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Parallel axiom and the 2-nd order differentiability of Busemann functions

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Abstract. Busemann has introduced a function to develop the theory of parallels in G-spaces. The function is called the Busemann function after him. Busemann functions have been used in Riemannian and Finslerian geometry at infinity. We study the relation of Parallel Axiom to the 2-nd order differentiability of Busemann functions.

1. Introduction

Parallel Axiom has been playing an important role in the development of geometry. After the discovery of non-Euclidean geometry, H. Busemann [8] proposed a new theory of parallels in G-spaces, and, nowadays, it is used in the study of many subjects in geometry, such as the geometry of rays, geodesic flows, geometry at infinity, etc.

Let $f^t : SM \to SM$ be a geodesic flow of a unit tangent bundle $\pi : SM \to M$. We have used the stable Jacobi vector fields along geodesics $\pi(f^t(v))$, $v \in SM$, in order to study some ergodic properties of the asymptotic behavior of orbits $\pi(f^t(U))$, $U \subset SM$. As a dual problem, we are interested in the asymptotic behavior of geodesic spheres $\pi(f^t(S_pM))$ with center $p \in M$ or the limit spheres. Since the distance spheres $S(p, R)$ with center p and radius R are the envelope of the distance spheres with radius $R - r$ and centers $q \in S(p, r)$, the distance

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defined on a Finsler manifold describes Huygens' principle. We study what happens on those spheres when the centers go to the infinity along a straight line. In the present paper we discuss a contribution of Parallel Axiom to the 2-nd order differentiability of limit spheres in a Finsler manifold.

Busemann introduced a function which has been called the Busemann function after him. Roughly speaking, the level sets of a Busemann function of a ray γ are limit sets of distance spheres with centers on γ. The distribution of points where a Busemann function is not differentiable is influenced by the topology and metric of the space (cf. [1], [2], [19], [20], [30], [32], [34]). In the study of geodesic flows, the second and higher order differentiability of Busemann functions are important and, therefore, have been studied (cf. [3], [4], [11], [12]). E. Hopf [17] ($n = 2$) and D. Burago–S. Ivanov [7] ($n \geq 3$) have proved that any Riemannian n-torus without conjugate points is flat. Although many results in Riemannian geometry can be generalized in Finsler geometry, the theorem of E. Hopf can not (see [8]). In fact, Busemann [8] and N. Zinoviev [36] have given Finsler metrics on tori without conjugate points which are not Minkowskian and flat. A condition on flatness of those metrics is given by [13]. Those geodesic flows are integrable. There is an important class of integrable geodesic flows which is not without conjugate points, i.e., the geodesic flows of surfaces of revolution. The behavior of geodesics in a 2-torus of revolution is determined (cf. [6], [10], [14], [25], [26]). We have studied the asymptotic behavior of distance spheres in [23] as a topological sub-mixing property.

Recall that any Randers metric $F = \alpha + \beta$ is pointwise projective to a Riemannian metric g, where α is the norm of the metric g, and β is any closed 1-form ([15], [31]). It means that all geodesics with respect to F and g are identified as point sets. This fact suggests that it is almost impossible to treat the rigidity problem in Finsler geometry as the geometry of geodesics without curvature condition. In fact, we easily obtain a class of non-Minkowskian Finsler manifolds satisfying Parallel Axiom (see Examples 2.1 and 2.2): Let (M, g_0) be the Euclid n-space, and f a smooth function on M. Let $F(x, y) = \sqrt{g_0(y, y)} + df(y)$ for $y \in T_xM$. It is known (cf. [31]) that F is a Finsler metric on M if $|df(y)| < 1$ for all $y \in T_xM$ with $\|y\|_{g_0} = 1$. Then (M, F) satisfies Parallel Axiom and all Busemann functions are smooth.

The distance spheres propagate according to Huygens' principle. A limit sphere S passing through a point p arises as their centers go to infinity along a ray from p. The limit sphere S is the envelope of limit spheres passing through points q on S of rays which are perpendicular to S at q. Those limit spheres are equal if and only if the asymptote relation is symmetric and transitive. This is not true for G-spaces without differentiability condition on the distance spheres ([8]). So far, the 2-nd order differentiability of Busemann functions are proved only when the asymptote relation is an equivalence relation with additional condition (cf. [12]).

Since the asymptotic behavior of trajectories of geodesic flows and the rigidity problem of Riemannian metrics have been focused on, we usually and conventionally have looked into the stable Jacobi tensor fields along a straight line, in studying the 2-nd order differentiability of a Busemann function. In this paper, we review and study it from the view point of distance geometry with Huygens' principle (see §3, 4), and use Taylor's theorem for the distance functions to the points along a straight line instead (see §5). Observing the behavior of geodesics in the universal covering space of a 2-torus of revolution (see §6), we propose a condition that the 2-nd order Taylor polynomials modified from the distance functions converge to that of a Busemann function as those points go to infinity. Since we can obtain some information of only the 1-st and 2-nd derivatives of distance functions in the intrinsic geometry, due to Huygens' principle, we may assume a certain boundedness of the 3-rd derivatives of distance functions (see §2). However, the assumption without anything else is not satisfactory for the convergence of Taylor polynomials. We will see that Parallel Axiom is closely connected to the existence and continuity of the 2-nd derivatives of a Busemann function. However, there exists a metric on a plane which does not satisfy Parallel Axiom but Busemann functions are of class C^2 (see §6 and Lemma 7.2). The main result of this paper is precisely stated in Theorem 2.3 after some definitions are provided.

The 2-nd order differentiability of a Busemann function was discussed in [22] and [24] for the plane convex billiards, and, however, their proofs are incomplete. They need a certain assumption that we adopt in this article.

2. Definitions, examples and statements

Let (M, F) be a *Finsler n-manifold* which is by definition a smooth n-manifold equipped with fundamental function $F : TM \to \mathbb{R}$ such that F is continuous on TM and smooth on $TM \setminus \{0\}$, $F(x, ty) = tF(x, y)$, for all $t > 0$ and $y \in T_xM$, and F is strictly convex on all tangent spaces T_xM. Here TM denotes the tangent bundle of M. We define, as usual, the length $L_F(c)$ of a piecewise smooth curve c in M with respect to F, and an intrinsic distance d on M

induced by F, i.e., $d(p,q)$ is the infimum of the lengths of all piecewise smooth curves from $p \in M$ to $q \in M$. The distance d is not symmetric, in general.

Let $p, q \in M$ with $p \neq q$. Let $T(p,q) : [0, d(p,q)] \to M$ denote a minimizing *geodesic going* from p to q with unit speed such that $T(p,q)(0) = p$ and $T(p,q)(d(p,q)) = q$. The minimizing *geodesic coming* from q to p is given by $T^{-}(q,p)(t) = T(q,p)(d(q,p)+t)$ for $t \in [-d(q,p), 0]$. This change of parametrization of $T(q,p)$ is that its domain is just translated from $[0, d(q,p)]$ to $[-d(q,p), 0]$, but it does not reverse the orientation. By definition, the length of $T(p,q)$ (resp., $T^{-}(q,p)$) equals $d(p,q)$ (resp., $d(q,p)$). Since the distance d may not be symmetric, $T(p,q) \neq T(q,p)$ as sets in general. Even if $d(p,q) = d(q,p)$, it may happen that $T(p,q) \neq T(q,p)$ as sets. Further, $T(p,q) = T(q,p)$ as a set, and $d(p,q) \neq d(q,p)$ may be true simultaneously.

In this paper, we say that (M,d) is *complete* if any Cauchy sequence of points p_j in M converges to a point in M. Here, a *Cauchy sequence* of points p_j by definition satisfies that for any $\varepsilon > 0$ there exists an integer j_0 such that $d(p_i, p_j) < \varepsilon$ for all $i, j > j_0$. We assume that (M,d) is complete. Then there exists a minimizing geodesic connecting any two points in M.

Let $\gamma : [0,\infty) \to M$ denote a *ray* which by definition satisfies $d(\gamma(0), \gamma(t)) = t$ for all $t \geq 0$. If (M,d) is non-compact, then there exists at least one ray from any point $p \in M$. A ray $\alpha : [0,\infty) \to M$ is called a *co-ray* to γ from $q = \alpha(0)$ if there exist sequences of points q_n converging to q and numbers t_n going to ∞ such that the sequence of minimizing geodesics $T(q_n, \gamma(t_n))$ converges to α.

Let $\beta : (-\infty, 0] \to M$ denote a *backward ray* which by definition satisfies $d(\beta(t), \beta(0)) = |t|$ for all $t \leq 0$. A backward ray $\alpha : (-\infty, 0] \to M$ is called a *backward co-ray* to β from $q = \alpha(0)$ if there exist sequences of points q_n converging to q and numbers t_n going to $-\infty$ such that the sequence of minimizing geodesics $T^{-}(\beta(t_n), q_n)$ converges to α.

Let $\gamma : (-\infty,\infty) \to M$ be a *straight line*, i.e., $d(\gamma(s), \gamma(t)) = t - s$ for all $-\infty < s < t < \infty$. We say that a straight line $\alpha : (-\infty,\infty) \to M$ is an *asymptote* to γ if $\alpha|_{[s,\infty)}$ is a co-ray to γ for all $s \in \mathbb{R}$. When α is an asymptote to γ, any sub-ray $\alpha|_{[s,\infty)}$ of α is the unique co-ray from $\alpha(s)$ to γ (cf. (22.19) Theorem, p. 136 of [8]). We say that a straight line $\alpha : (-\infty,\infty) \to M$ is a *parallel* to γ if $\alpha|_{[s,\infty)}$ is a co-ray to γ and $\alpha|_{(-\infty,s]}$ is a backward co-ray to γ for all $s \in \mathbb{R}$. If α is a parallel to γ, then α is the unique parallel to γ through any point $\alpha(s)$.

Let $\Omega(\gamma)$ (resp., $\Lambda(\gamma)$) be the set of all parallels (resp., asymptotes) to a straight line γ. We consider the following condition on $\Omega(\gamma)$ (resp., $\Lambda(\gamma)$) called Parallel (resp., Asymptote) Axiom:

(1) For any point $x \in M$ there passes $\alpha \in \Omega(\gamma)$ (resp., $\Lambda(\gamma)$) through x.

(2) If $\alpha \in \Omega(\gamma)$ (resp., $\Lambda(\gamma)$), then $\gamma \in \Omega(\alpha)$ (resp., $\Lambda(\gamma)$).

(3) If $\alpha \in \Omega(\gamma)$ (resp., $\Lambda(\gamma)$) and $\beta \in \Omega(\alpha)$ (resp., $\Lambda(\alpha)$), then $\beta \in \Omega(\gamma)$ (resp., $\Lambda(\gamma)$).

Since any sub-ray of an asymptote to γ is the unique co-ray to γ, it follows from (1) that an asymptote to γ through any point $x \in M$ is unique. In other words, $\Lambda(\gamma)$ and $\Omega(\gamma)$ simply cover M if they satisfy Asymptote Axiom and Parallel Axiom, respectively.

Let T^2 be a 2-torus of revolution which is not flat, and M its universal covering space which is homeomorphic to a plane. If γ is a lift of a minimal parallel circle in T^2 into M, then $\Omega(\gamma)$ is the set of all lifts of minimal parallel circles in T^2 and $\Lambda(\gamma)$ consists of all asymptotes through all points $x \in M$. Then, $\Omega(\gamma)$ satisfies (2) and (3) in Parallel Axiom, but $\Lambda(\gamma)$ does not satisfy them (cf. [1], [10], [25]). On the other hand, if α is a straight line such that $\alpha \notin \Lambda(\gamma)$ for any lift γ of any minimal parallel circle in T^2, then $\Omega(\alpha)$ satisfies Parallel Axiom (cf. same as above). In §6, we detail those Axioms for straight lines in M.

We can obtain Finsler manifolds satisfying Parallel Axiom, using a projective Randers change to the Euclid space.

Example 2.1. Let (M, G) be a Riemannian manifold. Hashiguchi and Ichijo ([15]) proved that any Randers metric $F = \alpha + \beta$ is pointwise projective to G, where α is the norm of G and β is any closed 1-form (cf. [31]). It means that all geodesics with respect to F and G are identified as point sets. Therefore, the asymptote and parallel relations are invariant under this change of a metric α.

As a special case, let (M, G_0) be the Euclid n-space, and f a smooth function on M. Let $F(x, y) = \sqrt{G_0(y, y)} + df(y)$ for $y \in T_xM$. It is known (cf. [31]) that F is a Finsler metric on M if $|df(y)| < 1$ for all $y \in T_xM$ with $\|y\|_{G_0} = 1$. The distance between two points p and q is given by $d_F(p, q) = \|q - p\|_{G_0} + f(q) - f(p)$ for all $p, q \in M$, since $d_F(p, q)$ is the infimum of all lengths of piecewise smooth curves from p to q. Since (M, F) is pointwise projective to (M, G_0), we see that (M, F) satisfies Parallel Axiom.

Using this example, we can obtain a Riemannian metric with conjugate points such that $\Omega(\gamma)$ satisfies Parallel Axiom for a certain straight line γ.

Example 2.2. Let γ be a straight line in (M, F) constructed in Example 2.1. Let $V(x)$ be a unit tangent vector at $x \in M$, i.e., $F(V(x)) = 1$, such that the straight line with initial tangent vector $V(x)$ is a parallel to γ. Obviously, V is a smooth vector field on M. Using the fundamental tensor g for F and the vector field V, we define a Riemannian metric $g_V(x) = g_{V(x)}(\cdot, \cdot)$ on $x \in M$. Here, we have, by definition,

$$g_{V(x)}(u,v)=\frac{1}{2}\frac{\partial^2}{\partial s\partial t}F(x,V(x)+su+tv)^2\Big|_{s=0,t=0},\qquad u,v\in T_xM.$$

Then all integral curves of V are geodesics with respect to g_V on M (cf. [29]). Thus the parallels to γ w.r.t. F are also parallels w.r.t. this Riemannian metric g_V in M. Therefore, $\Omega(\gamma)$ satisfies Parallel Axiom in (M, g_V). Other geodesics may not satisfy Parallel Axiom w.r.t this metric g_V and have a conjugate pair on it. In fact, if $f(x+z)=f(x)$ for all $x\in M$ and $z\in\mathbb{Z}^n$, then g_V is a lift of a Riemannian metric on a torus $M/\mathbb{Z}^n$ to M. The theorem of E. Hopf [17] and its higher dimensional case ([7], [11]) state that g_V is not without conjugate points.

In order to study asymptote and parallel relations, Busemann introduced a function called the Busemann function after him. A *(forward) Busemann function* $F_\gamma : M\to\mathbb{R}$ for a ray γ is defined by

$$F_\gamma(x)=\lim_{t\to\infty} d(x,\gamma(t))-t,\qquad x\in M.$$

We define a *backward Busemann function* $B_\beta : M\to\mathbb{R}$ for a backward ray β by

$$B_\beta(x)=\lim_{t\to-\infty} d(\beta(t),x)+t,\qquad x\in M.$$

Asymptote and Parallel Axioms reflect some properties of Busemann functions (see Lemma 3.4).

Let $d_p(\cdot)=d(p,\cdot)$, $d_p{}^-(\cdot)=d(\cdot,p)$, and let $f_p{}^\pm=\pm d_p{}^\pm$ for $p\in M$ (double-sign corresponds, the sign + often omitted). For a straight line $\gamma:(-\infty,\infty)\to M$, we define and use functions $f_{\gamma(t)}{}^-(x)=-d(x,\gamma(t))$ and $f_{\gamma(t)}(x)=d(\gamma(t),x)$ for all $x\in M$. In these notations, if γ is a straight line with $\gamma(0)=p$, then F_γ and B_γ are denoted by

$$F_\gamma(x)=\lim_{t\to\infty} -f_{\gamma(t)}{}^-(x)+f_{\gamma(t)}{}^-(p)\quad\text{and}\quad B_\beta(x)=\lim_{t\to-\infty} f_{\gamma(t)}(x)-f_{\gamma(t)}(p)$$

for $x\in M$.

For $p\in M$ and $v\in T_pM$, let γ_v denote a constant speed geodesic such that $\gamma_v(0)=p$ and $\dot\gamma_v(0)=v$. Since (M,d) is complete, γ_v is defined on $(-\infty,\infty)$. We define an *exponential map* $\exp_p : T_pM\to M$ by $\exp_p(v)=\gamma_v(1)$. Similarly, we define a *backward exponential map* $\exp_p{}^- : T_pM\to M$ by $\exp_p{}^-(v)=\gamma_{-v}(-1)$. Then, both $\exp_p$ and $\exp_p{}^-$ are smooth on $T_pM\smallsetminus\{0\}$ and of class C^1 at 0 (cf. [9]). Using the exponential maps, the distances between p and q are denoted by

$$f_p(q) = d(p,q) = \min\{F(p,v) \mid v \in {\exp_p}^{-1}(q)\},$$
$$-{f_p}^-(q) = d(q,p) = \min\{F(p,-v) \mid v \in ({\exp_p}^-)^{-1}(q)\}.$$

Let U_p be the interior of the set $\{v \in T_pM \setminus \{0\} \mid \exp_p(v) = q,\ d(p,q) = F(p,v)\}$, and ${U_p}^-$ the interior of $\{v \in T_pM \setminus \{0\} \mid {\exp_p}^-(v) = q,\ d(q,p) = F(p,-v)\}$. If $V_p = \exp_p(U_p)$ and ${V_p}^- = {\exp_p}^-({U_p}^-)$, then $\exp_p$ and ${\exp_p}^-$ are diffeomorphisms from U_p to V_p and from ${U_p}^-$ to ${V_p}^-$, respectively. In particular, f_p (resp., ${f_p}^-$) are smooth on V_p (resp., ${V_p}^-$). Furthermore, V_p and ${V_p}^-$ are open dense sets in M.

For our argument in this paper, we may assume that there exists a number $K > 0$ such that for any point $p \in M$ and $q \in {V_p}^{\mp}$ with $d(p,q) > 1$, the absolute values of 3-rd order partial derivatives of ${f_p}^{\mp}$ are bounded above by K for some local coordinate neighborhood around q. We mention the details of this assumption in the form we will use:

Let $(V; x^1, \dots, x^n)$ be a coordinate neighborhood around $q \in M$, and $B_q(r)$ a distance ball with center q and radius r. For a ray (or a backward ray) γ such that there exists a number $R > 0$ satisfying that ${f_{\gamma(t)}}^{\mp}$ is of class C^3 around q for any $t > R$, we define

$$D(\gamma,q)^{\mp} = \lim_{r\to 0} \liminf_{t\to\pm\infty} \sup\left\{\left|\frac{\partial^3 {f_{\gamma(t)}}^{\mp}}{\partial x^i \partial x^j \partial x^k}(x)\right| \mid x \in B_q(r),\ i,j,k = 1,\dots,n\right\}.$$

If $D(\gamma,q)^{\mp} < \infty$, then for any $\varepsilon > 0$, there exist a number $r_0 > 0$ and a sequence $|t_n|$ going to ∞ such that

$$\left|\frac{\partial^3 {f_{\gamma(t_n)}}^{\mp}}{\partial x^i \partial x^j \partial x^k}(x)\right| < D(\gamma,q)^{\mp} + \varepsilon \quad \text{for all } x \in B_q(r_0) \text{ and all } n.$$

Any distance function from or to p is smooth except for cut points of p. The 2-nd derivatives are bounded in any compact set if the centers p are far way, since the distance spheres satisfy Huygens' principle (see Lemma 5.3). For the 3-rd derivatives, we do not have any information from the distance geometry.

Let f be a function defined on an open set $U \subset \mathbb{R}^n$. We say that $g(y)$ is a *k-th order Taylor polynomial* at $x \in U$ of a function f if $f(x+y) - g(y) = o(\|y\|^k)$. Here $f(x+y) - g(y) = o(\|y\|^k)$ means that $|f(x+y) - g(y)|/\|y\|^k \to 0$ as $y \to 0$. From Lemma 7.1, f is of class C^2 in U if and only if f has a 2-nd order Taylor polynomial with continuous coefficients for $x \in U$.

In Lemma 5.1, we see that if $D(\gamma,\gamma(s))^{\mp}<\infty$, we then have the 2-nd order Taylor polynomials $P(x)$ and $Q(x)$ of F_γ and B_γ at any point $q=\gamma(s)$, i.e. $F_\gamma(x)-P(x)=o(\|x-q\|^2)$ and $B_\gamma(x)-Q(x)=o(\|x-q\|^2)$, respectively.

Theorem 2.3. *Let γ be a straight line in a complete non-compact Finsler manifold (M,F) such that $\Omega(\gamma)$ satisfies Parallel Axiom. If for any compact set N in M there exists a number K such that $D(\gamma,q)^{\mp}<K$ for any point $q\in N$, then the Busemann function F_γ is of class C^2.*

In order to understand the role of our assumption, we review in §6 what happens on the asymptotes $\Lambda(\gamma)$ and parallels $\Omega(\gamma)$ in the universal covering space M of a 2-torus T^2 of revolution, where γ is a lift of a minimal parallel circle in T^2 into M.

3. Co-rays and Busemann functions

Let (M,F) be a complete non-compact Finsler manifold. Let $\gamma:(-\infty,\infty)\to M$ be a straight line with $\gamma(0)=p$. Let $f_\gamma(x,t):=d(x,\gamma(t))-t=-f_{\gamma(t)}{}^-(x)+f_{\gamma(t)}{}^-(p)$ for $t\in(0,\infty]$, and $b_\gamma(x,s):=d(\gamma(s),x)+s=f_{\gamma(s)}(x)-f_{\gamma(s)}(p)$ for $s\in(-\infty,0)$. The following lemmas are well known for G-spaces (cf. [8]). The same proofs are valid for non-symmetric distances.

Lemma 3.1 (cf. [8, p. 131])**.** *The following are true:*

(1) *$f_\gamma(x,t)$ converges to $F_\gamma(x)$ uniformly on any compact set in M as $t\to\infty$.*

(2) *$b_\gamma(x,s)$ converges to $B_\gamma(x)$ uniformly on any compact set in M as $s\to-\infty$.*

(3) *$-d(y,x)\le F_\gamma(x)-F_\gamma(y)\le d(x,y)$ for all $x,y\in M$.*

(4) *$-d(x,y)\le B_\gamma(x)-B_\gamma(y)\le d(y,x)$ for all $x,y\in M$.*

Lemma 3.2 (cf. [8, (22.16) and (22.20)])**.** *The following are true:*

(1) *Let γ be a ray in M. Then, a unit speed curve $\alpha:[0,\infty)\to M$ is a co-ray to γ if and only if $F_\gamma(\alpha(t))=F_\gamma(\alpha(0))-t$ for all $t\in[0,\infty)$.*

(2) *Let γ be a backward ray in M. Then, a unit speed curve $\alpha:(-\infty,0]\to M$ is a backward co-ray to γ if and only if $B_\gamma(\alpha(t))=B_\gamma(\alpha(0))+t$ for all $t\in(-\infty,0]$.*

(3) *Let γ be a straight line in M. Then, a unit speed curve $\alpha:(-\infty,\infty)\to M$ is a parallel to γ if and only if $F_\gamma(\alpha(t))=F_\gamma(\alpha(s))-(t-s)$ for all $t>s$, and $B_\gamma(\alpha(t))=B_\gamma(\alpha(s))+(t-s)$ for all $t<s$. In particular, if a unit speed curve α is a parallel to γ, then $F_\gamma(\alpha(t))+B_\gamma(\alpha(t))$ is constant for all $t\in\mathbb{R}$.*

Lemma 3.3 (cf. [19]). *Let $\gamma : (-\infty, \infty) \to M$ be a straight line. If $\Lambda(\gamma)$ satisfies* (1) *of Asymptote Axiom, then the Busemann function F_γ is differentiable on M and its differential at $q \in M$ equal that of $-f_{\gamma_q(t)}{}^-$ for $t > 0$, i.e., $dF_\gamma = -df_{\gamma_q(t)}{}^-$, where $\gamma_q \in \Lambda(\gamma)$ with $\gamma_q(0) = q$.*

When we prove (2) of Lemma 4.2, we will use a similar computation for this lemma.

Lemma 3.4 (cf. [8]). *Let γ be a straight line such that $\Lambda(\gamma)$ satisfies Asymptote Axiom, and let $V(x)$ be the unit tangent vector of an asymptote through x to γ at all $x \in M$. Let $y \in T_xM$. Then the following are true:*

(1) *V is continuous on M.*

(2) *$F_\gamma - F_\alpha$ is constant $F_\gamma(\alpha(0))$ on M for any $\alpha \in \Lambda(\gamma)$.*

Furthermore, if $\Omega(\gamma)$ satisfies Parallel Axiom, then

(3) *$F_\gamma + B_\gamma$ is constant 0 on M.*

Proof. Assume that $\gamma : (-\infty, \infty) \to M$ is a straight line. Let $\beta : (-\infty, \infty) \to M$ be an asymptote to γ with $\beta(0) = x$. Hence, $\dot{\beta}(0) = V(x)$. From (1) of Asymptote and Parallel Axioms and (1), (2) in Lemma 3.2, if a sequence of asymptotes to γ converges a straight line β, then β is an asymptote to γ. Hence, $V(x)$ continuously depends on $x \in M$, proving (1).

Let $x \in M$ and $\gamma_x \in \Lambda(\gamma)$ with $\gamma_x(0) = x$. As was mentioned in Lemma 3.3, F_γ, F_α and B_γ are differentiable on M and $dF_\gamma = dF_\alpha = -dB_\gamma = -df_{\gamma_x(t)}{}^-$ at x. This proves (2) and (3). □

Since the level sets of a Busemann function are called the *limit spheres* centered at $\gamma(t)$ as $|t| \to \infty$, the items (2) and (3) of Lemma 3.4 play important roles in the argument for the 2-nd order differentiability.

4. The 2-nd order differentials of distance functions

Let γ be a straight line in a complete non-compact Finsler manifold (M, F). For $y \in T_pM$, let $c(u)$, $u \in (-\varepsilon, \varepsilon)$, be a curve such that $c(0) = p$ and $c'(0) = y$, where c' implies the differentiation of c by its parameter. Let $f_{\gamma(s)}{}^{\pm}(u) = \pm d_{\gamma(s)}{}^{\pm}(c(u))$ for $u \in (-\varepsilon, \varepsilon)$ (double-sign corresponds).

Lemma 4.1. *Let $\gamma : (-\infty, \infty) \to M$ be a straight line such that $\gamma(0) = p$. Let a and s be numbers such that $0 < a < s$. Then, $f_{\gamma(a)}{}^{-\prime\prime}(0) \leq f_{\gamma(s)}{}^{-\prime\prime}(0)$.*

Proof. From Lemma 3.3, we have ${f_{\gamma(a)}}^{-\prime}(0) = {f_{\gamma(s)}}^{-\prime}(0)$. We use the notation above. Since $d(c(u),\gamma(s)) \leq d(c(u),\gamma(a)) + d(\gamma(a),\gamma(s))$, we have

$$\begin{aligned} {f_{\gamma(s)}}^{-}(u) - {f_{\gamma(s)}}^{-}(0) &= -d(c(u),\gamma(s)) + s \\ &\geq -d(c(u),\gamma(a)) + a = {f_{\gamma(a)}}^{-}(u) - {f_{\gamma(a)}}^{-}(0). \end{aligned}$$

Therefore,

$$\begin{aligned} {f_{\gamma(s)}}^{-\prime\prime}(0) &= 2\lim_{u\to 0} \frac{{f_{\gamma(s)}}^{-}(u) - {f_{\gamma(s)}}^{-}(0) - {f_{\gamma(s)}}^{-\prime}(0)u}{u^2} \\ &\geq 2\lim_{u\to 0} \frac{{f_{\gamma(a)}}^{-}(u) - {f_{\gamma(a)}}^{-}(0) - {f_{\gamma(a)}}^{-\prime}(0)u}{u^2} = {f_{\gamma(a)}}^{-\prime\prime}(0). \qquad \square \end{aligned}$$

Let $\gamma : (-\infty,\infty) \to M$ be a straight line with $\gamma(0) = p$. Recall that $f_\gamma(x,t) = -{f_{\gamma(t)}}^{-}(x) + {f_{\gamma(t)}}^{-}(p)$ for $t \in (0,\infty]$, and $b_\gamma(x,s) = f_{\gamma(s)}(x) - f_{\gamma(s)}(p)$ for $s \in (-\infty,0)$.

Lemma 4.2. *Let $\gamma : (-\infty,\infty) \to M$ be a straight line with $\gamma(0) = p$. Let $s,t \in (-\infty,\infty)$ satisfy $s < 0 < t$. Then the following are true:*

(1) *$b_\gamma(x,s) + f_\gamma(x,t) \geq 0$, and equality holds if and only if $x = \gamma(u)$ for some $u \in [s,t]$.*

(2) *${f_{\gamma(s)}}''(0) \geq {f_{\gamma(t)}}^{-\prime\prime}(0)$.*

Proof. Let $s < 0 < t$. Since

$$t - s = d(\gamma(s),\gamma(t)) \leq d(\gamma(s),x) + d(x,\gamma(t)),$$

we have

$$\begin{aligned} b_\gamma(x,s) + f_\gamma(x,t) &= (d(\gamma(s),x) + s) + (d(x,\gamma(t)) - t) \\ &\geq d(\gamma(s),\gamma(t)) - (t - s) = 0. \end{aligned}$$

This proves (1).

Let $y \in T_pM$, and let $c(u)$ be a curve such that $c(0) = p$ and $c'(0) = y$. Let $f(u) = b_\gamma(c(u),s) + f_\gamma(c(u),t)$. Differentiate $f(u)$ at $u = 0$, and we have

$$f'(0) = {f_{\gamma(s)}}'(0) - {f_{\gamma(t)}}^{-\prime}(0) = 0.$$

Furthermore, we have

$$f''(0) = {f_{\gamma(s)}}''(0) - {f_{\gamma(t)}}^{-\prime\prime}(0) \geq 0, \tag{4.1}$$

because $f(0) = 0$ is a minimum of f on M, proving (2). $\square$

5. The 2-nd order differentiability of a Busemann function

Let γ be a straight line in a complete non-compact Finsler manifold (M,F), and let $\gamma_x \in \Lambda(\gamma)$ pass through $x \in M$ with $\gamma_x(0)=x$. Let $(V,x_1,\dots,x_n)$ be a local coordinate system around x which is compatible with the differentiable structure of M. Let $y=\sum_{i=1}^n y^i\partial/\partial x^i \in T_xM$. It follows from Lemmas 4.1 and 4.2 that

$$\sum_{i,j=1}^n \frac{\partial^2 {f_{\gamma_x(b)}}^-}{\partial x^i\partial x^j}(x)y^iy^j \le \sum_{i,j=1}^n \frac{\partial^2 {f_{\gamma_x(t)}}^-}{\partial x^i\partial x^j}(x)y^iy^j \le \sum_{i,j=1}^n \frac{\partial^2 {f_{\gamma_x(a)}}^-}{\partial x^i\partial x^j}(x)y^iy^j$$

for any $a<0<b<t$. Hence, we have

$$\left(\frac{\partial^2 {f_{\gamma_x(b)}}^-}{\partial x^i\partial x^j}(x)\right) \le \left(\frac{\partial^2 {f_{\gamma_x(t)}}^-}{\partial x^i\partial x^j}(x)\right) \le \left(\frac{\partial^2 {f_{\gamma_x(a)}}^-}{\partial x^i\partial x^j}(x)\right), \tag{5.1}$$

where $(\cdot)$ is a symmetric matrix whose (i,j)-entries are the second partial derivatives by the i-th and j-th coordinates. Since the symmetric matrices in the middle are monotone non-decreasing, there exists the limit

$$-H_x = \lim_{t\to\infty}\left(\frac{\partial^2 {f_{\gamma_x(t)}}^-}{\partial x^i\partial x^j}(x)\right), \quad -H_{x,ij} = \lim_{t\to\infty}\frac{\partial^2 {f_{\gamma_x(t)}}^-}{\partial x^i\partial x^j}(x). \tag{5.2}$$

Lemma 5.1. *In the notation above, if $D(\gamma_x,x)^{\mp}<\infty$, we then have the 2-nd order Taylor polynomial of F_{γ_x} at x:*

$$F_{\gamma_x}(x+y) = F_{\gamma_x}(x) + \sum_{i=1}^n y^iD_{x,i}(x) + \sum_{i,j=1}^n y^iy^jH_{x,ij} + \frac{1}{2}\sum_{i,j,k=1}^n y^iy^jy^kR_{x,ijk}(y), \tag{5.3}$$

where

$$D_{x,i} = -\frac{\partial {f_{\gamma_x(t)}}^-}{\partial x^i}(x), \quad R_{x,ijk}(y) = -\lim_{t\to\infty}\int_0^1 (1-u)^2\frac{\partial^3 {f_{\gamma_x(t)}}^-}{\partial x^i\partial x^j\partial x^k}(c(u))\,du.$$

Proof. Applying Taylor's theorem to $-{f_{\gamma_x(t)}}^- - t$ at x, we have

$$-{f_{\gamma_x(t)}}^-(x+y) - t = -{f_{\gamma_x(t)}}^-(x) - t - \sum_{i=1}^n y^i\frac{\partial {f_{\gamma_x(t)}}^-}{\partial x^i}(x) - \sum_{i,j=1}^n y^iy^j\frac{\partial^2 {f_{\gamma_x(t)}}^-}{\partial x^i\partial x^j}(x) - \frac{1}{2}\sum_{i,j,k=1}^n y^iy^jy^k\int_0^1 (1-u)^2\frac{\partial^3 {f_{\gamma_x(t)}}^-}{\partial x^i\partial x^j\partial x^k}(c(u))\,du.$$

Because of $D_{x,i}$ is constant for t and the definitions of H_x and R_x, we have equation (5.3) as $t \to \infty$. Here, $R_{x,ijk}(y) = O(1)$, meaning that $|R_{x,ijk}(y)| \leq C$ as $y \to 0$ for some $C \geq 0$. □

We will see a relation of H_x to the second order differential of F_γ at x.

Lemma 5.2. *Let N be a compact set in M. Let $\alpha_{x,t} : (-\infty, \infty) \to M$ be a geodesic which is the extension of $T(x, \gamma(t))$ for every $t > 0$. Then, the sequence of points $\alpha_{x,t}(b)$ uniformly converges to $\gamma_x(b)$ on $x \in N$ as $t \to \infty$ for any constant $b \in \mathbb{R}$.*

Proof. Suppose for indirect proof that there exist $\varepsilon > 0$ and a sequence of points $x_j \in N$ and a sequence t_j going to ∞ such that $d(\gamma_{x_j}(b), \alpha_{x_j,t_j}(b)) > \varepsilon$. Since N is compact, there exists a subsequence x_i of x_j converging to a point $x \in N$. Then, γ_{x_i} and α_{x_i,t_i} converges to γ_x, because γ_x is the unique asymptote to γ passing through x, meaning that $d(\gamma_{x_i}(b), \alpha_{x_i,t_i}(b)) \to 0$ as $t_i \to \infty$, a contradiction. □

Lemma 5.3. *Let γ be a straight line in (M, F) such that $\Lambda(\gamma)$ satisfies Asymptote Axiom. Then, the absolute values of the second derivatives of ${f_{\gamma(t)}}^{\pm}$ are bounded in all compact set N in M as $t \to \pm\infty$.*

Proof. From Lemma 5.2, we can set

$$A = \left(\frac{\partial^2 {f_{\gamma_x(b)}}^-}{\partial x^i \partial x^j}(x) \right) = \lim_{t\to\infty} \left(\frac{\partial^2 {f_{\alpha_{x,t}(b)}}^-}{\partial x^i \partial x^j}(x) \right),$$
$$B = \left(\frac{\partial^2 {f_{\gamma_x(a)}}^-}{\partial x^i \partial x^j}(x) \right) = \lim_{t\to\infty} \left(\frac{\partial^2 {f_{\alpha_{x,t}(a)}}^-}{\partial x^i \partial x^j}(x) \right).$$

Further, we note that inequality (5.1) is true for a minimizing geodesic $\beta : (c, d) \to M$ with $c < a < 0 < b < t < d$ even if its extension is not minimizing in a whole real line. Applying these facts to minimizing geodesics containing $T(x, \gamma(t))$, we can find, from (5.1), a number $t_1 > 0$ such that

$$A < \left(\frac{\partial^2 {f_{\gamma(t)}}^-}{\partial x^i \partial x^j}(x) \right) < B,$$

for all $x \in N$ and all $t > t_1$. □

Set $-H_{{f_{\gamma(t)}}^-}(x) = \left(\frac{\partial^2 {f_{\gamma(t)}}^-}{\partial x^i \partial x^j}(x) \right)$.

Lemma 5.4. *Let γ be a straight line in a complete Finsler manifold (M, F) such that $\Lambda(\gamma)$ satisfies Asymptote Axiom. If for any compact set N in M there exists a number K such that $D(\gamma, q)^{\mp} < K$ for all points $q \in N$, then $H_{{f_{\gamma(t)}}^-}(x)$ converges to H_x for all $x \in M$ as $t \to \infty$.*

PROOF. Let $(U; x^1, \dots, x^n)$ be a coordinate neighborhood around q such that $q = (q^1, \dots, q^n)$ and let $c(u) = (q^1 + uy^1, \dots, q^n + uy^n)$ denote a curve in U. We then have

$$f_{\gamma(t)}{}^-(q+y) - f_{\gamma(t)}{}^-(q) = \sum_{i=1}^n y^i \frac{\partial f_{\gamma(t)}{}^-}{\partial x^i}(q) + \sum_{i,j=1}^n y^i y^j \frac{\partial^2 f_{\gamma(t)}{}^-}{\partial x^i \partial x^j}(q) + \frac{1}{2} \sum_{i,j,k=1}^n y^i y^j y^k \int_0^1 (1-u)^2 \frac{\partial^3 f_{\gamma(t)}{}^-}{\partial x^i \partial x^j \partial x^k}(c(u))\, du, \tag{5.4}$$

and hence,

$$\begin{aligned}
&(f_{\gamma(t)}{}^-(q+y) - f_{\gamma_q(t)}{}^-(q+y)) - (f_{\gamma(t)}{}^-(q) - f_{\gamma_q(t)}{}^-(q)) \\
&= \sum_{i=1}^n y^i \left(\frac{\partial f_{\gamma(t)}{}^-}{\partial x^i}(q) - \frac{\partial f_{\gamma_q(t)}{}^-}{\partial x^i}(q) \right) + \sum_{i,j=1}^n y^i y^j \left(\frac{\partial^2 f_{\gamma(t)}{}^-}{\partial x^i \partial x^j}(q) - \frac{\partial^2 f_{\gamma_q(t)}{}^-}{\partial x^i \partial x^j}(q) \right) \\
&\quad + \frac{1}{2} \sum_{i,j,k=1}^n y^i y^j y^k \int_0^1 (1-u)^2 \times \left(\frac{\partial^3 f_{\gamma(t)}{}^-}{\partial x^i \partial x^j \partial x^k}(c(u)) - \frac{\partial^3 f_{\gamma_q(t)}{}^-}{\partial x^i \partial x^j \partial x^k}(c(u)) \right) du.
\end{aligned} \tag{5.5}$$

From (2) of Lemma 3.4,

$$\begin{aligned}
&\lim_{t\to\infty} (f_{\gamma(t)}{}^-(q+y) - f_{\gamma_q(t)}{}^-(q+y)) - (f_{\gamma(t)}{}^-(q) - f_{\gamma_q(t)}{}^-(q)) \\
&\quad = (-F_\gamma(q+y) + F_{\gamma_q}(q+y)) + (F_\gamma(q) - F_{\gamma_q}(q)) = 0.
\end{aligned} \tag{5.6}$$

Therefore, from Lemmas 3.4 and 5.3, and since the absolute 2-nd and 3-rd derivatives of $f_{\gamma(t)}{}^-$ are bounded above by K, we have

$$\lim_{t\to\infty} \sum_{i=1}^n y^i \left(\frac{\partial f_{\gamma(t)}{}^-}{\partial x^i}(q) - \frac{\partial f_{\gamma_q(t)}{}^-}{\partial x^i}(q) \right) = 0, \tag{5.7}$$

$$\lim_{t\to\infty} \sum_{i,j=1}^n y^i y^j \left(\frac{\partial^2 f_{\gamma(t)}{}^-}{\partial x^i \partial x^j}(q) - \frac{\partial^2 f_{\gamma_q(t)}{}^-}{\partial x^i \partial x^j}(q) \right) = 0, \tag{5.8}$$

$$\begin{aligned}
&\lim_{t\to\infty} \sum_{i,j,k=1}^n y^i y^j y^k \int_0^1 (1-u)^2 \\
&\quad \times \left(\frac{\partial^3 f_{\gamma(t)}{}^-}{\partial x^i \partial x^j \partial x^k}(c(u)) - \frac{\partial^3 f_{\gamma_q(t)}{}^-}{\partial x^i \partial x^j \partial x^k}(c(u)) \right) du = 0.
\end{aligned} \tag{5.9}$$

Equation (5.7) implies that $df_{\gamma(t)}{}^-(q)$ converges to $df_{\gamma_q}{}^-(q)$ as $t \to \infty$. Equation (5.8) implies that $H_{f_{\gamma(t)}{}^-}(q)$ converges to H_q as $t \to \infty$ because of (5.2). $\square$

From the limit situation of equation (5.4) as $t \to \infty$, we can have the 2-nd order Taylor polynomial of the Busemann function F_γ at q. We proceed to prove that they have continuous coefficients for q. In the argument, we need to assume Parallel Axiom.

Lemma 5.5. *Let γ be a straight line in (M, F) such that $\Omega(\gamma)$ satisfies Parallel Axiom. If for any compact set N in M there exists a number K such that $D(\gamma, q)^{\mp} < K$ at all points $q \in N$, then $H_x = H_x{}^-$ for all $x \in M$.*

Proof. Let $(U; x^1, \dots, x^n)$ be a coordinate neighborhood around q such that $q = (q^1, \dots, q^n)$, and let $c(u) = (q^1 + uy^1, \dots, q^n + uy^n)$ denote a curve in U. We then have

$$f_{\gamma_q(t)}{}^-(q+y) - f_{\gamma_q(t)}{}^-(q) = \sum_{i=1}^n y^i \frac{\partial f_{\gamma_q(t)}{}^-}{\partial x^i}(q) + \sum_{i,j=1}^n y^i y^j \frac{\partial^2 f_{\gamma_q(t)}{}^-}{\partial x^i \partial x^j}(q) + \frac{1}{2} \sum_{i,j,k=1}^n y^i y^j y^k \int_0^1 (1-u)^2 \frac{\partial^3 f_{\gamma_q(t)}{}^-}{\partial x^i \partial x^j \partial x^k}(c(u))\, du, \tag{5.10}$$

and hence, for $t > 0$,

$$\begin{aligned}&(-f_{\gamma_q(t)}{}^-(q+y) + f_{\gamma_q(-t)}(q+y)) - (-f_{\gamma_q(t)}{}^-(q) + f_{\gamma_q(-t)}(q)) \\ &= \sum_{i=1}^n y^i \left(-\frac{\partial f_{\gamma_q(t)}{}^-}{\partial x^i}(q) + \frac{\partial f_{\gamma_q(-t)}}{\partial x^i}(q)\right) + \sum_{i,j=1}^n y^i y^j \left(-\frac{\partial^2 f_{\gamma_q(t)}{}^-}{\partial x^i \partial x^j}(q) + \frac{\partial^2 f_{\gamma_q(-t)}}{\partial x^i \partial x^j}(q)\right) \\ &\quad + \frac{1}{2} \sum_{i,j,k=1}^n y^i y^j y^k \int_0^1 (1-u)^2 \times \left(-\frac{\partial^3 f_{\gamma_q(t)}{}^-}{\partial x^i \partial x^j \partial x^k}(c(u)) + \frac{\partial^3 f_{\gamma_q(-t)}}{\partial x^i \partial x^j \partial x^k}(c(u))\right) du.\end{aligned} \tag{5.11}$$

From (3) of Lemma 3.4,

$$\begin{aligned}&\lim_{t\to\infty} (-f_{\gamma_q(t)}{}^-(q+y) + f_{\gamma_q(-t)}(q+y)) - (-f_{\gamma_q(t)}{}^-(q) + f_{\gamma_q(-t)}(q)) \\ &= (F_{\gamma_q}(q+y) + B_{\gamma_q}(q+y)) - (F_{\gamma_q}(q) + B_{\gamma_q}(q)) = 0.\end{aligned} \tag{5.12}$$

Therefore, we have

$$\lim_{t\to\infty} \sum_{i=1}^n y^i \left(\frac{\partial f_{\gamma_q(t)}{}^-}{\partial x^i}(q) - \frac{\partial f_{\gamma_q(-t)}}{\partial x^i}(q)\right) = 0, \tag{5.13}$$

$$\lim_{t\to\infty} \sum_{i,j=1}^n y^i y^j \left(\frac{\partial^2 f_{\gamma_q(t)}{}^-}{\partial x^i \partial x^j}(q) - \frac{\partial^2 f_{\gamma_q(-t)}}{\partial x^i \partial x^j}(q)\right) = 0, \tag{5.14}$$

$$\lim_{t\to\infty}\sum_{i,j,k=1}^{n} y^i y^j y^k \int_0^1 (1-u)^2 \times\left(\frac{\partial^3 f_{\gamma_q(t)}{}^-}{\partial x^i\partial x^j\partial x^k}(c(u)) - \frac{\partial^3 f_{\gamma_q(-t)}}{\partial x^i\partial x^j\partial x^k}(c(u))\right) du = 0. \tag{5.15}$$

Equation (5.13) implies that $df_{\gamma_q(t)}{}^-(q) = df_{\gamma_q(-t)}(q)$. Equation (5.14) implies that $H_q{}^- = H_q$, because of (5.2). □

Proof of Theorem 2.3. Let N be a compact set in M and $\varepsilon > 0$. Let $q \in N$. From the definition of H_q, $H_q{}^-$ and Lemma 5.5, there exists a number $t_0 > 0$ such that

$$\|H_{f_{\gamma_q(t_0)}{}^-}(q) - H_q(q)\| < \varepsilon/9, \qquad \|H_{f_{\gamma_q(-t_0)}}(q) - H_q{}^-(q)\| < \varepsilon/9,$$
$$\|H_q - H_q{}^-\| < \varepsilon/9.$$

Hence, $\|H_{f_{\gamma_q(t_0)}{}^-}(q) - H_{f_{\gamma_q(-t_0)}}(q)\| < \varepsilon/3$. Set $p_0 = \gamma_q(-t_0)$ and $p_1 = \gamma_q(t_0)$. There exist neighborhoods $U_q{}'$ of q, U_{p_0} of p_0 and U_{p_1} of p_1 such that

$$\|H_{f_{y_1}{}^-}(x) - H_{f_{y_0}}(x)\| < \varepsilon/3, \tag{5.16}$$
$$\|H_{f_{\gamma_x(t_0)}{}^-}(x) - H_{f_{y_1}{}^-}(x)\| < \varepsilon/3, \tag{5.17}$$
$$\|H_{f_{\gamma_x(-t_0)}}(x) - H_{f_{y_0}}(x)\| < \varepsilon/3, \tag{5.18}$$

for any $x \in U_q{}'$, $y_0 \in U_{p_0}$ and $y_1 \in U_{p_1}$. Moreover, we can find a number $t(q) > 0$ and a neighborhood $U_q \subset U_q{}'$ such that the maximal minimizing geodesic containing $T(x, \gamma(t))$ intersects both U_{p_0} and U_{p_1} for any $x \in U_q$ and any number $t > t(q)$, because $T(x, \gamma(t))$ converges to a sub-ray of γ_x as $t \to \infty$. We may assume that $\gamma_x(-t_0) \in U_{p_0}$ and $\gamma_x(t_0) \in U_{p_1}$ for all $x \in U_q$.

Let $x \in U_q$ be given, and let $\alpha_{x,t} : (-\infty, \infty) \to M$ be a unit speed geodesic satisfying $\alpha_{x,t}(0) = x$ and $\alpha_{x,t}(d(x, \gamma(t))) = \gamma(t)$ for $t > t(q)$. Then $\alpha_{x,t}$ converges to the unique parallel γ_x as $t \to \infty$. We may assume that $y_0(t) = \alpha_{x,t}(-t_0) \in U_{p_0}$ and $y_1(t) = \alpha_{x,t}(t_0) \in U_{p_1}$. In the setting above, from Lemmas 4.1 and 4.2, we have

$$\|H_{f_{\gamma_x(t_0)}{}^-}(x) - H_{f_{\gamma_x(-t_0)}}(x)\| < \varepsilon/3, \tag{5.19}$$
$$H_{f_{\gamma_x(t_0)}{}^-}(x) < H_x = H_x{}^- < H_{f_{\gamma_x(-t_0)}}(x), \tag{5.20}$$
$$H_{f_{y_1(t)}{}^-}(x) < H_{f_{\gamma(t)}{}^-}(x) < H_{f_{y_0(t)}}(x). \tag{5.21}$$

From (5.20) and (5.21),

$$H_{f_{y_1(t)}{}^-}(x) - H_{f_{\gamma_x(-t_0)}}(x) < H_{f_{\gamma(t)}{}^-}(x) - H_x < H_{f_{y_0(t)}}(x) - H_{f_{\gamma_x(t_0)}{}^-}(x).$$

The norm of the right hand side is evaluated by (5.18) and (5.19):

$$\begin{aligned}&\|H_{f_{y_0(t)}}(x) - H_{f_{\gamma_x(t_0)^-}}(x)\| \\ &\quad < \|H_{f_{y_0(t)}}(x) - H_{f_{\gamma_x(-t_0)}}(x)\| + \|H_{f_{\gamma_x(-t_0)}}(x) - H_{f_{\gamma_x(t_0)^-}}(x)\| < \varepsilon,\end{aligned}$$

and, similarly, by (5.17) and (5.19):

$$\begin{aligned}&\|H_{f_{\gamma_x(-t_0)}}(x) - H_{f_{y_1(t)^-}}(x)\| \\ &\quad < \|H_{f_{\gamma_x(-t_0)}}(x) - H_{f_{\gamma_x(t_0)^-}}(x)\| + \|H_{f_{\gamma_x(t_0)^-}}(x) - H_{f_{y_1(t)^-}}(x)\| < \varepsilon.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\|H_{f_{\gamma(t)^-}}(x) - H_x\| \\ &\quad < \max\{\|H_{f_{y_0(t)}}(x) - H_{f_{\gamma_x(t_0)^-}}(x)\|, \|H_{f_{\gamma_x(-t_0)}}(x) - H_{f_{y_1(t)^-}}(x)\|\} < \varepsilon.\end{aligned}$$

This implies that $H_{f_{\gamma(t)^-}}(x)$ uniformly converges to H_x in $x \in U_q$. Therefore, H_x is continuous. Moreover, F_γ has the 2-nd order Taylor polynomial with continuous coefficients. Applying Lemma 7.1, we conclude that F_γ is of class C^2. □

6. Parallel Axiom for the covering space of a torus of revolution

In this section, we review the property of straight lines in the universal covering space (M, ds^2) of a Riemannian 2-torus of revolution. The same results shown in this section are true (see [25]) even if M is a Finsler 2-torus of revolution but not Riemannian. However, we deal with a Riemannian case, since the purpose of this section is to see what happens to Asymptote and Parallel Axiom. Riemannian geometry is more familiar to us than Finsler geometry.

Let $M = \mathbb{R}^2$ and $g : ds^2 = f(y)^2 dx^2 + dy^2$, where $f(y) = f(x, y) > 0$ and $f(y + k) = f(y)$ for all $y \in \mathbb{R}$ and integers $k \in \mathbb{Z}$. Let $\tau_s(x, y) = (x + s, y)$ and $\psi(x, y) = (x, y + 1)$ for all $(x, y) \in \mathbb{R}^2$ and any $s \in \mathbb{R}$. Then, all τ_s and ψ^n are isometries on (M, g). If Φ is a group of isometries generated by τ_1 and ψ, then $T^2 = M/\Phi$ is a Riemannian 2-torus of revolution. In order to classify the geodesics in a surface of revolution, Clairaut's relation is most useful.

In the following lemma, we do not have to assume that $f(y + k) = f(y)$ for all $k \in \mathbb{Z}$.

Lemma 6.1 (Clairaut, cf. [33, p. 213])**.** *Let $\gamma(t) = (x(t), y(t))$, $t \in (-\infty, \infty)$, be a unit speed geodesic in (M, g). Then, $g(\dot\gamma(t), \partial/\partial x)$ is constant for $t \in (-\infty, \infty)$. If we denote the angle of $\dot\gamma(t)$ with $\partial/\partial x$ by $\varphi(t)$, then $f(y(t)) \cos\varphi(t)$ is constant for $t \in (-\infty, \infty)$.*

Proof. Here we show this lemma, using a property of a Jacobi vector field. Since τ_s is a 1-parameter group of isometries, $\partial/\partial x$ is a Jacobi vector field along $\gamma(t)$. Hence, there exist constants A, B such that $g(\dot\gamma(t), \partial/\partial x) = At + B$ for all $t \in (-\infty, \infty)$. The values in the left hand side around $t = t_0$ are calculated locally for any $t_0 \in (-\infty, \infty)$, in other words, in a thin strip S bounded by two trajectories of τ_s containing $\gamma(t_0)$. The constants A, B are determined there. Let g_1 be a Riemannian metric on $\mathbb{R}^2$ such that a bounded function $f_1(y)$ on $\mathbb{R}$ with $f_1(y) = f(y)$ for $(x, y) \in S$ replaces $f(y)$. Then, for a geodesic $\gamma_1(t)$ which is identified with $\gamma(t)$ around $t = t_0$, we have the same equation $g_1(\dot\gamma_1(t), \partial/\partial x) = At+B$ for all $t \in (-\infty, \infty)$. Since the left hand side is bounded, we have $A = 0$. □

The behavior of geodesics in a 2-torus of revolution is determined (cf. [6], [10], [14], [25], [26]). We mention the classification of straight lines in M from the viewpoint of asymptote and parallel relations. In the following argument, we use Lemma 6.1 without saying anything.

If $p_0 = (x_0, y_0)$ is a point such that $f(y_0) = \min f$, then all geodesics through p_0 defined on $\mathbb{R}$ are straight lines. Let $\gamma(t) = (x(t), y(t))$, $t \in \mathbb{R}$, be a straight line with $\gamma(0) = p_0$ such that $y(t)$ is not constant for $t \in \mathbb{R}$. Then, $|x(t)|, |y(t)| \to \infty$ as $|t| \to \infty$. If $\gamma_s = \tau_s \circ \gamma$ for all $s \in \mathbb{R}$, then all γ_s are parallel to each other and simply covers M. Namely, $\Omega(\gamma) = \{\gamma_s \,|\, s \in \mathbb{R}\}$ satisfies Parallel Axiom.

We consider the case where $y(t) = y_0$ for all $t \in \mathbb{R}$. Let $\alpha_p(t) = \tau_t(p)$ for all $t \in \mathbb{R}$. The speed of α_p is not 1, in general, but constant. As was mentioned above, for $p = (x, y)$, α_p is a straight line if $f(y) = \min f$. Next, for any point $q = (x_3, y_3)$ with $f(y_3) \neq \min f$, let $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ be points such that $y_1 < y_3 < y_2$, $f(y_1) = f(y_2) = \min f$ and $\mathbb{R} \times (y_1, y_2)$ contains no minimum point of f. Then there exist two straight lines ${\gamma_q}^u$ and ${\gamma_q}^l$ such that they pass through $q = {\gamma_q}^u(0)$ and ${\gamma_q}^u$ (resp., ${\gamma_q}^l$) is asymptotic to α_{p_2} (resp., α_{p_1}). Further, the backward straight lines ${\gamma_q}^{u-}$ and ${\gamma_q}^{\ell-}$ are asymptotes to the backward straight lines ${\alpha_{p_1}}^-$ and ${\alpha_{p_2}}^-$, respectively. Therefore, $\Omega(\alpha_{p_0}) = \{\alpha_p \,|\, p = (x, y), \text{ where } f(y) = \min f\}$, and it does not satisfy Parallel Axiom if M is not flat.

However, in $\mathbb{R} \times (y_1, y_2)$, if we set $\Omega^u(\alpha_{p_2}, y_1, y_2) = \{{\gamma_q}^u \,|\, q = (x, y), y_1 < y < y_2\}$ and $\Omega^\ell(\alpha_{p_1}, y_1, y_2) = \{{\gamma_q}^\ell \,|\, q = (x, y), y_1 < y < y_2\}$, then they satisfy Parallel Axiom. Using this fact, we find what $\Lambda(\alpha_{p_0})$ is. Let $\{\dots, z_{-1}, z_0 = y_0, z_1, \dots\}$ be the set of all minimum points of f such that $z_k < z_{k+1}$. Then, $\Lambda(\alpha_{p_0}) = \cup_{k=0}^\infty \Omega^u(\alpha_{p_0}, z_{-k-1}, z_{-k}) \cup \cup_{k=0}^\infty \Omega^\ell(\alpha_{p_0}, z_k, z_{k+1}) \cup \Omega(\alpha_{p_0})$. The set $\Lambda(\alpha_{p_0})$ simply covers M, but it does not satisfy Asymptote Axiom.

In the other case, when a straight line γ does not pass through any minimum point of f, we take y_1 and y_2 such that $f(y_1) = f(y_2) = \min f$, $f(y) \neq \min f$

for $y \in (y_1, y_2)$, and γ lies in $S = \mathbb{R} \times (y_1, y_2)$. Then, either $\gamma = {\gamma_{\gamma(0)}}^u$ or $\gamma = {\gamma_{\gamma(0)}}^\ell$. If $p_1 = (x, y_1)$ and $p_2 = (x, y_2)$, then $\Lambda(\gamma) = \Lambda(\alpha_{p_2})$ if $\gamma = {\gamma_{\gamma(0)}}^u$, and $\Lambda(\gamma) = \Lambda(\alpha_{p_1})$ if $\gamma = {\gamma_{\gamma(0)}}^\ell$. Thus, we see that one of $F_\gamma - F_{\alpha_{p_2}}$ and $F_\gamma - F_{\alpha_{p_1}}$ is constant on M.

From these facts, we find the gradient vector fields of Busemann functions on M and discuss their differentiability. Let $\gamma(t) = (x(t), y(t))$, $t \in \mathbb{R}$, be a straight line. We may assume that it passes through a minimum point $p_0 = (x_0, y_0)$ of f. Recall that either (1) $y((-\infty, \infty)) = \mathbb{R}$ or (2) $y(t)$ is constant for all $t \in \mathbb{R}$.

Let V be the gradient vector field of F_γ on M. Since $\alpha \in \Lambda(\gamma)$ implies $\tau_s \circ \alpha \in \Lambda(\gamma)$ for all $s \in \mathbb{R}$, we have $(\tau_s)_* V(q) = V(\tau_s(q))$ for all $q \in M$ and $s \in \mathbb{R}$. From this, the angle $\varphi(q)$ of $V(q)$ and $\partial/\partial x$ at q depends only on the y-coordinate of q. Therefore, we can write $g(-V(q), \partial/\partial x) = f(y)\cos\varphi(y)$, and $-V(q) = (\cos\varphi(y)/f(y), \sin\varphi(y))$ for $q = (x, y) \in M$.

In case (1), from Clairaut's relation, $f(y)\cos\varphi(y) = f(y_0)\cos\varphi(y_0)$. Since $\varphi(y_0) \neq 0, \pi$, we have $\varphi(y) \neq 0$ for all $y \in \mathbb{R}$. From these equations, φ is smooth, proving that V and F_γ are smooth on M. Some precise equations of differentials are seen in the argument for case (2).

In case (2), we examine the differentiability of the tangent vector field given by $\dot\gamma^u(0) = -V((x, y))$ for $y < y_0$. Let $\gamma^u(t) = (x(t), y(t))$ and $\dot\gamma^u(t) = (x'(t), y'(t))$ for all $t \in \mathbb{R}$. Then, $x'(t) = \cos\varphi(y(t))/f(y(t))$ and $y'(t) = \sin\varphi(y(t))$ for all $t \in \mathbb{R}$.

We may first suppose $y'(t) > 0$ for all $t \in \mathbb{R}$, meaning that $\varphi(y(t)) \neq 0, \pi$. Then, the straight line can be reparametrized by y, and we have $f(y)\cos\varphi(y) =$ const. Hence, along γ^u, differentiating both sides by y, we have

$$f'\cos\varphi - f\varphi'\sin\varphi = 0 \quad \text{and} \quad \varphi' = \frac{f'\cos\varphi}{f\sin\varphi}. \tag{6.1}$$

This implies that the vector field generated by $\gamma^u(0)$ is differentiable at all points (x, y) with $f(y) \neq \min f$ or equivalently $\varphi(y) \neq 0, \pi$. At those points, $\varphi' = 0$ if and only if $f' = 0$, since $\varphi \neq \pi/2$.

We evaluate the value of $\varphi'(y)$ when $\varphi(y + h)$ approaches to $0 = \varphi(y)$. We assume that $\varphi(y) = 0$. Then, $f'(y) = 0$, because $f(y) = \min f$. Let $K(y)$ be the Gauss curvature of M at (x, y). Then it satisfies $f''(y) + K(y)f(y) = 0$. As we see in Lemma 7.2 later, if $K(y) < 0$, we then have

$$\varphi'(y \pm 0) = \pm\sqrt{-K(y)} \quad \text{and} \quad \varphi''(y \pm 0) = -\frac{K'(y)}{3\varphi'(y \pm 0)},$$

where the double-sign corresponds. This means that the gradient vector field V is not differentiable when $\varphi(y) = 0$, and its second derivatives are bounded.

Since $\varphi'(y \pm 0)$ and $\varphi''(y \pm 0)$ have the opposite signs when we use $\gamma^\ell(0)$, i.e., $\varphi(y+h) < 0$, we conclude that if $p_0 = (x_0, y_0)$ with $f(y_0) = \min f$, then the Busemann function $F_{\alpha_{p_0}}$ is of class C^1 on M, not differentiable twice at $p = (x, y)$ with $f(y) = \min f$ and $y \neq y_0$. However, its 3-rd derivatives are bounded on M when $K(y) < 0$ for y with $f(y) = \min f$.

Furthermore, when $K(y_0) = 0$ for y_0 with $f(y_0) = \min f$, we see from Lemma 7.2 that a Busemann function $F_{\alpha_{p_0}}$ is at least of class C^3 on M, although $\Lambda(\gamma)$ does not satisfy Asymptote Axiom for any geodesic $\gamma(t) = (x(t), y_0)$.

7. Taylor polynomials and higher order differentiability

In this section, we prove two lemmas which were used in the previous sections. We note that a function f may not be differentiable twice at p under the assumption that there exists a Taylor polynomial of f at single point p.

Lemma 7.1. *Let U be an open set in $\mathbb{R}^n$, and $f(x)$ a function defined on U. Assume that $f(x)$ has the 2-nd order Taylor polynomial with continuous coefficients. Then, $f(x)$ is of class C^2.*

Proof. Let $e_1, \dots, e_n$ be a natural basis of $\mathbb{R}^n$ and $h_i = he_i$ for $h \in \mathbb{R}$. We denote the 2-nd order Taylor polynomial of $f(x)$ at $x \in U$ by

$$f(x+y) = a(x) + \sum_{i=1}^{n} b_i(x) y^i + \sum_{i,j=1}^{n} c_{ij}(x) y^i y^j + o(\|y\|^2), \tag{7.1}$$

where the matrix function $(c_{ij}(x))$ is symmetric. From the assumption, $a(x)$, $b_i(x)$ and $c_{ij}(x)$ are continuous for all $x \in U$. Setting $y = 0$, we have $a(x) = f(x)$. Setting $y = h_i$, we then have

$$\frac{f(x+h_i) - f(x) - b_i(x)h}{h} = c_{ii}(x)h + \frac{o(h^2)}{h}.$$

As $h \to 0$, we see $f_i(x) = b_i(x)$ for all $x \in U$. Since $b_i(x)$ are continuous, $f(x)$ is continuously differentiable on U.

Next, setting $y = h_i + h_j$, we have

$$\begin{aligned} f(x+h_i+h_j) &= f(x+h_i) + f_j(x+h_i)h + c_{jj}(x+h_i)h^2 + o(h^2) \\ &= f(x) + f_i(x)h + c_{ii}(x)h^2 + o(h^2) \\ &\quad + f_j(x+h_i)h + c_{jj}(x+h_i)h^2 + o(h^2), \end{aligned} \tag{7.2}$$

and

$$f(x+h_i+h_j) = f(x) + (f_i(x)h + f_j(x))h + (c_{ii}(x) + 2c_{ij}(x) + c_{jj}(x))h^2 + o(h^2). \tag{7.3}$$

Subtracting (7.3) from (7.2) in both sides, we have

$$0 = (f_j(x+h_i) - f_j(x) - 2c_{ij}(x)h)h + (c_{jj}(x+h_i) - c_{jj}(x))h^2 + o(h^2). \tag{7.4}$$

Hence,

$$\frac{f_j(x+h_i) - f_j(x) - 2c_{ij}(x)h}{h} = -(c_{jj}(x+h_i) - c_{jj}(x)) + \frac{o(h^2)}{h^2}.$$

Since $c_{jj}(x)$ are continuous, we see $f_j(x)$ have partial derivatives and $f_{ji}(x) = 2c_{ij}(x)$. Since they are continuous on U, $f(x)$ is of class C^2. □

Lemma 7.2. *Let $f(h)$ be a smooth function on a interval $(-a,a)$ such that $f(h) > 0$ and $f(0) = \min f =: c$. Let $K(h) := -f''(h)/f(h)$ for $h \in (-a,a)$. We define a continuous function $\varphi(h)$ on $(-a,a)$ satisfying that $f(h)\cos\varphi(h) = c$, $\varphi(h) \geq 0$ and $\varphi(0) = 0$. If $K(0) < 0$, then*

$$\varphi'(\pm 0) = \pm\sqrt{-K(0)} \quad \text{and} \quad \varphi''(\pm 0) = -\frac{\varepsilon K'(0)}{3\sqrt{-K(0)}}.$$

If $K(0) = 0$, we then have $\varphi'(0) = 0$ and $\varphi''(0) = \sqrt{\frac{-K''(0)}{3}}$.

Proof. From the definition of $\varphi(h)$, we have

$$\sin\varphi(h) = \frac{\sqrt{f(h)^2 - c^2}}{f(h)}.$$

When $h \neq 0$, $\varphi(h)$ is differentiable and

$$\varphi'(h) = \frac{c^2 f'(h)}{f(h)^2 \cos\varphi(h)\sqrt{f(h)^2 - c^2}} = \frac{c f'(h)}{f(h)\sqrt{f(h)^2 - c^2}}. \tag{7.5}$$

Let $k \geq 1$ be the minimum integer such that $f^{(k)}(0) \neq 0$. Since $f(0) = c$ is the minimum of f, the integer k is even and $f^{(k)}(0) > 0$. The functions in the right hand side of (7.5) have their higher order Taylor polynomials as in the following:

$$f(h) = f(0) + \frac{f^{(k)}(0)}{k!}h^k + \frac{f^{(k+1)}(0)}{(k+1)!}h^{k+1} + o(h^{k+1}), \tag{7.6}$$

$$f'(h) = \frac{f^{(k)}(0)}{(k-1)!}h^{k-1} + \frac{f^{(k+1)}(0)}{k!}h^k + o(h^k). \tag{7.7}$$

From (7.6), we have

$$f(h)^2 = f(0)^2 + \frac{2f(0)f^{(k)}(0)}{k!}h^k + \frac{2f(0)f^{(k+1)}(0)}{(k+1)!}h^{k+1} + o(h^{k+1}).$$

Therefore, because of $\sqrt{1+h} = 1 + \frac{1}{2}h + o(h)$,

$$\sqrt{f(h)^2 - f(0)^2} = |h|^{\frac{k}{2}}\sqrt{\frac{2f(0)f^{(k)}(0)}{k!}}\left(1 + \frac{f^{(k+1)}(0)}{2(k+1)f^{(k)}(0)}h + o(h)\right).$$

The denominator in the right hand side of (7.5) is denoted by

$$\begin{aligned} &f(h)\sqrt{f(h)^2 - f(0)^2} \\ &= |h|^{\frac{k}{2}}f(0)\sqrt{\frac{2f(0)f^{(k)}(0)}{k!}}\left(1 + \frac{f^{(k+1)}(0)}{2(k+1)f^{(k)}(0)}h + o(h)\right). \end{aligned} \tag{7.8}$$

Substituting (7.7) and (7.8) into (7.5), we have, if $|h| = \varepsilon h$,

$$\varphi'(h) = \varepsilon^{\frac{k}{2}}h^{\frac{k}{2}-1}\sqrt{\frac{k!}{2f(0)f^{(k)}(0)}}\left(\frac{f^{(k)}(0)}{(k-1)!} + \frac{f^{(k+1)}(0)(k+2)}{2(k+1)!}h + o(h)\right). \tag{7.9}$$

Assume that $K(0) \neq 0$. Then $k = 2$. From $f''(0) = -K(0)f(0)$ and $f^{(3)}(0) = -K'(0)f(0)$, we have

$$\varphi'(h) = \varepsilon\sqrt{-K(0)} - \frac{\varepsilon K'(0)}{3\sqrt{-K(0)}}h + o(h).$$

Thus, we have $\varphi'(\pm 0) = \varepsilon\sqrt{-K(0)}$ and $\varphi''(\pm 0) = -\frac{\varepsilon K'(0)}{3\sqrt{-K(0)}}$.

Assume that $K(0) = 0$. Then $k \geq 4$ and $\varphi'(0) = 0$. If $k \geq 6$, we then have $\varphi''(0) = 0$, $K'(0) = 0$ and $K''(0) = 0$. If $k = 4$, we then have

$$\varphi'(h) = \sqrt{\frac{-K''(0)}{3}}h + o(h).$$

Hence we have $\varphi''(0) = \sqrt{\frac{-K''(0)}{3}}$. □

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