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Matchings in hypergraphs and Castelnuovo–Mumford regularity

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Abstract. In this paper, we introduce and generalize to hypergraphs some combinatorial invariants of graphs such as matching number and induced matching number. Then we compare them and present some upper bounds for the regularity of the Stanley–Reisner ring of $\Delta_{\mathcal{H}}$ for certain hypergraphs $\mathcal{H}$ in terms of the introduced generalizations of matching numbers.

Introduction

There is a natural correspondence between simplicial complexes and hypergraphs. We associate to a hypergraph $\mathcal{H}$ the simplicial complex whose faces are the independent sets of vertices of $\mathcal{H}$, i.e., the sets which do not contain any edge of $\mathcal{H}$. This simplicial complex is called the *independence complex* of $\mathcal{H}$ and is denoted by $\Delta_{\mathcal{H}}$. Also, we associate to a simplicial complex Δ a squarefree ideal I_Δ, called the Stanley–Reisner ideal of Δ, which is generated by monomials $x_{i_1},\dots,x_{i_s}$ where $\{x_{i_1},\dots,x_{i_s}\}$ is not a face of Δ. So squarefree monomial ideals can be studied using these combinatorial ideas. Recently, edge ideals of graphs, as the easiest class of squarefree monomial ideals, has been studied by many researchers. Nice characterizations of the algebraic invariants, in terms of data from graphs, have been studied (cf. [11], [12], [13], [15], [18] and [22]). Extending the concepts in graphs to hypergraphs and finding more general results in hypergraphs, which will cover all squarefree monomial ideals, are of great interest, see, for example, [6], [8], [9], [16] and [20]. One graph invariant, which has been

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studied extensively, is the matching number or other related graph parameters (cf. [14]). In this paper, we are going to extend the notion of matchings and induced matchings to hypergraphs. Then we will examine connections between algebraic invariants and these extensions of matching numbers.

The *Castelnuovo–Mumford regularity* (or simply regularity) of an R-module M is defined as

$$\mathrm{reg}(M) := \max\{j - i \mid \beta_{i,j}(M) \neq 0\},$$

where $\beta_{i,j}(M)$ is the (i,j)-th Betti number of M. Bounding the Castelnuovo–Mumford regularity of $R/I_{\Delta_{\mathcal{H}}}$ in terms of invariants of $\mathcal{H}$ has been studied extensively by many authors. In the case that $\mathcal{H}$ is a graph, in certain circumstances, $\mathrm{reg}(R/I_{\Delta_{\mathcal{H}}})$ is characterized precisely. For instance, in [8], [13] and [18], respectively for chordal graphs, C_5-free vertex-decomposable graphs and sequentially Cohen–Macaulay bipartite graphs G, it was shown that $\mathrm{reg}(R/I(G)) = c_G$, where $I(G)$ is the edge ideal of G and c_G is the induced matching number of G. Furthermore, combinatorial characterizations of the Castelnuovo–Mumford regularity of the edge ideal of hypergraphs has been the subject of many works. Indeed, in [9], the authors introduced the concept of 2-collage in a simple hypergraph as a generalization of the matching number in a graph, and proved that the Castelnuovo–Mumford regularity of the edge ideal of a simple hypergraph is bounded above in terms of 2-collages. In this paper, we provide a counterexample for [9, Lemma 3.4] which is a main tool to gain this bound (see also [10]). Also, Morey and Villarreal, in [16], gave a lower bound for the regularity of the edge ideal of any simple hypergraph in terms of an induced matching of the hypergraph. Moreover, in [8], for d-uniform properly-connected hypergraphs a lower bound for the regularity is given. For more results, see [3], [4], [7], [17], [21].

In this paper, we also study the regularity of the Stanley–Reisner ring of $\Delta_{\mathcal{H}}$ for some families of hypergraphs, and relate it to some combinatorial concepts and generalize or improve some results, which had been gained for graphs, such as [13, Theorem 2.4].

The paper proceeds as follows. After reviewing some hypergraph terminologies in the first section, in Section 2, we define the induced matching and semi-induced matching numbers for a hypergraph $\mathcal{H}$, which we denote by $c_{\mathcal{H}}$ and $c'_{\mathcal{H}}$, respectively. We compare them under different conditions. Also, we present a class of hypergraphs $\mathcal{H}$, consisting simple graphs such that $c_{\mathcal{H}} = c'_{\mathcal{H}}$.

In the light of [16, Corollary 3.9(a)], $c_{\mathcal{H}}$ is a lower bound for $\mathrm{reg}(R/I_{\Delta_{\mathcal{H}}})$, when $\mathcal{H}$ is a hypergraph. In Section 3, we are going to obtain some upper bounds for $\mathrm{reg}(R/I_{\Delta_{\mathcal{H}}})$ for a hypergraph $\mathcal{H}$. In Theorem 3.6, it is proved that if a vertex decomposable hypergraph $\mathcal{H}$ is (C_2, C_5)-free, then $\mathrm{reg}(R/I_{\Delta_{\mathcal{H}}}) \leq c'_{\mathcal{H}} \leq \dim \Delta_{\mathcal{H}} + 1$.

This extends a result on graphs from [13] to hypergraphs, which states that for a C_5-free vertex-decomposable graph G, $\operatorname{reg}(R/I(G)) = c_G$.

1. A review of hypergraph terminology

In this section, we present some preliminaries on hypergraphs from [1] and [2].

Definition 1.1. A *hypergraph* is a pair $(V, \mathcal{E})$, where V is a finite set of vertices, and $\mathcal{E}$ is a collection of edges (or hyperedges). A hypergraph is called *d-uniform* if all of its edges have the same cardinality d. So, every simple graph is a 2-uniform hypergraph.

Throughout this paper, we assume that $\mathcal{H} = (V(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ is a *simple hypergraph*. That means that no element of $\mathcal{E}(\mathcal{H})$ contains another. A vertex of $\mathcal{H}$ is called *isolated* if it is not contained in any edge of $\mathcal{H}$.

Definition 1.2. Assume that $\mathcal{H}$ is a hypergraph. For any vertex $x \in V(\mathcal{H})$, $\mathcal{H} \backslash x$ is a hypergraph with vertex set $V(\mathcal{H}) \backslash \{x\}$ and edge set $\{E \in \mathcal{E}(\mathcal{H}) : x \notin E\}$. Moreover, $\mathcal{H}/x$ is a hypergraph with vertex set $V(\mathcal{H}) \setminus \{x\}$ whose edges are the non-empty minimal elements (with respect to inclusion) of the set $\{E \setminus \{x\} : E \in \mathcal{E}(\mathcal{H})\}$. It is clear that $\mathcal{H} \setminus x$ and $\mathcal{H}/x$ are two simple hypergraphs. They are called *deletion* and *contraction* of $\mathcal{H}$ by x, respectively. Moreover, for a simplicial complex Δ with vertex set X and $x \in X$, the link of x in Δ is defined as

$$\mathrm{lk}_{\Delta}(x) = \{G \in \Delta : x \notin G, G \cup \{x\} \in \Delta\},$$

and the deletion of x is the simplicial complex

$$\mathrm{del}_{\Delta}(x) = \{G \in \Delta : x \notin G\}.$$

One can easily see that $\mathrm{lk}_{\Delta}(x)$ and $\mathrm{del}_{\Delta}(x)$ are also simplicial complexes on $X \setminus \{x\}$.

Note that for a vertex $x \in V(\mathcal{H})$, $\mathrm{del}_{\Delta_{\mathcal{H}}}(x) = \Delta_{\mathcal{H} \setminus x}$ and $\mathrm{lk}_{\Delta_{\mathcal{H}}}(x) = \Delta_{\mathcal{H}/x}$.

Lemma 1.3. *Assume that $\mathcal{H}$ is a hypergraph and x is a shedding vertex. Then for each facet T of $\Delta_{\mathcal{H} \setminus x}$ there is an edge F in $\mathcal{H}$ containing x such that $F \setminus \{x\}$ is contained in T.*

Proof. Since x is a shedding vertex, T is not a facet of $\Delta_{\mathcal{H}/x}$. So T is not an independent set of vertices in $\mathcal{H}/x$. This ensures that there is an edge F of $\mathcal{H}$ such that $F \setminus \{x\} \subseteq T$. Since T is an independent set of vertices in $\mathcal{H} \setminus x$, we should have $x \in F$. □

Definition 1.4 (see [2]). A *chain* in $\mathcal{H}$ is a sequence $v_0, E_1, v_1, \dots, E_k, v_k$, where $v_i \in E_i$ for $1 \leq i \leq k, v_i \in E_{i+1}$ for $0 \leq i \leq k-1$, and $E_1, \dots, E_k$ are edges in $\mathcal{H}$. For our convenience, we denote this chain by $E_1, \dots, E_k$, if there is no ambiguity. We say that $\mathcal{H}$ is $\mathcal{C}_k$*-free* if it doesn't contain any chain $v_0, E_1, v_1, \dots, E_k, v_0$ with $k > 1$ and distinct v_is and E_is.

The following lemma can be easily gained from Definitions 1.2 and 1.4.

Lemma 1.5. *Assume that $k > 1$ is an integer, $\mathcal{H}$ is a $\mathcal{C}_k$-free hypergraph and $x \in V(\mathcal{H})$. Then $\mathcal{H} \setminus x$ and $\mathcal{H}/x$ are also $\mathcal{C}_k$-free hypergraphs.*

Definition 1.6. A d-uniform hypergraph $\mathcal{H}$ is called *strongly connected* if for each pair of distinct edges E and E', there is a chain $E = E_0, E_1, \dots, E_{k-1}, E_k = E'$ of edges of $\mathcal{H}$ such that for each $i := 0, 1, \dots, k-1$, $|E_i \cap E_{i+1}| = d - 1$.

2. Matching numbers of hypergraphs

In this section, firstly, inspired by the definition of an induced matching in [16], we introduce the concepts of induced matching number and semi-induced matching number of a hypergraph. Then we give some equalities and inequalities between these invariants.

Definition 2.1. A set $\{E_1, \dots, E_k\}$ of edges of a hypergraph $\mathcal{H}$ is called a *semi-induced matching* if the only edges contained in $\bigcup_{\ell=1}^k E_\ell$ are $E_1, \dots, E_k$. A semi-induced matching where all of its elements are mutually disjoint is called an *induced matching.* Also, we set

$$c_{\mathcal{H}} := \max\left\{ |\bigcup_{\ell=1}^{k} E_\ell| - k : \{E_1, \dots, E_k\} \text{ is an induced matching in } \mathcal{H} \right\},$$
$$c'_{\mathcal{H}} := \max\left\{ |\bigcup_{\ell=1}^{k} E_\ell| - k : \{E_1, \dots, E_k\} \text{ is a semi induced matching in } \mathcal{H} \right\},$$

and we call them the *induced matching number* and the *semi-induced matching number* of $\mathcal{H}$, respectively.

The following lemma is an immediate consequence of Definition 2.1.

Lemma 2.2. *Assume that $\mathcal{H}$ is a hypergraph and $x \in V(\mathcal{H})$. Then $c'_{\mathcal{H}\setminus x} \leq c'_{\mathcal{H}}$.*

Example 2.3. If G is a graph, then the set of all edges of a graph is a perfectly good semi-induced matching, although the semi-induced matching number will usually not be achieved by this. For example, if G is a graph on $\{1,2,3,4\}$ with edge set $\{E_1 = \{1,2\}, E_2 = \{2,3\}, E_3 = \{3,4\}, E_4 = \{2,4\}\}$, then $\{E_1, E_2\}$ is a semi-induced matching that induces $c'_G = 1$.

Let $E_1, \ldots, E_k$ be an induced matching of $\mathcal{H}$ and for each $1 \leq i \leq k$, $v_i \in E_i$. Then it is clear that $\bigcup(E_i \setminus \{v_i\})$ is an independent set of vertices in $\mathcal{H}$, and so $c_{\mathcal{H}} \leq \dim(\Delta_{\mathcal{H}}) + 1$. But in the following theorem we gain more. In fact, it compares the invariants $c_{\mathcal{H}}$, $c'_{\mathcal{H}}$ and $\dim(\Delta_{\mathcal{H}})$ for an arbitrary hypergraph $\mathcal{H}$.

Theorem 2.4. *For any hypergraph $\mathcal{H}$, we have the following inequalities:*

$$c_{\mathcal{H}} \leq c'_{\mathcal{H}} \leq \dim(\Delta_{\mathcal{H}}) + 1$$

Proof. It is clear that every induced matching of $\mathcal{H}$ is a semi-induced matching. So, we have $c_{\mathcal{H}} \leq c'_{\mathcal{H}}$. To prove the last inequality, suppose that $\{E_1, \ldots, E_k\}$ is a semi-induced matching in $\mathcal{H}$ such that $c'_{\mathcal{H}} = |\bigcup_{\ell=1}^{k} E_\ell| - k$. Set $S_0 = \emptyset$ and for each $1 \leq i \leq k$, if $E_i \cap S_{i-1} \neq \emptyset$, then set $S_i = S_{i-1}$; else, choose a vertex $x_i \in E_i$ and set $S_i = S_{i-1} \cup \{x_i\}$. Now, consider the set $G = (\bigcup_{\ell=1}^{k} E_\ell) \setminus S_k$. We claim that G is an independent set of vertices in $\mathcal{H}$. By contrary, assume that $E \subseteq G$ for some $E \in \mathcal{E}(\mathcal{H})$. Then $E \cap S_k = \emptyset$ and $E \subseteq \bigcup_{\ell=1}^{k} E_\ell$. So $E = E_i$ for some $1 \leq i \leq k$, since $\{E_1, \ldots, E_k\}$ is a semi-induced matching in $\mathcal{H}$. From the choice of x_is, it is clear that $x_j \in E_i \cap S_k$ for some $1 \leq j \leq i$, which is a contradiction. Therefore, G is contained in a facet F of $\Delta_{\mathcal{H}}$. Since $|S_k| \leq k$, we have $c'_{\mathcal{H}} \leq |G| \leq |F| \leq \dim(\Delta_{\mathcal{H}}) + 1$, which completes the proof. □

The following example illustrates that the inequalities in Theorem 2.4 can be strict.

Example 2.5. Let $\mathcal{H}$ be a hypergraph with vertex set $V = \{x_1, \ldots, x_6\}$ and edges $E_1 = \{x_1, x_2, x_3\}, E_2 = \{x_2, x_3, x_4\}$ and $E_3 = \{x_4, x_5, x_6\}$. Then one can see that $c_{\mathcal{H}} = 2$ and $c'_{\mathcal{H}} = 3$. So $c_{\mathcal{H}} < c'_{\mathcal{H}}$.

Assume that G is a star graph with vertex set $V = \{x_1, \ldots, x_4\}$ and edges $\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}$. Then one can easily see that $c'_G = 1$, but $\dim(\Delta_G) = 2$. So, even when $\mathcal{H}$ is a graph, the second inequality in Theorem 2.4 can be strict.

Remark 2.6. It is easily seen that when $\mathcal{H}$ is a graph, the induced matching number $c_{\mathcal{H}}$, as we have defined it, coincides with the well-known definition in graph theory (cf. [5]); i.e., the maximum number of 3-disjoint edges in $\mathcal{H}$. Also, one may find that a semi-induced matching corresponds to any subset A of non-isolated vertices of $\mathcal{H}$. So, the semi-induced matching number is the maximum of the number of vertices in A, minus the number of edges contained in A.

In the following proposition, we provide conditions under which $c_{\mathcal{H}} = c'_{\mathcal{H}}$.

Proposition 2.7. *Assume that $\mathcal{H}$ is a d-uniform hypergraph such that for each pair of distinct edges E and E', $E \cap E' \neq \emptyset$ implies that $|E \cap E'| = d - 1$. Then $c_{\mathcal{H}} = c'_{\mathcal{H}}$. So $c'_{\mathcal{H}} \leq m_{\mathcal{H}}$, for the special class of hypergraphs consisting simple graphs. (Note that the mentioned condition in this proposition is different from the property of* strongly connectedness *for hypergraphs.)*

Proof. In view of Theorem 2.4, it is enough to show that $c'_{\mathcal{H}} \leq c_{\mathcal{H}}$. To this end, assume that $\{E_1, \dots, E_k\}$ is a semi-induced matching in $\mathcal{H}$ such that $|\bigcup_{\ell=1}^{k} E_\ell| - k = c'_{\mathcal{H}}$. It is sufficient to show that there is a subset S of $\{1, \dots, k\}$ such that $\{E_\ell : \ell \in S\}$ is an induced matching in $\mathcal{H}$ and

$$|\bigcup_{\ell=1}^{k} E_\ell| - k \leq |\bigcup_{\ell \in S} E_\ell| - |S|.$$

We use induction on k. The result is clear when $k = 1$. So assume inductively that $k > 1$, and the result is true for smaller values of k. We may consider the following cases.

Case 1. Suppose that there is an integer $1 \leq i \leq k$ such that $E_i \cap (\bigcup_{\ell=1, \ell \neq i}^{k} E_\ell) = \emptyset$. Then by inductive hypothesis, there is a subset S of $\{1, \dots, i-1, i+1, \dots, k\}$ such that $\{E_\ell : \ell \in S\}$ is an induced matching in $\mathcal{H}$, and we have

$$|\bigcup_{\ell=1, \ell \neq i}^{k} E_\ell| - (k-1) \leq |\bigcup_{\ell \in S} E_\ell| - |S|.$$

Now, set $S' = S \cup \{i\}$. It is obvious that $\{E_\ell : \ell \in S'\}$ is an induced matching in $\mathcal{H}$, and we have

$$\begin{aligned} |\bigcup_{\ell=1}^{k} E_\ell| - k &= |\bigcup_{\ell=1, \ell \neq i}^{k} E_\ell| - (k-1) + |E_i| - 1 \\ &\leq |\bigcup_{\ell \in S} E_\ell| - |S| + |E_i| - 1 = |\bigcup_{\ell \in S'} E_\ell| - |S'| \end{aligned}$$

as desired.

Case 2. Suppose that there is an integer $1 \leq i \leq k$ such that $0 < |E_i \cap (\bigcup_{\ell=1, \ell \neq i}^{k} E_\ell)| < |E_i|$. Then inductive hypothesis implies that there is a subset S

of $\{1, \dots, i-1, i+1, \dots, k\}$ such that $\{E_\ell : \ell \in S\}$ is an induced matching in $\mathcal{H}$, and

$$|\bigcup_{\ell=1,\ell\neq i}^{k} E_\ell| - (k-1) \leq |\bigcup_{\ell\in S} E_\ell| - |S|.$$

On the other hand, by our assumption on $\mathcal{H}$, we should have $|E_i \cap (\bigcup_{\ell=1,\ell\neq i}^{k} E_\ell)| = d-1$. Now, we have

$$\begin{aligned}|\bigcup_{\ell=1}^{k} E_\ell| - k &= |\bigcup_{\ell=1,\ell\neq i}^{k} E_\ell| - (k-1) + |E_i| - |E_i \cap (\bigcup_{\ell=1,\ell\neq i}^{k} E_\ell)| - 1\\ &\leq |\bigcup_{\ell\in S} E_\ell| - |S| + d - (d-1) - 1 = |\bigcup_{\ell\in S} E_\ell| - |S|\end{aligned}$$

as desired.

Case 3. Suppose that for each $1 \leq i \leq k$, $E_i \subseteq \bigcup_{\ell=1,\ell\neq i}^{k} E_\ell$. Then by inductive hypothesis, there is a subset S of $\{1, \dots, k-1\}$ such that $\{E_\ell : \ell \in S\}$ is an induced matching in $\mathcal{H}$, and

$$|\bigcup_{\ell=1}^{k-1} E_\ell| - (k-1) \leq |\bigcup_{\ell\in S} E_\ell| - |S|.$$

So, we have

$$\begin{aligned}|\bigcup_{\ell=1}^{k} E_\ell| - k &= |\bigcup_{\ell=1}^{k-1} E_\ell| - (k-1) - 1\\ &\leq |\bigcup_{\ell\in S} E_\ell| - |S| - 1 \leq |\bigcup_{\ell\in S} E_\ell| - |S|\end{aligned}$$

as desired. □

Remark 2.8. Note that although the conditions of Proposition 2.7 look like so extremely, it contains the following types of hypergraphs:

(i) $\mathcal{H}$ is a graph;

(ii) $\mathcal{H}$ is a hypergraph whose every connected component has either the vertex set $V = \{x_1, \dots, x_{d-1}, y_1, \dots, y_n\}$ and edge set $\{E_i = \{x_1, \dots, x_{d-1}, y_i\} : 1 \leq i \leq n\}$; or the vertex set $V = \{x_1, \dots, x_{d+1}\}$ and its edge set is a subset of the set of all d-subsets of V.

So, one may reduce the proof in each case separately.

As an immediate consequence of Proposition 2.7, we obtain the following.

Corollary 2.9. *For a simple graph G, we have $c_G = c'_G$.*

3. Regularity of edge ideals of certain hypergraphs

In this section, we show that for a hypergraph $\mathcal{H}$, the introduced invariants in Section 2 give bounds for $\mathrm{reg}(R/I_{\Delta_{\mathcal{H}}})$, and for some families of hypergraphs we are able to calculate $\mathrm{reg}(R/I_{\Delta_{\mathcal{H}}})$ exactly in terms of these numbers. We begin by the following remark.

As a main result of this paper, we are going to show that $c'_{\mathcal{H}}$ is an upper bound for $\mathrm{reg}(R/I_{\Delta_{\mathcal{H}}})$ for a certain class of hypergraphs. To this end, we need to recall the following definition.

Definition 3.1. Let Δ be a simplicial complex on the vertex set $V=\{x_1,\dots,x_n\}$. Then Δ is *vertex-decomposable* if either:

(1) The only facet of Δ is $\{x_1,\dots,x_n\}$, or $\Delta=\emptyset$.

(2) There exists a vertex $x\in V$ such that $\mathrm{del}_\Delta(x)$ and $\mathrm{lk}_\Delta(x)$ are vertex-decomposable, and every facet of $\mathrm{del}_\Delta(x)$ is a facet of Δ.

A vertex $x\in V$ for which every facet of $\mathrm{del}_\Delta(x)$ is a facet of Δ is called a *shedding vertex* of Δ. Note that this is equivalent to say that no facet of $\mathrm{lk}_\Delta(x)$ is a facet of $\mathrm{del}_\Delta(x)$.

A hypergraph $\mathcal{H}$ is called vertex-decomposable, if the independence complex $\Delta_{\mathcal{H}}$ is vertex-decomposable, and a vertex of $\mathcal{H}$ is called a shedding vertex if it is a shedding vertex of $\Delta_{\mathcal{H}}$. So the next lemma immediately follows.

Lemma 3.2. *Let $\mathcal{H}$ be a hypergraph. Then*

(i) *if x is a shedding vertex of $\mathcal{H}$ and $\{E_1,\dots,E_k\}$ is the set of all edges of $\mathcal{H}$ containing x, then every facet of $\mathcal{H}\setminus x$ contains $E_i\setminus\{x\}$ for some $1\le i\le k$.*

(ii) *the vertex x is a shedding vertex of $\mathcal{H}$ if and only if for every maximal independent set I of $\mathcal{H}\setminus x$, there is an edge E of $\mathcal{H}$ containing x, such that I contains $E\setminus\{x\}$.*

For our main result, we also need to illustrate the relations between $c'_{\mathcal{H}}$, $c'_{\mathcal{H}\setminus x}$ and $c'_{\mathcal{H}/x}$ for a vertex x of $\mathcal{H}$. Note that it is obvious that $c_{\mathcal{H}\setminus x}\le c_{\mathcal{H}}$ and $c'_{\mathcal{H}\setminus x}\le c'_{\mathcal{H}}$. Now, suppose that $\{E_1\setminus\{x\},\dots,E_k\setminus\{x\}\}$ is a semi-induced matching in $\mathcal{H}/x$ such that $c'_{\mathcal{H}/x}=|\bigcup_{\ell=1}^k(E_\ell\setminus\{x\})|-k$. The following example shows that it is not necessarily true that $\{E_1,\dots,E_k\}$ is a semi-induced matching in $\mathcal{H}$.

Example 3.3. Let $\mathcal{H}$ be a hypergraph with $V(\mathcal{H})=\{x_1,\dots,x_5\}$ and $\mathcal{E}(\mathcal{H})=\{E_1=\{x_1,x_2,x_3\},E_2=\{x_2,x_3,x_4\},E_3=\{x_4,x_5\}\}$. Then $\mathcal{E}(\mathcal{H}/x_1)=\{E_1\setminus\{x_1\},E_3\setminus\{x_1\}\}$. It is clear that $\{E_1\setminus\{x_1\},E_3\setminus\{x_1\}\}$ is a semi-induced matching in $\mathcal{H}/x_1$, but $\{E_1,E_3\}$ is not a semi-induced matching in $\mathcal{H}$.

Now, the following two lemmas provide conditions under which we can get to a semi-induced matching in $\mathcal{H}$ from one in $\mathcal{H}/x$, for a vertex x of $\mathcal{H}$.

Lemma 3.4. *Assume that $\mathcal{H}$ is a $\mathcal{C}_2$-free hypergraph, x is a vertex of $\mathcal{H}$ and k is the smallest integer such that there exists a semi-induced matching $\{E_1 \setminus \{x\}, \dots, E_k \setminus \{x\}\}$ in $\mathcal{H}/x$ such that $c'_{\mathcal{H}/x} = |\bigcup_{\ell=1}^{k}(E_\ell \setminus \{x\})| - k$. Then $\{E_1, \dots, E_k\}$ is a semi-induced matching in $\mathcal{H}$, and $c'_{\mathcal{H}/x} \leq c'_{\mathcal{H}}$. Moreover if $x \in E_i$ for some $1 \leq i \leq k$, we have $c'_{\mathcal{H}/x} + 1 \leq c'_{\mathcal{H}}$.*

Proof. Suppose that there is an edge E of $\mathcal{H}$ such that $E \subseteq \bigcup_{\ell=1}^{k} E_\ell$. Then $E \setminus \{x\} \subseteq \bigcup_{\ell=1}^{k}(E_\ell \setminus \{x\})$, and by the definition of semi-induced matching, when $E \setminus \{x\}$ is an edge, then $E \setminus \{x\} = E_i \setminus \{x\}$ for some $1 \leq i \leq k$. Now, we have three cases.

Case 1. If $x \in E$, then $E \setminus \{x\} = E_i \setminus \{x\}$, for some $1 \leq i \leq k$. If $x \notin E_i$, then E strictly contains E_i, which is a contradiction. So, $x \in E_i$, and hence $E = E_i$ as desired.

Case 2. If $x \notin E$ and E is an edge of $\mathcal{H}/x$, then $E = E_i \setminus \{x\}$, for some $1 \leq i \leq k$. If $x \in E_i$, then E_i strictly contains E, which is a contradiction. So, $x \notin E_i$, which implies that $E = E_i$ as desired.

Case 3. If $x \notin E$ and E is not an edge of $\mathcal{H}/x$, then there is an edge E' of $\mathcal{H}$ containing x such that $E' \setminus \{x\} \subset E$ and $E' \setminus \{x\}$ is an edge of $\mathcal{H}/x$. So, $E \cap E' = E' \setminus \{x\}$. Since $\mathcal{H}$ is $\mathcal{C}_2$-free, $|E' \setminus \{x\}| = 1$. Since $E' \setminus \{x\} \subseteq \bigcup_{\ell=1}^{k}(E_\ell \setminus \{x\})$, we get that $E' \setminus \{x\} = E_i \setminus \{x\}$ for some $1 \leq i \leq k$. Thus $|E_i \setminus \{x\}| = 1$. Moreover, $E_i \setminus \{x\} \nsubseteq \bigcup_{\ell=1, \ell \neq i}^{k}(E_\ell \setminus \{x\})$, since otherwise $E_i \setminus \{x\} \subseteq E_j \setminus \{x\}$ for some $j \neq i$, which is impossible. Therefore, $\{E_\ell \setminus \{x\}, 1 \leq \ell \leq k, \ell \neq i\}$ is a semi inducing matching in $\mathcal{H}/x$ and $|\bigcup_{\ell=1, \ell \neq i}^{k}(E_\ell \setminus \{x\})| - (k-1) = |\bigcup_{\ell=1}^{k}(E_\ell \setminus \{x\})| - 1 - (k-1) = c'_{\mathcal{H}/x}$, which contradicts our assumption on k. So this case cannot occur.

Hence, $\{E_1, \dots, E_k\}$ is a semi-induced matching in $\mathcal{H}$. Now, if $x \in E_i$ for some $1 \leq i \leq k$, we have

$$c'_{\mathcal{H}/x} = |\bigcup_{\ell=1}^{k}(E_\ell \setminus \{x\})| - k = |\bigcup_{\ell=1}^{k} E_\ell| - k - 1 \leq c'_{\mathcal{H}} - 1,$$

which completes the proof. □

Lemma 3.5. *Assume that $\mathcal{H}$ is a $(\mathcal{C}_2, \mathcal{C}_5)$-free hypergraph, x is a shedding vertex of $\mathcal{H}$ and $\{E_1 \setminus \{x\}, \dots, E_k \setminus \{x\}\}$ is a semi-induced matching in $\mathcal{H}/x$ such that $x \notin E_\ell$ for all $1 \leq \ell \leq k$. Then there is an edge F of $\mathcal{H}$ containing x such that $\{E_1, \dots, E_k, F\}$ is a semi-induced matching in $\mathcal{H}$. Moreover, $c'_{\mathcal{H}/x} + 1 \leq c'_{\mathcal{H}}$.*

Proof. By definition of semi-induced matching, $\{E_1,\dots,E_k\}$ forms a semi-induced matching in $\mathcal{H}$. Suppose, in contrary, that there is no edge F containing x such that $\{E_1,\dots,E_k,F\}$ is a semi-induced matching. Then for each edge F containing x, there is an edge F' such that $F'\notin\{E_1,\dots,E_k,F\}$, $F'\cap F\neq\emptyset$ and $F'\setminus F$ is contained in $\bigcup_{\ell=1}^k E_\ell$.

Since $\mathcal{H}$ is $\mathcal{C}_2$-free, the intersection of any two edges contains at most one vertex. Thus, $F\cap F'$ contains a single vertex y, and y is the unique vertex of F' that is not contained in $\bigcup_{\ell=1}^k E_\ell$. Also, if F and G are both edges containing x, then $F\cap G=\{x\}$. It follows that F' does not contain x, and that F' and G' are distinct.

Now, set $S=\bigcup_{x\in F}(F'\setminus F)$. We claim that S is an independent set of vertices in $\mathcal{H}/x$. Suppose, in contrary, that S is not independent. Then, since S is contained in the vertex set of the semi-induced matching $\{E_1,\dots,E_k\}$, there is some $E_\ell\in\{E_1,\dots,E_k\}$ with $E_\ell\subseteq S$. In particular, for some edges F,G containing x, the edge E_ℓ intersects non-trivially with both F' and G'. By the distinctness of F,G,F',G', the edges $E_\ell--F'--F--G--G'--E_\ell$ form a $\mathcal{C}_5$ in $\mathcal{H}$ (note that for each pair of integers $1\leq i,j\leq s$, $F_j\cap F'_j\not\subseteq F'_i\setminus F_i$, because otherwise, since $F'_i\setminus F_i$ and $F'_j\setminus F_j$ are contained in $\bigcup_{\ell=1}^k E_\ell$, we should have $F'_j\subseteq\bigcup_{\ell=1}^k E_\ell$, which is a contradiction. Hence, $E_\ell\cap F'_i\neq F_j\cap F'_j$ for all $1\leq\ell\leq k$). This is a contradiction, so we conclude that S is an independent set.

Finally, let T be a maximal independent set of vertices in $\mathcal{H}/x$ containing S. Since x is a shedding vertex in $\mathcal{H}$, by Lemma 1.3 there is an edge F of $\mathcal{H}$ with $x\in F$ such that $F\setminus\{x\}\subseteq T$. But then $F'\subseteq S\cup(F\setminus\{x\})\subseteq T$, a contradiction.

We conclude that there is an edge F such that $\{E_1,\dots,E_k,F\}$ is a semi-induced matching in $\mathcal{H}$. By hypothesis, $F\setminus\{x\}$ is not contained in $\bigcup_{\ell=1}^k E_\ell$, so

$$c'_{\mathcal{H}/x}+1\leq|F\cup\bigcup_{\ell=1}^k E_\ell|-(k+1)\leq c'_{\mathcal{H}}. \qquad \square$$

Now, we are ready to state our main result of this section.

Theorem 3.6. *Let $\mathcal{H}$ be a $(\mathcal{C}_2,\mathcal{C}_5)$-free vertex-decomposable hypergraph. Then*

$$\operatorname{reg}(R/I_{\Delta_{\mathcal{H}}})\leq c'_{\mathcal{H}}.$$

Proof. We use induction on $|V(\mathcal{H})|$. If $|V(\mathcal{H})|=2$, the result is clear. Suppose, inductively, that the result has been proved for smaller values of $|V(\mathcal{H})|$. Assume that x is a shedding vertex of $\mathcal{H}$. In the light of Lemma 1.5, $\mathcal{H}\setminus x$ and $\mathcal{H}/x$ are $(\mathcal{C}_2,\mathcal{C}_5)$-free vertex-decomposable hypergraphs. By inductive hypothesis, we have

$$\mathrm{reg}(R/I_{\Delta_{\mathcal{H}\setminus x}}) \le c'_{\mathcal{H}\setminus x} \text{ and } \mathrm{reg}(R/I_{\Delta_{\mathcal{H}/x}}) \le c'_{\mathcal{H}/x}.$$

On the other hand, by [9, Theorem 4.2], we have the inequality

$$\mathrm{reg}(R/I_{\Delta_{\mathcal{H}}}) \le \max\{\mathrm{reg}(R/I_{\Delta_{\mathcal{H}\setminus x}}), \mathrm{reg}(R/I_{\Delta_{\mathcal{H}/x}}) + 1\}.$$

Hence

$$\mathrm{reg}(R/I_{\Delta_{\mathcal{H}}}) \le \max\{c'_{\mathcal{H}\setminus x}, c'_{\mathcal{H}/x} + 1\}.$$

Now, the result immediately follows from Lemmas 2.2, 3.4 and 3.5. □

Remark 3.7. Note that, in the light of Theorems 2.4 and 3.6, if $\mathcal{H}$ is a $(\mathcal{C}_2, \mathcal{C}_5)$-free vertex-decomposable hypergraph, then we have

$$\mathrm{reg}(R/I_{\Delta_{\mathcal{H}}}) \le c'_{\mathcal{H}} \le \dim(\Delta_{\mathcal{H}}) + 1.$$

Recall that a subset C of the edges of a hypergraph $\mathcal{H}$ is called a *2-collage* for $\mathcal{H}$ if for each edge E of $\mathcal{H}$ we can delete a vertex v such that $E \setminus \{v\}$ is contained in some edge of C.

Example 3.8. Assume that $d \ge 3$, and $\mathcal{H}$ is a d-uniform simple hypergraph with vertex set $V(\mathcal{H}) = \bigcup_{i=1}^{k} \bigcup_{j=1}^{d-1} \{x_{i,j}\} \cup \{x\}$ and edge set $\mathcal{E}(\mathcal{H}) = \{E_i = \{x_{i,1}, \dots, x_{i,d-1}, x\} : 1 \le i \le k\}$. One can easily check that $\mathcal{H}$ is a $(\mathcal{C}_2, \mathcal{C}_5)$-free vertex-decomposable hypergraph, and $\{E_1, \dots, E_k\}$ is the semi-induced matching of $\mathcal{H}$ such that $c'_{\mathcal{H}} = |\bigcup_{i=1}^{k} E_i| - k$. Hence, by Theorem 3.6, we have

$$\mathrm{reg}(R/I_{\Delta_{\mathcal{H}}}) \le k(d-2) + 1.$$

We remark that this is an improvement over the bound obtained for this hypergraph by [9, Theorem 1.2]. For one may find out that the only 2-collage of $\mathcal{H}$ is $\{E_1, \dots, E_k\}$. So, in the light of [9, Theorem 1.2], we have

$$\mathrm{reg}(R/I_{\Delta_{\mathcal{H}}}) \le k(d-1).$$

The above example illustrates that for $d \ge 3$ and large enough values of k, the upper bound on $\mathrm{reg}(R/I_{\Delta_{\mathcal{H}}})$ presented in [9, Theorem 1.2] in this special case is much larger than the actual value of $\mathrm{reg}(R/I_{\Delta_{\mathcal{H}}})$, and our upper bound in Theorem 3.6 is better than the one given in [9, Theorem 1.2]. Note that the proof of [9, Lemma 3.4] has some flaws, and so the proof of [9, Theorem 1.2] will be uncertain. The following example shows this defect.

Example 3.9. Assume that $\mathcal{H}$ is a hypergraph with vertex set $V = \{a, b, c, d, e, f\}$ and edge set $\{\{a, b\}, \{a, c\}, \{d, f\}, \{e, f\}, \{b, c, d, e\}\}$. With the notations in [9], for each edge E of $\mathcal{H}$, let $\mathcal{H}_E$ be the hypergraph whose edge set consists of the minimal (under inclusions) members of $\{E \cup E' : E' \neq E$ is an edge of $\mathcal{H}\}$. So, by considering $E = \{a, b\}$, the edge set of $\mathcal{H}_E$ is $\{\{a, b, c\}, \{a, b, d, f\}, \{a, b, e, f\}\}$. Now, one can easily see that $\{E_0 = \{b, c, d, e\}\}$ is a 2-collage for $\mathcal{H}$, but $\{E \cup E_0\}$ is not even an edge of $\mathcal{H}_E$ and it doesn't contain any 2-collage of $\mathcal{H}_E$. Also, for each choice of $E \neq E_0$, the above assertion holds. This shows the mentioned defect of the proof of [9, Lemma 3.4].

In the situation when $c_{\mathcal{H}} = c'_{\mathcal{H}}$, by [16, Corollary 3.9(a)] together with Theorem 3.6, we have that

$$\operatorname{reg}(R/I_{\Delta_{\mathcal{H}}}) = c_{\mathcal{H}} = c'_{\mathcal{H}}.$$

It is thus natural to ask: for what hypergraphs does the equality $c_{\mathcal{H}} = c'_{\mathcal{H}}$ hold? With this point of view, [16, Corollary 3.9(a)], Proposition 2.7 and Theorems 2.4 and 3.6 imply the following corollaries, which characterize $\operatorname{reg}(R/I_{\Delta_{\mathcal{H}}})$ precisely in terms of combinatorial invariants.

Corollary 3.10 (compare [13, Theorem 2.4]). *If $\mathcal{H}$ is a $(\mathcal{C}_2, \mathcal{C}_5)$-free vertex-decomposable hypergraph such that $c_{\mathcal{H}} = c'_{\mathcal{H}}$, then*

$$\operatorname{reg}(R/I_{\Delta_{\mathcal{H}}}) = c_{\mathcal{H}}.$$

In particular, we recover the following.

Corollary 3.11 (see [13, Theorem 2.4]). *If G is a simple C_5-free vertex-decomposable graph, then*

$$\operatorname{reg}(R/I_{\Delta_G}) = c_G.$$

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