Publ. Math. Debrecen
91/3-4 (2017), 441–454
DOI: 10.5486/PMD.2017.7761

Weakly stretch Finsler metrics

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Abstract. In this paper, we introduce a new non-Riemannian quantity named mean stretch curvature. A Finsler metric with vanishing mean stretch curvature is called weakly stretch metric. This class of Finsler metrics contains the class of stretch metrics. First, we show that every complete weakly stretch Finsler manifold with bounded mean Cartan torsion is a weakly Landsberg manifold. Then, we prove a rigidity theorem stating that every compact weakly stretch manifold with negative flag curvature reduces to a Riemannian manifold. Finally, we show that every generalized Berwald Randers metric with a Killing form β with respect to α is a weakly stretch metric if and only if it is a Berwald metric.

1. Introduction

In Finsler geometry, there are several important non-Riemannian quantities: the Cartan torsion $\mathbf{C}$, the Berwald curvature $\mathbf{B}$, the Landsberg curvature $\mathbf{L}$, the mean Landsberg curvature $\mathbf{J}$ and the stretch curvature $\mathbf{\Sigma}$, etc. (see, [9], [21], [17] and [28]). They all vanish for Riemannian metrics, hence they are said to be non-Riemannian. These non-Riemannian geometric quantities describe the difference between Finsler geometry and Riemann geometry. The study of these quantities is benefit for us to make out their distinction and the nature of Finsler geometry.

Let (M, F) be a Finsler manifold. There are two basic tensors on Finsler manifolds: the fundamental metric tensor $\mathbf{g}_y$ and the Cartan torsion $\mathbf{C}_y$, which are second and third order derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$, respectively. The rate of change of the Cartan torsion along geodesics, $\mathbf{L}_y$ is said to be Landsberg

Mathematics Subject Classification: 53B40, 53C60.
Key words and phrases: stretch metric, Landsberg metric, generalized Berwald metric, Randers metric, flag curvature.

curvature. Taking trace with respect to $\mathbf{g}_y$ in first and second variables of $\mathbf{C}_y$ and $\mathbf{L}_y$ gives rise to mean Cartan torsion $\mathbf{I}_y$ and mean Landsberg curvature $\mathbf{J}_y$, respectively. The mean Landsberg curvature is the rate of change of the mean Cartan torsion along geodesics.

In [3], L. Berwald introduced a non-Riemannian curvature so-called stretch curvature and denoted it by $\mathbf{\Sigma}_y$. He showed that this tensor vanishes if and only if the length of a vector remains unchanged under the parallel displacement along an infinitesimal parallelogram. Then, this curvature investigated by Shibata [16] and Matsumoto [6]. A Finsler metric is said to be stretch metric if $\mathbf{\Sigma} = 0$. Taking trace with respect to $\mathbf{g}_y$ in first and second variables of $\mathbf{\Sigma}_y$ gives rise to *mean stretch curvature* $\bar{\mathbf{\Sigma}}_y$. A Finsler metric is said to be a *weakly stretch metric* if $\bar{\mathbf{\Sigma}} = 0$. By definition, every weakly Landsberg metric is a weakly stretch metric. It is interesting to find some topological condition on the manifold M such that every weakly stretch metric on M reduces to a weakly Landsberg metric.

Theorem 1.1. *Every complete weakly stretch manifold with bounded mean Cartan torsion is weakly Landsbergian.*

In Finsler geometry, it is natural to find the geometric assumptions under which a Finsler manifold reduces to a Riemannian manifold. The best result towards this question is due to H. Akbar-Zadeh, who proved that every compact Finsler manifold with negative constant flag curvature is Riemannian [1]. Recently, Shen proved that a closed Finsler manifold with negative flag curvature and constant S-curvature must be Riemannian [15]. He also proved that if a weakly Landsberg manifold is of non-zero constant flag curvature, then it must be Riemannian [15]. Then Wu extended this result and proved that any closed weakly Landsberg manifold with negative flag curvature is Riemannian [31]. In this paper, we generalize their results to weakly stretch metrics as follows.

Theorem 1.2. *Every compact weakly stretch manifold with negative flag curvature is a Riemannian manifold.*

A Finsler manifold (M, F) is called a generalized Berwald manifold if there exists a covariant derivative ∇ on M such that the parallel translations induced by ∇ preserve the Finsler function F (see [19], [29] and [30]). If the covariant derivative ∇ is also torsion-free, then (M, F) is called a Berwald manifold. In [30], Vincze showed that a Randers manifold is a generalized Berwald manifold if and only if the dual vector field of the perturbating term is of constant Riemannian length. In this paper, we study generalized Berwald Randers metric and prove the following.

Theorem 1.3. *Let $F=\alpha+\beta$ be a Randers metric on a manifold M, and let β be a Killing 1-form with respect to α. Suppose that F is generalized Berwald metric. Then F is a weakly stretch metric if and only if F is a Berwald metric.*

Throughout this paper, we use the Berwald connection on Finsler manifolds (see [4], [5], [18] and [20]). The h- and v-covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively.

2. Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x\in M$, and by $TM=\cup_{x\in M}T_xM$ the tangent bundle of M. A Finsler metric on M is a function $F: TM\to[0,\infty)$ which has the following properties:

(i) F is C^∞ on $TM_0:=TM\setminus\{0\}$;

(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM,

(iii) for each $y\in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u,v):=\frac{1}{2}\frac{\partial^2}{\partial s\partial t}\left[F^2(y+su+tv)\right]|_{s,t=0},\quad u,v\in T_xM.$$

Let $x\in M$ and $F_x:=F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y: T_xM\otimes T_xM\otimes T_xM\to\mathbb{R}$ by

$$\mathbf{C}_y(u,v,w):=\frac{1}{2}\frac{d}{dt}\left[\mathbf{g}_{y+tw}(u,v)\right]|_{t=0},\quad u,v,w\in T_xM.$$

The family $\mathbf{C}:=\{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C}=0$ if and only if F is Riemannian. For $y\in T_xM_0$, define the mean Cartan torsion $\mathbf{I}_y$ by $\mathbf{I}_y(u):=I_i(y)u^i$, where $I_i:=g^{jk}C_{ijk}$.

There is a notion of distortion $\tau=\tau(x,y)$ on TM associated with the Busemann–Hausdorff volume form $dV=\sigma(x)dx$ of the Finsler metric F, which is defined by

$$\tau(x,y)=\ln\frac{\sqrt{\det(g_{ij}(x,y))}}{\sigma(x)}.$$

It is well known that $I_i=\frac{\partial\tau}{\partial y^i}$ (see [23]).

Let $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ be a Riemannian metric, and $\beta=b_i(x)y^i$ be a 1-form on M with $||\beta||_\alpha:=b=<1$. Then $F=\alpha+\beta$ is called a Randers metric. To characterize Randers metric among Finsler metrics, we introduce the Matsumoto torsion. Let (M, F) be an n-dimensional Finsler manifold. For $y \in T_xM_0$, define the Matsumoto torsion $\mathbf{M}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{M}_y(u, v, w) := M_{ijk}(y)u^i v^j w^k$, where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\Big\{I_i h_{jk} + I_j h_{ik} + I_k h_{ij}\Big\},$$

and $h_{ij} := g_{ij} - \frac{1}{F^2} g_{ip} y^p g_{jq} y^q$ is the angular metric. A Finsler metric F is said to be C-reducible if $\mathbf{M}_y = 0$. Matsumoto proves that every Randers metric satisfies $\mathbf{M}_y = 0$. Later on, Matsumoto–Hōjō prove that the converse is true, too.

Lemma 2.1 ([7]). *A Finsler metric F on a manifold of dimension $n \geq 3$ is a Randers metric if and only if $\mathbf{M}_y = 0$, $\forall y \in TM_0$.*

The horizontal covariant derivatives of $\mathbf{C}$ along geodesics give rise to the Landsberg curvature $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ defined by $\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^i v^j w^k$, where $L_{ijk} := C_{ijk|s} y^s$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y \in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L=0}$. The horizontal covariant derivatives of $\mathbf{I}$ along geodesics give rise to the mean Landsberg curvature $\mathbf{J}_y(u) := J_i(y)u^i$, where $J_i := g^{jk} L_{ijk} = I_{i|s} y^s$. A Finsler metric is said to be weakly Landsbergian if $\mathbf{J} = 0$ (see [11] and [12]).

Define the stretch curvature $\mathbf{\Sigma}_y : T_xM \otimes T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{\Sigma}_y(u, v, w, z) := \Sigma_{ijkl}(y)u^i v^j w^k z^l$, where

$$\Sigma_{ijkl} := 2(L_{ijk|l} - L_{ijl|k}).$$

A Finsler metric is said to be stretch metric if $\mathbf{\Sigma} = 0$ [3]. Every Landsberg metric is a stretch metric. It is well known that $\mathbf{\Sigma} = 0$ if and only if the length of a vector remains unchanged under the parallel displacement along an infinitesimal parallelogram. Taking an average on the two first indices of the stretch curvature, we get a new non-Riemannian curvature, namely, *mean stretch curvature*.

Definition 2.1. For $y \in T_xM_0$, define $\bar{\mathbf{\Sigma}}_y : T_xM \otimes T_xM \to \mathbb{R}$ by $\bar{\mathbf{\Sigma}}_y(u, v) := \bar{\Sigma}_{ij}(y)u^i v^j$, where $\bar{\Sigma}_{ij} := g^{kl}\Sigma_{klij}$. A Finsler metric is said to be a *weakly stretch metric* if $\bar{\mathbf{\Sigma}} = 0$.

It is easy to see that every Landsberg metric or stretch metric is a weakly stretch metric.

Given a Finsler manifold (M, F), a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$, where

$$G^i(x, y) := \frac{1}{4} g^{il}\Bigg\{\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l}\Bigg\}.$$

The vector field $\mathbf{G}$ is called the associated spray to (M, F). In local coordinates, a curve $c = c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$ (see [10]).

The notion of Riemann curvature for Riemann metrics can be extended to Finsler metrics and sprays. For a vector $y \in T_xM_0$, the Riemann curvature $\mathbf{R}_y : T_xM \to T_xM$ is defined by

$$\mathbf{R}_y(u) = R^i_k(y)u^k \frac{\partial}{\partial x^i},$$

where

$$R^i_k(y) := 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}$$

Take an arbitrary plane $P \subset T_xM$ (flag) and a non-zero vector $y \in P$ (flag pole), the flag curvature $\mathbf{K} = \mathbf{K}(P, y)$ is defined by

$$\mathbf{K}(P, y) := \frac{\mathbf{g}_y\big(\mathbf{R}_y(v), v\big)}{\mathbf{g}_y(y, y)\mathbf{g}_y(v, v) - \mathbf{g}_y(v, y)\mathbf{g}_y(v, y)},$$

where v is an arbitrary vector in P such that $P = \mathrm{span}\{y, v\}$. F is said to be of scalar curvature if, for any non-zero vector $y \in T_xM_0$ and any flag $P \subset T_xM$, $x \in M$, with $y \in P$, $\mathbf{K}(P, y) = \lambda(x, y)$ is independent of P, or equivalently,

$$\mathbf{R}_y = \lambda(x, y)F^2(y)\big\{I - g_y(y, .)y\big\}, \quad y \in T_xM, \quad x \in M,$$

where $I : T_xM \to T_xM$ denotes the identity map and $\mathbf{g}_y(y, .) = \frac{1}{2}[F^2]_{y^i}dy^i$. It is said to be of constant curvature λ if the above identity holds for the constant λ.

For a vector $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $\mathbf{E}_y : T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y)u^jv^kw^l \frac{\partial}{\partial x^i}|_x$ and $\mathbf{E}_y(u, v) := E_{jk}(y)u^jv^k$, where

$$B^i{}_{jkl}(y) := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}(y), \quad E_{jk}(y) := \frac{1}{2}B^m{}_{jkm}(y),$$

$u = u^i\frac{\partial}{\partial x^i}|_x$, $v = v^i\frac{\partial}{\partial x^i}|_x$ and $w = w^i\frac{\partial}{\partial x^i}|_x$. $\mathbf{B}$ and $\mathbf{E}$ are called the Berwald curvature and mean Berwald curvature, respectively (see [8]). A Finsler metric is called a Berwald metric and a mean Berwald metric if $\mathbf{B} = 0$ and $\mathbf{E} = 0$, respectively.

3. Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. For this, we need the following.

Lemma 3.1. *Let (M, F) be a Finsler manifold. Suppose that F is a weakly stretch metric. Then, for any geodesic $c = c(t)$ and any parallel vector field $V = V(t)$ along c, the function $\mathbf{I}(t) := \mathbf{I}_{\dot{c}}(V(t))$ must be in the following form:*

$$\mathbf{I}(t) = t\ \mathbf{J}(0) + \mathbf{I}(0). \tag{1}$$

Proof. By definition, we have

$$\bar{\Sigma}_{ij} = 2(J_{i|j} - J_{j|i}).$$

By assumption, F is weakly stretch metric. Then

$$J_{i|j} = J_{j|i}. \tag{2}$$

Contracting (2) with y^j, we have

$$J_{i|j}y^j = 0. \tag{3}$$

Let

$$J(t) := J_{\dot{c}}(V(t)). \tag{4}$$

From our definition of $\mathbf{J}_y$, we have $\mathbf{J}(t) = \mathbf{I}^{'}(t)$. Then by (4), we obtain

$$\mathbf{I}^{''}(t) = \mathbf{J}^{'}(t) = J_{i|l}(\dot{c}(t))\dot{c}^l(t)V^i(t) = 0. \tag{5}$$

Then (5) yields (1). □

Remark 3.1. Suppose that F is a weakly stretch metric, i.e., $J_{i|j} = J_{j|i}$. Then, we have $J_{i|j}y^j = 0$, which means that the rate of change of the mean Landsberg curvature is constant along any geodesic.

Remark 3.2. Let (M, F) be a Finsler space and $c : [a, b] \to M$ be a geodesic. For a parallel vector field $V(t)$ along c,

$$\mathbf{g}_{\dot{c}}\big(V(t), V(t)\big) = \text{constant}. \tag{6}$$

PROOF OF THEOREM 1.1. Let (M, F) be a complete Finsler manifold. Suppose that F is a weakly stretch metric. Take an arbitrary unit vector $y \in T_xM$ and an arbitrary vector $v \in T_xM$. Let $c = c(t)$ be the geodesic with $c(0) = x$ and $\dot{c}(0) = y$, and $V(t)$ be the parallel vector field along c with $V(0) = v$. Then by Lemma 3.1, we get

$$\mathbf{I}(t) = t\ \mathbf{J}(0) + \mathbf{I}(0). \tag{7}$$

Suppose that I_y is bounded, i.e., there is a constant $A < \infty$ such that

$$||\mathbf{I}||_x := \sup_{y \in T_xM_0} \sup_{v \in T_xM} \frac{I_y(v)}{[\mathbf{g}_y(v,v)]^{\frac{3}{2}}} \leq A. \tag{8}$$

By Remark 3.2, we have $|\mathbf{I}(t)| \leq AQ^{\frac{3}{2}} < \infty$ for some constant Q. Therefore, $\mathbf{I}(t)$ is a bounded function on $(-\infty, \infty)$. Thus, letting $t \to \pm\infty$ in (7) implies that $\mathbf{J}_y(v) = \mathbf{J}(0) = 0$. Hence, F is a weakly Landsberg metric. □

4. Proof of Theorem 1.2

In this section, we are going to prove Theorem 1.2. First, we must mention that weakly stretch curvature has a delicate relation with the flag curvature. Indeed, we discover a relation between the distortion and the flag curvature on weakly stretch manifolds of scalar flag curvature $\mathbf{K} = \mathbf{K}(x, y)$. Indeed, we have

Proposition 4.1. *Let (M, F) be an n-dimensional Finsler manifold. Suppose that F is a weakly stretch metric of non-zero scalar flag curvature $\mathbf{K} = \mathbf{K}(x, y)$. Then $\mathbf{K}$ is given by following*

$$\mathbf{K} = \epsilon e^{\frac{n+1}{3}(\sigma - \tau)}, \tag{9}$$

where $\epsilon = \pm 1$ depending on the sign of $\mathbf{K}$, $\sigma = \sigma(x)$ is a scalar function on M and τ is the distortion of F.

PROOF. For Finsler metric F of scalar flag curvature $\mathbf{K}$, the following holds:

$$L_{ijk|s}y^s + \mathbf{K}F^2C_{ijk} + \frac{1}{3}F^2\Big\{h_{ij}\mathbf{K}_k + h_{jk}\mathbf{K}_i + h_{ki}\mathbf{K}_j\Big\} = 0, \tag{10}$$

where $\mathbf{K}_i = \frac{\partial \mathbf{K}}{\partial y^i}$ (for more details, see [1]). Multiplying (10) with g^{jk} and using $J_{i|s}y^s = 0$, we get

$$\mathbf{K}I_i + \frac{3}{n+1}\mathbf{K}_i = 0. \tag{11}$$

Taking into account $I_i = \frac{\partial \tau}{\partial y^i}$, we get the following:

$$\left\{\tau + \frac{3}{n+1}\ln(\epsilon\mathbf{K})\right\}_{y^i} = 0. \tag{12}$$

Thus, for some scalar function $\sigma = \sigma(x)$ on M, we have

$$\tau + \frac{3}{n+1}\ln(\epsilon\mathbf{K}) = \sigma, \tag{13}$$

and consequently we get (9). □

Proof of Theorem 1.2. Let F be a weakly stretch metric of negative flag curvature on a compact manifold M. Define $f := F^2 I_i I^i$, where I_i is the mean Cartan torsion of F. The scalar function f is homogeneous of degree zero on TM_0. It is known that $\pounds_G(F) = 0$, i.e., F is constant on every geodesic. Therefore, we have

$$\pounds_G(f) = f_{|s}y^s = F^2 I^i I_{i|s}y^s + F^2 I^i{}_{|s}y^s I_i = 2F^2 J^i I_i = 0, \tag{14}$$

where we have used Theorem 1.1. (14) means that f is constant along geodesics of F. Using a Ricci identity given in [15], we get

$$f_{,p}R^p_i + f_{,i|p|q}y^p y^q = 0. \tag{15}$$

Let $c : \mathbb{R} \to M$ be an arbitrary unit speed geodesic, and put

$$\phi(t) := f_{,i}f_{,j}g^{ij}\big(c(t), \dot{c}(t)\big).$$

It is easy to see that

$$\phi''(t) = 2f_{,i|p|q}f_{,j}\dot{c}^p\dot{c}^q g^{ij} + 2f_{,i|p}f_{,j|q}\dot{c}^p\dot{c}^q g^{ij}. \tag{16}$$

Plugging (15) into (16), we get

$$\phi''(t) = -2R^i_k f_{,i}f_{,j}g^{jk} + 2f_{,i|p}f_{,j|q}\dot{c}^p\dot{c}^q g^{ij}. \tag{17}$$

Since F has negative flag curvature, we have $\phi''(t) \geq 0$. It means that ϕ is a convex function. Therefore, for every t_0, we have

$$\phi(t) \geq \phi(t_0) + \phi'(t_0)(t - t_0), \quad \forall t \in \mathbb{R}. \tag{18}$$

If $\phi'(t_0) \neq 0$ for some t_0, then letting $t \to \infty$ or $t \to -\infty$ implies that ϕ is an unbounded function which is a contradiction with compactness of M. Thus, ϕ' is zero function, and consequently $\phi'' = 0$. It follows form (17) that

$$R^i_k f_{,i} f_{,j} g^{jk} = f_{,i|p} f_{,j|q} \dot{c}^p \dot{c}^q g^{ij} = 0. \tag{19}$$

The non-negatively curved condition and the arbitrariness of the geodesic c imply that $f_{,i} = 0$. It means that f is a function of position. From (14), we get

$$\frac{\partial f}{\partial x^i} y^i = 0.$$

Thus, $\frac{\partial f}{\partial x^i} = 0$, and as a result f is a constant. We recall that $I_i = \frac{\partial \tau}{\partial y^i}$. For a fixed point $x_0 \in M$, the distortion attains its extremum on indicatrix of F at x_0. At this point f vanishes, and constancy of f implies that $f = 0$. The proof follows from Deicke's theorem. □

One can relax the topological condition from Theorem 1.2 and still get the same result under a stronger condition on the flag curvature. More precisely, we have the following.

Corollary 4.1. *Every weakly stretch manifold with non-zero constant flag curvature is a Riemannian manifold.*

Proof. Multiplying (10) with g^{jk} and using $J_{i|s} y^s = 0$ and $\mathbf{K}_i = 0$, we get $\mathbf{K} I_i = 0$. By Deicke's theorem, it follows that F is Riemannian. □

A Finsler space is said to be R-quadratic if the Riemannian curvature $\mathbf{R}_y$ of Berwald connection is quadratic in $y \in T_x M$. Here, we prove that every R-quadratic Finsler manifold is a stretch metric and get the following.

Corollary 4.2. *Every R-quadratic Finsler manifold of non-zero scalar flag curvature with dimension $n \geq 3$ is a Riemannian manifold of constant curvature.*

Proof. Indeed, a Finsler metric is R-quadratic if and only if the h-curvature of the Berwald connection depends on position only in the sense of Bácsó–Matsumoto [2]. We have the following Bianchi identity

$$R^h{}_{mij,k} = B^h{}_{mjk|i} - B^h{}_{mik|j}. \tag{20}$$

Contracting (20) with y_h yields

$$y_h R^h{}_{mij,k} = y_h B^h{}_{mjk|i} - y_h B^h{}_{mik|j} = -2L_{mjk|i} + 2L_{mik|j} = \Sigma_{mikj}. \tag{21}$$

Then, every R-quadratic Finsler metric is a stretch metric. In [8], it is proved that every R-quadratic Finsler manifold of scalar flag curvature with dimension $n \geq 3$ is of constant flag curvature. By Corollary 4.1, we get the proof. □

5. Proof of Theorem 1.3

In this section, we are going to prove Theorem 1.3. Suppose that $F=\alpha+\beta$ is a Randers metric, where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta=b_i(x)y^i$ is a 1-form on a manifold M. First, we find the stretch curvature and the mean stretch curvature of F. Then, we get the following.

Lemma 5.1. *Let $F=\alpha+\beta$ be a Randers metric. Then the stretch and the mean stretch curvatures are given by following:*

$$\begin{aligned}\Sigma_{ijkm} &= \frac{2}{n+1}\Big\{J_{i|m}h_{jk}-J_{i|k}h_{jm}+J_{j|m}h_{ik}-J_{j|k}h_{im}\Big\}\\ &\quad-\frac{2}{n+1}\Big\{(J_{m|k}-J_{k|m})h_{ij}+2J_kL_{ijm}-2J_mL_{ijk}\Big\}, \qquad (22)\\ \bar{\Sigma}_{km} &= 2(J_{k|m}-J_{m|k}). \qquad (23)\end{aligned}$$

Proof. By Lemma 2.1, F is C-reducible:

$$C_{ijk}=\frac{1}{n+1}\Big\{I_ih_{jk}+I_jh_{ik}+I_kh_{ij}\Big\}. \qquad (24)$$

Taking a horizontal derivation of (24) implies that

$$L_{ijk}=\frac{1}{n+1}\Big\{J_ih_{jk}+J_jh_{ik}+J_kh_{ij}\Big\}. \qquad (25)$$

Since $h_{ij}=g_{ij}-F^{-2}y_iy_j$, $F_{|l}=0$ and $y_{i|l}=0$, we have

$$h_{ij|l}=g_{ij|l}=-2L_{ijl}.$$

Then by taking a horizontal derivation of (25), we get

$$\begin{aligned}L_{ijk|m} &= \frac{1}{n+1}\Big\{J_{i|m}h_{jk}+J_{j|m}h_{ik}+J_{k|m}h_{ij}\Big\}\\ &\quad-\frac{2}{n+1}\Big\{J_iL_{jkm}+J_jL_{ikm}+J_kL_{ijm}\Big\}. \qquad (26)\end{aligned}$$

By definitions of stretch and mean stretch curvatures, and by relations (25) and (26), we get (22) and (23). □

It is known that for a Randers metric $\mathbf{L}=0$ if and only if $\mathbf{J}=0$ if and only if $\mathbf{B}=0$. On the other hand, Randers metrics have bounded Cartan and mean Cartan torsions [24]. Therefore, by Theorem 1.1, we have the following.

Corollary 5.1. *Let $F=\alpha+\beta$ be a complete Randers metric on a manifold M. Then F is a weakly stretch metric if and only if F is a Berwald metric.*

Remark 5.1. In [27], it is proved that if a Randers metric F is a Douglas metric, then F is a stretch metric if and only if F is a Berwald metric.

Let $\nabla\beta = b_{i;j}dx^i\otimes dx^j$ be the covariant derivative of β with respect to α. Put

$$s_{ij}=\frac{1}{2}\Big(b_{i;j}-b_{j;i}\Big),\quad r_{ij}=\frac{1}{2}\Big(b_{i;j}+b_{j;i}\Big),$$
$$s^i_j=a^{ik}s_{kj},\quad s_j=b^is_{ij},\quad r_j=b^ir_{ij},\quad r_{i0}:=r_{ij}y^j,$$
$$s_{i0}:=s_{ij}y^j,\quad r_0:=r_jy^j,\quad s_0:=s_jy^j,\quad e_{ij}:=r_{ij}+b_is_j+b_js_i.$$

The mean Landsberg curvature of a Randers metric $F=\alpha+\beta$ on an n-dimensional manifold M is given by the following:

$$J_i=\frac{n+1}{4\alpha^2F^2}\bigg\{2\alpha\Big[(e_{i0}\alpha^2-y_ie_{00})-2\beta(s_i\alpha^2-y_is_0)+s_{i0}(\alpha^2+\beta^2)\Big]+4\alpha^2\beta s_{i0}$$
$$+\alpha^2(e_{i0}\beta-b_ie_{00})+\beta(e_{i0}\alpha^2-y_ie_{00})-2(s_i\alpha^2-y_is_0)(\alpha^2+\beta^2)\bigg\}. \tag{27}$$

Lemma 5.2. *Let $F=\alpha+\beta$ be a Randers metric on a manifold M. Then F is a generalized Berwald metric with a Killing form β with respect to α if and only if $r_{ij}=0$ and $s_i=0$.*

Proof. Since β is a Killing form, it has a skew-symmetric covariant derivative with respect to the Levi–Civita connection of α, i.e., $r_{ij}=0$. Suppose that F is a generalized Berwald metric. Then, [30, Theorem 2] implies that the dual vector field of β is of constant length, that is, $r_i+s_i=0$. Using that $r_{ij}=0$ and contracting r_{ij} with b^j, one can get $s_i=0$.

Conversely, $r_{ij}=0$ and $s_i=0$ imply that F is a generalized Berwald manifold with a Killing form β with respect to α. □

Proof of Theorem 1.3. By Lemma 5.2, we have $r_{ij}=0$ and $s_i=0$. In this case, the mean Landsberg curvature and spray of F are given by

$$G^i=\bar{G}^i+\alpha s^i_0, \tag{28}$$
$$J_i=\frac{n+1}{2}\alpha^{-1}s_{i0}, \tag{29}$$

where the index 0 means contracting with y^j. By taking a horizontal derivation of (29), we get

$$\frac{2}{n+1}J_{i|j} = (\alpha^{-1})_{|j}\ s_{i0} + \alpha^{-1}s_{ik|j}y^k. \tag{30}$$

A direct computation shows that

$$(\alpha^{-1})_{|j} = \alpha^{-2}s_{0j}, \tag{31}$$

and

$$s_{ik|j}y^k = y^k\frac{\partial s_{ik}}{\partial x^j} - s_{t0}G^t_{ij} - s_{it}G^t_j. \tag{32}$$

Substituting (31) and (32) into (30) yields

$$\frac{2}{n+1}J_{i|j} = \alpha^{-2}s_{0j}s_{i0} + \alpha^{-1}\left(y^k\frac{\partial s_{ik}}{\partial x^j} - s_{t0}G^t_{ij} - s_{it}G^t_j\right). \tag{33}$$

Putting (28) and (33) into (23), we get

$$\frac{\alpha^2}{n+1}\bar{\Sigma}_{ij} = \alpha\left[y^k\Big(\frac{\partial s_{ik}}{\partial x^j} - \frac{\partial s_{jk}}{\partial x^i}\Big) + s_{it}\bar{G}^t_j - s_{jt}\bar{G}^t_i\right] + t_{j0}y_i - t_{i0}y_j, \tag{34}$$

where $t_{ij} := s_{ik}s^k_j$ and $t_{i0} := t_{ij}y^j$. Decomposing (34) into its rational and irrational parts with respect to (y^i) implies the following

$$y^k\Big(\frac{\partial s_{ik}}{\partial x^j} - \frac{\partial s_{jk}}{\partial x^i}\Big) + s_{it}\bar{G}^t_j - s_{jt}\bar{G}^t_i = 0, \tag{35}$$

$$t_{j0}y_i - t_{i0}y_j = 0. \tag{36}$$

Contracting (36) with b^i and using the assumption $s_i = 0$, we get $t_{j0}\beta = 0$, which implies either $t_{ij} = 0$ or $\beta = 0$. The former and $s_{ij} = 0$ imply that β is parallel with respect to α, and then F is a Berwald metric. This completes the proof. □

Randers metrics belong to a class of Finsler metrics named (α, β)-metrics. An (α, β)-metric is a Finsler metric defined by $F := \alpha\Phi(s)$, $s := \beta/\alpha$, where Φ is a smooth function on a symmetric interval $(-b_0, b_0)$ with certain regularity, α is a Riemannian metric and β is a 1-form on the base manifold (see [22], [25] and [26]). These metrics form an important class of Finsler metrics, appearing iteratively in formulating Physics and Seismology, Biology, Ecology, Control Theory, etc.

If the function

$$A_\Phi(s) := \Phi'(-s)\Phi(s) + \Phi(-s)\Phi'(s)$$

has a fixed sign on a symmetric interval $(-b_0, b_0)$, then we say that the (α,β)-metric $F = \alpha\,\Phi(s)$, $s = \beta/\alpha$, satisfies the sign property [19]. In [19], the second author with Barzegari extend Vincze's Theorem for the class of (α,β)-metrics with sign property. More precisely, they show that every (α,β)-Finsler function with sign property is a generalized Berwald manifold if and only if $\beta^\sharp$ is of constant Riemannian length. Next, we would like to study the class of generalized Berwald (α,β)-metrics with sign property and vanishing mean stretch curvature. Are they Berwaldian metric? This problem remains open.

Acknowledgements. The authors would like to cordially thank Professor László Kozma for his constant encouragement and exact comments. Also, we would like to thank the referees for their careful reading of the manuscript and helpful suggestions.

References

[1] H. Akbar-Zadeh, Sur les espaces de finsler à courbures sectionnelles constantes, *Acad. Roy. Belg. Bull. Cl. Sci. (5)* **74** (1988), 281–322.

[2] S. Bácsó and M. Matsumoto, Finsler spaces with h-curvature tensor dependent on position alone, *Publ. Math. Debrecen* **55** (1999), 199–210.

[3] L. Berwald, Über Parallelübertragung in Räumen mit allgemeiner Maßbestimmung, *Jber. Deutsch. Math.-Verein* **34** (1926), 213–220.

[4] B. Bidabad and A. Tayebi, A classification of some Finsler connections and their applications, *Publ. Math. Debrecen* **71** (2007), 253–266.

[5] B. Bidabad and A. Tayebi, Properties of generalized Berwald connections, *Bull. Iranian Math. Soc.* **35** (2009), 235–252.

[6] M. Matsumoto, An improved proof of Numata and Shibata's theorems on Finsler spaces of scalar curvature, *Publ. Math. Debrecen* **64** (2004), 489–500.

[7] M. Matsumoto and S. Hōjō, A conclusive theorem on C-reducible Finsler spaces, *Tensor (N.S)* **32** (1978), 225–230.

[8] B. Najafi, B. Bidabad and A. Tayebi, On R-quadratic Finsler metrics, *Iran. J. Sci. Technol. Trans. A Sci.* **31** (2007), 439–443.

[9] B. Najafi, Z. Shen and A. Tayebi, Finsler metrics of scalar flag curvature with special non-Riemannian curvature properties, *Geom. Dedicata* **131** (2008), 87–97.

[10] B. Najafi and A. Tayebi, A new quantity in Finsler geometry, *C. R. Math. Acad. Sci. Paris* **349** (2011), 81–83.

[11] E. Peyghan and A. Tayebi, Generalized Berwald metrics, *Turkish J. Math* **36** (2012), 475–484.

[12] E. Peyghan, A. Tayebi and B. Najafi, Doubly warped product Finsler manifolds with some non-Riemannian curvature properties, *Ann. Polon. Math* **105** (2012), 293–311.

[13] L.-I. Pişcoran, From Finsler geometry to noncommutative geometry, *Gen. Math* **12** (2004), 29–38.

[14] L.-I. Pişcoran and V. N. Mishra, S-curvature for a new class of (α,β)-metrics, *Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Math. RACSAM* **111** (2017), 1187-1200.

[15] Z. Shen, Finsler manifolds with nonpositive flag curvature and constant S-curvature, *Math. Z.* **249** (2005), 625–639.

[16] C. Shibata, On the curvature R_{hijk} of Finsler spaces of scalar curvature, *Tensor (N.S.)* **32** (1978), 311–317.

[17] A. Tayebi, On the class of generalized Landsberg manifolds, *Period. Math. Hungar.* **72** (2016), 29–36.

[18] A. Tayebi, A. Azizpour and E. Esrafilian, On a family of connections in Finsler geometry, *Publ. Math. Debrecen* **72** (2008), 1–15.

[19] A. Tayebi and M. Barzegari, Generalized Berwald spaces with (α,β)-metrics, *Indag. Math. (N.S.)* **27** (2016), 670–683.

[20] A. Tayebi and B. Najafi, Shen's processes on Finslerian connections, *Bull. Iranian Math. Soc.* **36** (2010), 57–73.

[21] A. Tayebi and B. Najafi, On isotropic Berwald metrics, *Ann. Polon. Math* **103** (2012), 109–121.

[22] A. Tayebi and A. Nankali, On generalized Einstein Randers metrics, *Int. J. Geom. Methods Mod. Phys.* **12** (2015), 1550105, 14 pp.

[23] A. Tayebi and M. Rafie Rad, S-curvature of isotropic Berwald metrics, *Sci. China Ser. A* **51** (2008), 2198–2204.

[24] A. Tayebi and H. Sadeghi, On Cartan torsion of Finsler metrics, *Publ. Math. Debrecen* **82** (2013), 461–471.

[25] A. Tayebi and H. Sadeghi, On generalized Douglas–Weyl (α,β)-metrics, *Acta Math. Sin. (Engl. Ser.)* **31** (2015), 1611–1620.

[26] A. Tayebi and H. Sadeghi, Generalized P-reducible (α,β)-metrics with vanishing S-curvature, *Ann. Polon. Math* **114** (2015), 67–79.

[27] A. Tayebi and T. Tabatabaeifar, Douglas–Randers manifolds with vanishing stretch tensor, *Publ. Math. Debrecen* **86** (2015), 423–432.

[28] A. Tayebi and T. Tabatabaeifar, Unicorn metrics with almost vanishing $\mathbf{H}$- and $\mathbf{\Xi}$-curvatures, *Turkish J. Math.* **41** (2017), 98–1008.

[29] Cs. Vincze, On generalized Berwald manifolds with semi-symmetric compatible linear connections, *Publ. Math. Debrecen* **83** (2013), 741–755.

[30] Cs. Vincze, On Randers manifolds with semi-symmetric compatible linear connections, *Indag. Math. (N.S.)* **26** (2015), 363–379.

[31] B. Wu, A global rigidity theorem for weakly Landsberg manifolds, *Sci. China Ser. A* **50** (2007), 609–614.

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(Received June 22, 2016; revised January 3, 2017)