^{\frac{i}{n}}\sum_{k=2^p}^{n} a_k\lambda_k\chi_k(x)\,dx\right| = O(1)\left(\sum_{k=1}^{2^p} a_k^2\lambda_k\right)^{\frac{1}{2}}.$$

Hence Theorem 4.2 is completely proved. □

References

[1] G. Alexits, Convergence Problems of Orthogonal Series, Translated from the English by A. V. Efimov, Translation edited by P. L. Ul'janov, *Izdat. Inostran. Lit.*, *Moscow*, 1963 (in *Russian*).

[2] S. Banach, Sur la divergence des séries orthogonales, *Studia Math.* **9** (1940), 139–155.

[3] A. M. Olevskiĭ, Orthogonal series in terms of complete systems, *Mat. Sb. (N.S.)* **58 (100)** (1962), 707–748 (in *Russian*).

[4] L. Gogoladze and V. Tsagareishvili, Some classes of functions and Fourier coefficients with respect to general orthonormal systems. *Tr. Mat. Inst. Steklova.* **280** (2013), *Ortogonalnye Ryady, Teoriya Priblizhenii i Smezhnye Voprosy*, 162–174, (in *Russian*); translation in *Proc. Steklov Inst. Math.* **280** (2013), 156–168.

[5] N. J. Fine, On the Walsh functions, *Trans. Amer. Math. Soc.* **65** (1949), 372–414.

[6] S. Kačmaž and G. Šteĭngauz, Theory of Orthogonal Series, *Gosudarstv. Izdat. Fiz.-Mat. Lit.*, *Moscow*, 1958 (in *Russian*).

LARRY GOGOLADZE
I. JAVAKHISHVILI TBILISI STATE UNIVERSITY
DEPTARTMENT OF MATHEMATICS
13 UNIVERSITY ST.
TBILISI 0186
GEORGIA

E-mail: lgogoladze1@hotmail.com

VAKHTANG TSAGAREISHVILI
I. JAVAKHISHVILI TBILISI STATE UNIVERSITY
DEPTARTMENT OF MATHEMATICS
13 UNIVERSITY ST.
TBILISI 0186
GEORGIA

E-mail: cagare@ymail.com

(Received May 20, 2016; revised January 6, 2017)