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B-spectral theory of linear relations in complex Banach spaces

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Abstract. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two complex Banach spaces. Let A be a multi-valued linear operator (a linear relation) from $\mathfrak{X}$ to $\mathfrak{Y}$, and let B be an everywhere defined bounded operator also from $\mathfrak{X}$ to $\mathfrak{Y}$. Operator B plays the role of a transition operator from $\mathfrak{X}$ to $\mathfrak{Y}$. It is the main goal of the present note to study the basic spectral properties of A linked to the transition operator B.

1. Introduction

Let A and B two closed linear operators in a Banach space $\mathfrak{X}$ with $\operatorname{dom} A \subset \operatorname{dom} B$, where $\operatorname{dom} A$ and $\operatorname{dom} B$ stand for the domains of the definition of A and B, respectively. The set

$$\{\lambda \in \mathbb{C} : \lambda B - A \,\text{has a single valued and bounded inverse on}\, \mathfrak{X}\}$$

is called the B modified resolvent set of A (or simply the B resolvent set of A) and is denoted by $\rho_B(A)$. The bounded operator $(\lambda B - A)^{-1}$ is called the B modified resolvent of A (or simply the B resolvent of A). These notions have been used in the study of degenerate equations on Banach spaces (see [7] and the references therein).

However, a large number of partial differential equations arising in physics and in applied sciences can be only modeled by using two different Banach spaces, let say $\mathfrak{X}$ and $\mathfrak{Y}$, and two different (possible multi-valued) linear operators, let

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say A and B from $\mathfrak{X}$ to $\mathfrak{Y}$. More precisely, assume that A is a multi-valued linear operator (a linear relation) from $\mathfrak{X}$ to $\mathfrak{Y}$, and B is an everywhere defined bounded operator also from $\mathfrak{X}$ to $\mathfrak{Y}$; the operator B can be seen as a transition operator from $\mathfrak{X}$ to $\mathfrak{Y}$. The main goal of the present note consists in the study of the basic spectral properties of A linked to the transition operator B. Section 2 contains some basic material concerning closed multi-valued linear operators (linear relations) in Banach spaces (more details can be found for instance in [1],[6]). In the next section, the notions of B-regular points of A and the B-resolvent set of A are introduced and studied. In Section 4, the B-pseudo-resolvent of A is defined, and some links with the previous notions are established. Finally, the B-spectrum of A is discussed in Section 5.

The results obtained in this note complete the corresponding ones in [2], and they are strongly related to concepts from various spectral problems in applied sciences (for related works see, for instance, [7], [8], [14], [16]). In particular, the study of different types of degenerate equations on Banach complex spaces could be done using the concepts and results obtained in the present note, cf. [7].

Examples to reveal the applicability of our theoretical treatment will be provided in [13]. More precisely, the main results of this note will be applied to study various perturbations of linear relations in Banach spaces in the spirit of the results obtained in [3], [4], [5], [8], [9], [10], [11], [12], [15]. In particular, finite B-rank perturbations and B-compact perturbations of closed linear relations will be studied.

2. Linear relations in complex Banach spaces

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two complex Banach spaces and provide the Cartesian product $\mathfrak{X} \times \mathfrak{Y}$ with the product topology, so that the Cartesian product $\mathfrak{X} \times \mathfrak{Y}$ is also a complex Banach space. A linear relation, or relation for short, A from $\mathfrak{X}$ to $\mathfrak{Y}$ is a linear subspace of the space $\mathfrak{X} \times \mathfrak{Y}$. The notation $L[\mathfrak{X}, \mathfrak{Y}]$ will stand for the class of all linear relations from $\mathfrak{X}$ to $\mathfrak{Y}$. The notations $\operatorname{dom} A$, $\operatorname{ran} A$, $\ker A$ and $\operatorname{mul} A$ stand for the domain, the range, the kernel and the multi-valued part of A:

$$\begin{aligned} \operatorname{dom} A &= \{x \in \mathfrak{X} : \{x, y\} \in A\}, \quad & \operatorname{ran} A &= \{y \in \mathfrak{Y} : \{x, y\} \in A\}, \\ \ker A &= \{x \in \mathfrak{X} : \{x, 0\} \in A\}, \quad & \operatorname{mul} A &= \{y \in \mathfrak{Y} : \{0, y\} \in A\}. \end{aligned}$$

The inverse A^{-1} is a linear relation from $\mathfrak{Y}$ to $\mathfrak{X}$ given by

$$A^{-1} = \{\{y, x\} : \{x, y\} \in A\},$$

so that

$$\mathrm{dom}\, A^{-1} = \mathrm{ran}\, A, \quad \mathrm{ran}\, A^{-1} = \mathrm{dom}\, A, \quad \ker A^{-1} = \mathrm{mul}\, A, \quad \mathrm{mul}\, A^{-1} = \ker A.$$

For linear relations A_1 and A_2 from $\mathfrak{X}$ to $\mathfrak{Y}$, the operator-like sum $A_1 + A_2$ is the linear relation from $\mathfrak{X}$ to $\mathfrak{Y}$ defined by

$$A_1 + A_2 = \{\{x, y_1 + y_2\} : \{x, y_1\} \in A_1,\ \{x, y_2\} \in A_2\},$$

so that $\mathrm{dom}(A_1 + A_2) = \mathrm{dom}\, A_1 \cap \mathrm{dom}\, A_2$ and $\mathrm{mul}(A_1 + A_2) = \mathrm{mul}\, A_1 + \mathrm{mul}\, A_2$. For $\lambda \in \mathbb{C}$, the linear relation λA from $\mathfrak{X}$ to $\mathfrak{Y}$ is defined by

$$\lambda A = \{\{x, \lambda y\} : \{x, y\} \in A\}.$$

Assume that $\mathfrak{Z}$ is also a complex Banach space. For a linear relation A_1 from $\mathfrak{X}$ to $\mathfrak{Z}$ and a linear relation A_2 from $\mathfrak{Z}$ to $\mathfrak{Y}$, the product $A_2 A_1$ is defined as the linear relation from $\mathfrak{X}$ to $\mathfrak{Y}$ by

$$A_2 A_1 = \{\{x, y\} \in \mathfrak{X} \times \mathfrak{Y} : \{x, z\} \in A_1,\ \{z, y\} \in A_2, \text{ for some } z \in \mathfrak{Z}\}.$$

For $\lambda \in \mathbb{C}$, the notation λA agrees in this sense with $(\lambda I)A$. The product of linear relations is associative.

A relation A from $\mathfrak{X}$ to $\mathfrak{Y}$ is closed if A is closed as a subset of $\mathfrak{X} \times \mathfrak{Y}$. It is easy to see that $\ker A$ and $\mathrm{mul}\, A$ are closed linear subspaces of $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. The notation $LC[\mathfrak{X}, \mathfrak{Y}]$ will stand for the class of all closed linear relations from $\mathfrak{X}$ to $\mathfrak{Y}$. The closure of $A \in L[\mathfrak{X}, \mathfrak{Y}]$ will be denoted by $\mathrm{clos}\, A$.

A linear operator B from $\mathfrak{X}$ to $\mathfrak{Y}$ with $\mathrm{dom}\, B \subset \mathfrak{X}$ and $\mathrm{ran}\, B \subset \mathfrak{Y}$ can be seen as a relation if it is identified with its graph: $\{\{x, Bx\} \in \mathfrak{X} \times \mathfrak{Y} : x \in \mathrm{dom}\, B\}$. The operator B is closed if its graph is closed, and it is closable if the closure of its graph is the graph of an operator. Equivalently, an operator B is closable if $\{0, y\} \in \mathrm{clos}\, B$ implies that $y = 0$. An operator B is bounded if it has a bounded norm, that is

$$\|B\| = \sup\{\|Bx\| : x \in \mathrm{dom}\, B,\ \|x\| = 1\} < \infty.$$

The closed graph theorem asserts that a closed linear operator B with $\mathrm{dom}\, B = \mathfrak{X}$ is bounded. The notation $[\mathfrak{X}, \mathfrak{Y}]$ will stand for the class of all linear bounded everywhere defined operators from $\mathfrak{X}$ to $\mathfrak{Y}$. The following well-known result, whose proof can be founded, for instance, in [6], is often useful.

Lemma 2.1. *Let B be a bounded linear operator from the Banach space $\mathfrak{X}$ to the Banach space $\mathfrak{Y}$. Then the following statements hold true:*

(i) *The operator B is closed if and only if* $\operatorname{dom} B$ *is closed.*

(ii) *The operator B is closable and* $\operatorname{clos} B$ *is a bounded operator. Furthermore,* $\|\operatorname{clos} B\| = \|B\|$.

(iii) dom (clos B) = clos (dom B).

Lemma 2.2. *Assume that $\mathfrak{X}$ and $\mathfrak{Y}$ are two complex Banach spaces, $A \in LC[\mathfrak{X}, \mathfrak{Y}]$ and $B \in [\mathfrak{X}, \mathfrak{Y}]$. Then the relations $A - \lambda B$ and $(A - \lambda B)^{-1}$ are also closed.*

Proof. Let $\{x_n, y_n\} \in A - \lambda B$ such that $\{x_n, y_n\} \to \{x, y\} \in \mathfrak{X} \times \mathfrak{Y}$. Then $\{x_n, y_n + \lambda B x_n\} \in A$, $y_n + \lambda B x_n \to y + \lambda B x$. Since A is closed, it follows that $\{x, y + \lambda B x\} \in A$. This implies that $\{x, y\} \in A - \lambda B$. Thus $A - \lambda B$ is a closed relation. Then its inverse $(A - \lambda B)^{-1}$ is also closed. □

3. *B*-resolvent set

Assume that $\mathfrak{X}$ and $\mathfrak{Y}$ are two complex Banach spaces, $A \in L[\mathfrak{X}, \mathfrak{Y}]$ and $B \in [\mathfrak{X}, \mathfrak{Y}]$. The set $\gamma_B(A)$ of B*-regular points* of A is defined by

$$\gamma_B(A) = \{\lambda \in \mathbb{C} : (A - \lambda B)^{-1} \text{ is a bounded operator}\}.$$

Clearly, $\lambda \in \gamma_B(A)$ if and only if there exists a number $r > 0$ depending on λ such that

$$\|y - \lambda B x\| \geq r \cdot \|x\|, \quad \text{for } \{x, y\} \in A,$$

in which case $\|(A - \lambda B)^{-1}\| \leq \dfrac{1}{r}$.

If A is closed and $\lambda \in \gamma_B(A)$, then $(A - \lambda B)^{-1}$ is a closed bounded operator so that $\operatorname{ran}(A - \lambda B) = \operatorname{dom}(A - \lambda B)^{-1}$ is closed by Lemma 2.1.

Conversely, if $\operatorname{ran}(A - \lambda B)$ is closed for some $\lambda \in \gamma_B(A)$, then the bounded operator $(A - \lambda B)^{-1}$ is closed by Lemma 2.1, which implies that A is closed. Moreover, $\gamma_B(\operatorname{clos} A) = \gamma_B(A)$, which is a consequence of the following identity:

$$\operatorname{clos}(A - \lambda B)^{-1} = (\operatorname{clos} A - \lambda B)^{-1}.$$

Theorem 3.1. *Assume that $\mathfrak{X}$ and $\mathfrak{Y}$ are two complex Banach spaces, $A \in LC[\mathfrak{X}, \mathfrak{Y}]$ and $B \in [\mathfrak{X}, \mathfrak{Y}]$. Let $\mu \in \gamma_B(A)$, and let $\lambda \in \mathbb{C}$ such that $|\lambda - \mu| \cdot \|B\| \cdot \|(A - \mu B)^{-1}\| < 1$. Then*

(i) $\lambda \in \gamma_B(A)$ *and*

$$\|(A-\lambda B)^{-1}\| \leq \frac{\|(A-\mu B)^{-1}\|}{1-|\lambda-\mu|\cdot\|(A-\mu B)^{-1}\|\cdot\|B\|}. \tag{3.1}$$

In particular, $\gamma_B(A)$ *is open.*

(ii) $\overline{\operatorname{ran}}\,(A-\lambda B)$ *is not a proper subset of* $\overline{\operatorname{ran}}\,(A-\mu B)$.

Proof. Let $\mu \in \gamma_B(A)$, and let $\{x,y\} \in A$. Since $(A-\mu B)^{-1}$ is a bounded operator, it follows from the identity $(A-\mu B)^{-1}(y-\mu Bx) = x$ that

$$\|(A-\mu B)^{-1}\| \cdot \|y-\mu Bx\| \geq \|x\|. \tag{3.2}$$

For each $\lambda \in \mathbb{C}$ one has

$$\begin{aligned}\|y-\lambda Bx\| &= \|(y-\mu Bx)-(\lambda-\mu)Bx\| \\ &\geq \|y-\mu Bx\| - |\lambda-\mu|\cdot\|Bx\| \\ &\geq \|y-\mu Bx\| - |\lambda-\mu|\cdot\|B\|\cdot\|x\|. \end{aligned}\tag{3.3}$$

A combination of (3.2) and (3.3) leads to

$$\begin{aligned}\|(A-\mu B)^{-1}\|\cdot\|y-\lambda Bx\| &\geq \|(A-\mu B)^{-1}\|\cdot\|y-\mu Bx\| \\ &\qquad - |\lambda-\mu|\cdot\|(A-\mu B)^{-1}\|\cdot\|B\|\cdot\|x\| \\ &\geq \|x\| - |\lambda-\mu|\cdot\|(A-\mu B)^{-1}\|\cdot\|B\|\cdot\|x\| \\ &= (1-|\lambda-\mu|\cdot\|(A-\mu B)^{-1}\|\cdot\|B\|)\cdot\|x\|. \end{aligned}\tag{3.4}$$

Since $\{y-\lambda Bx, x\} \in (A-\lambda B)^{-1}$, inequality (3.4) shows that $(A-\lambda B)^{-1}$ is a bounded operator, whose norm is estimated by (3.1).

(ii) Assume, by contradiction, that $\overline{\operatorname{ran}}\,(A-\lambda B)$ is a proper subset of $\overline{\operatorname{ran}}\,(A-\mu B)$. Let $\alpha \in \mathbb{R}$ such that $|\lambda-\mu|\cdot\|(A-\mu B)^{-1}\|\cdot\|B\| < \alpha < 1$. Using Riesz' Lemma, it follows that there exists an element $y_0 \in \overline{\operatorname{ran}}\,(A-\mu B)$ such that $\|y_0\| = 1$ and $\|y-y_0\| \geq \alpha$ for all $y \in \overline{\operatorname{ran}}\,(A-\lambda B)$. Let $\{y_n\} \subset \operatorname{ran}(A-\mu B)$ be such that $y_n \to y_0$. Then there exists $\{x_n\}$ such that $\{x_n, y_n\} \in A-\mu B$, so that $\{x_n, y_n+(\mu-\lambda)Bx_n\} \in A-\lambda B$. Then

$$\begin{aligned}\alpha &\leq \|y_0-(y_n+(\mu-\lambda)Bx_n)\| \\ &= \|(y_0-y_n)+(\lambda-\mu)Bx_n\| \leq \|y_0-y_n\| + |\lambda-\mu|\cdot\|Bx_n\| \\ &\leq \|y_0-y_n\| + |\lambda-\mu|\cdot\|(A-\mu B)^{-1}\|\cdot\|B\|\cdot\|y_n\|.\end{aligned}$$

Letting $n \to \infty$ in this inequality one has

$$\alpha \leq |\lambda - \mu| \cdot \|(A - \mu B)^{-1}\| \cdot \|B\|.$$

The last inequality contradicts the hypothesis. Hence, $\overline{\operatorname{ran}}\,(A-\lambda B)$ is not a proper subset of $\overline{\operatorname{ran}}\,(A - \mu B)$. □

The *B-resolvent set* $\rho_B(A)$ of $A \in L[\mathfrak{X}, \mathfrak{Y}]$ is defined by

$$\rho_B(A) = \{\lambda \in \mathbb{C} : \overline{\operatorname{ran}}\,(A - \lambda B) = \mathfrak{Y} \text{ and } (A - \lambda B)^{-1} \text{ is a bounded operator}\}.$$

Assume that $\rho_B(A) \neq \emptyset$. Then A is closed if and only if $\operatorname{ran}(A - \lambda B) = \mathfrak{Y}$, for some, and hence for all $\lambda \in \rho_B(A)$. Furthermore, $\rho_B(\operatorname{clos} A) = \rho_B(A)$.

Lemma 3.2. *Assume that $\mathfrak{X}$ and $\mathfrak{Y}$ are two complex Banach spaces, $A \in LC[\mathfrak{X}, \mathfrak{Y}]$ and $B \in [\mathfrak{X}, \mathfrak{Y}]$. If $\mu \in \rho_B(A)$ and $|\lambda - \mu| \cdot \|(A - \mu B)^{-1}\| \cdot \|B\| < 1$, then $\lambda \in \rho_B(A)$. In particular, $\rho_B(A)$ is open.*

Proof. Since $\mu \in \rho_B(A)$, one has $\mu \in \gamma_B(A)$ and $\overline{\operatorname{ran}}\,(A - \mu B) = \mathfrak{Y}$. Hence, $\lambda \in \gamma_B(A)$ and $\overline{\operatorname{ran}}\,(A - \lambda B)$ is not a proper subset of $\overline{\operatorname{ran}}\,(A - \mu B) = \mathfrak{Y}$, cf. Theorem 3.1. Therefore, $\overline{\operatorname{ran}}\,(A - \lambda B) = \mathfrak{Y}$, so that $\lambda \in \rho_B(A)$. □

Let now $A \in LC[\mathfrak{X}, \mathfrak{Y}]$. Then $\rho_B(A)$ is the set of all $\lambda \in \mathbb{C}$ for which $A - \lambda B$ is invertible, in the sense that $\operatorname{ran}(A - \lambda B) = \mathfrak{Y}$ and $\ker(A - \lambda B) = \{0\}$. For each $\lambda \in \rho_B(A)$ it follows that $(A - \lambda B)^{-1} \in [\mathfrak{Y}, \mathfrak{X}]$. This operator is called the *B-resolvent operator* of A.

Theorem 3.3. *Assume that $\mathfrak{X}$ and $\mathfrak{Y}$ are two complex Banach spaces, $A \in LC[\mathfrak{X}, \mathfrak{Y}]$ and $B \in [\mathfrak{X}, \mathfrak{Y}]$.*

(i) *If λ, $\mu \in \mathbb{C}$, then*

$$(A - \lambda B)^{-1} - (A - \mu B)^{-1} = (\lambda - \mu)(A - \lambda B)^{-1} \cdot B \cdot (A - \mu B)^{-1}. \tag{3.5}$$

Furthermore, the B-resolvent operator $(A - \lambda B)^{-1}$ is holomorphic for $\lambda \in \rho_B(A)$.

(ii) *If $\mu \in \rho_B(A)$ and $|\lambda - \mu| \cdot \|(A - \mu B)^{-1}\| \cdot \|B\| < 1$, then*

$$(A - \lambda B)^{-1} = \sum_{j=0}^{\infty} (\lambda - \mu)^j \cdot (A - \mu B)^{-1} \cdot \left(B \cdot (A - \mu B)^{-1}\right)^j. \tag{3.6}$$

Proof. (i) Assume that $\{x, y\} \in (A-\lambda B)^{-1}-(A-\mu B)^{-1}$, so that $\{x, y_1\} \in (A-\lambda B)^{-1}$ and $\{x, y_2\} \in (A-\mu B)^{-1}$ for some y_1, $y_2 \in \mathfrak{Y}$ with $y_1 - y_2 = y$. One has $\{y_1, x\} \in A - \lambda B$ and

$$\{y_2, x\} \in A - \mu B = A - \lambda B + (\lambda - \mu)B.$$

Then $\{y_2, x - (\lambda - \mu)By_2\} \in A - \lambda B$, so that

$$\{y, (\lambda - \mu)By_2\} = \{y_1, x\} - \{y_2, x - (\lambda - \mu)By_2\} \in A - \lambda B.$$

This implies that $\{(\lambda - \mu)By_2, y\} \in (A - \lambda B)^{-1}$, which shows that $\{y_2, y\} \in (\lambda - \mu)(A - \lambda B)^{-1}B$. Hence,

$$\{x, y\} \in (\lambda - \mu)(A - \lambda B)^{-1} \cdot B \cdot (A - \mu B)^{-1}, \tag{3.7}$$

which leads to

$$(A - \lambda B)^{-1} - (A - \mu B)^{-1} \subseteq (\lambda - \mu)(A - \lambda B)^{-1} \cdot B \cdot (A - \mu B)^{-1}. \tag{3.8}$$

Conversely, let $\{x, y\} \in (\lambda - \mu)(A - \lambda B)^{-1} \cdot B \cdot (A - \mu B)^{-1}$, so that $\{x, z\} \in (A - \mu B)^{-1}$, $\{z, w\} \in (\lambda - \mu)B$ and $\{w, y\} \in (A - \lambda B)^{-1}$ for some $z \in \mathfrak{X}$ and $w \in \mathfrak{Y}$. It follows from $\{x, z\} \in (A - \mu B)^{-1}$ that $\{z, x\} \in A - \mu B$, so that

$$\{z, x + (\mu - \lambda)Bz\} \in A - \lambda B. \tag{3.9}$$

Since $w = (\lambda - \mu)Bz$, relation (3.9) implies that $\{z, x - w\} \in A - \lambda B$, so that $\{x - w, z\} \in (A - \lambda B)^{-1}$. Consequently,

$$\{x, z + y\} = \{x - w, z\} + \{w, y\} \in (A - \lambda B)^{-1}.$$

Finally,

$$\{x, y\} = \{x, z + y\} - \{x, z\} \in (A - \lambda B)^{-1} - (A - \mu B)^{-1},$$

so that

$$(\lambda - \mu)(A - \lambda B)^{-1} \cdot B \cdot (A - \mu B)^{-1} \subseteq (A - \lambda B)^{-1} - (A - \mu B)^{-1}. \tag{3.10}$$

A combination of (3.8) and (3.10) leads to (3.5).

Assume now that λ, $\mu \in \rho_B(A)$, $\lambda \neq \mu$. It follows from (3.5) that

$$\frac{(A - \lambda B)^{-1} - (A - \mu B)^{-1}}{\lambda - \mu} = (A - \lambda B)^{-1} \cdot B \cdot (A - \mu B)^{-1}. \tag{3.11}$$

This identity further implies that the resolvent operator $(A-\lambda B)^{-1}$ is holomorphic for $\lambda \in \rho_B(A)$.

(ii) With the notation $R_B(\lambda) = (A-\lambda B)^{-1}$ it follows by induction from (3.5) that

$$\begin{aligned} R_B(\lambda) = &\sum_{j=0}^{n} (\lambda-\mu)^j \cdot R_B(\mu) \cdot (B \cdot R_B(\mu))^j \\ &+ (\lambda-\mu)^{n+1} \cdot R_B(\lambda) \cdot (B \cdot R_B(\mu))^{n+1} . \end{aligned} \tag{3.12}$$

From the estimation

$$\|(\lambda-\mu)^{n+1} \cdot R_B(\lambda) \cdot (B \cdot R_B(\mu))^{n+1}\| \le \|R_B(\lambda)\| \cdot (|\lambda-\mu| \cdot \|R_B(\mu)\| \cdot \|B\|)^{n+1} ,$$

and the inequality $|\lambda-\mu| \cdot \|R_B(\mu)\| \cdot \|B\| < 1$, it follows that the rest term in (3.12) tends to 0 as $n \to \infty$. This completes the proof. □

Equation (3.5) with λ, $\mu \in \rho_B(A)$ is called the B-resolvent identity of A. In the case $\mathfrak{X} = \mathfrak{Y}$ and $B = I$, the classical notion of resolvent identity is obtained.

4. *B*-pseudo-resolvents

Let $B \in [\mathfrak{X}, \mathfrak{Y}]$, and let $\Omega \subset \mathbb{C}$. Assume that for each λ, $\mu \in \Omega$ there exists an operator $R_B(\cdot) \in [\mathfrak{Y}, \mathfrak{X}]$ such that

$$R_B(\lambda) - R_B(\mu) = (\lambda-\mu) \cdot R_B(\lambda) \cdot B \cdot R_B(\mu). \tag{4.1}$$

Such a family of operators $(R_B(\lambda))_{\lambda \in \Omega}$ is called a *B-pseudo-resolvent*.

Theorem 4.1. *Assume that $\mathfrak{X}$ and $\mathfrak{Y}$ are two complex Banach spaces and $B \in [\mathfrak{X}, \mathfrak{Y}]$. Let $\{R_B(\lambda)\}_{\lambda \in \Omega}$ be a B-pseudo-resolvent. Then there exists a unique linear relation $A \in LC[\mathfrak{X}, \mathfrak{Y}]$ such that $\Omega \subset \rho_B(A)$ and $R_B(\lambda) = (A-\lambda B)^{-1}$, $\lambda \in \Omega$. In particular, the B-pseudo-resolvent $R_B(\lambda)$ has a unique maximal extension to $\rho_B(A)$.*

Proof. The linear relation $R_B(\lambda)^{-1} + \lambda B$ does not depend on $\lambda \in \Omega$. To see this, let $\{x, y\} \in R_B(\lambda)^{-1} + \lambda B$, so that $\{x, y - \lambda Bx\} \in R_B(\lambda)^{-1}$. Then $R_B(\lambda)(y - \lambda Bx) = x$. Using (4.1), one has

$$R_B(\mu)(y - \lambda Bx) = (I + (\mu-\lambda) R_B(\mu) B) R_B(\lambda)(y - \lambda Bx),$$

which implies that

$$R_B(\mu)(y - \lambda Bx) = x + (\mu - \lambda)R_B(\mu)Bx.$$

Then $R_B(\mu)(y - \mu Bx) = x$, so that $\{x, y\} \in R_B(\mu)^{-1} + \mu B$. Hence it follows that $R_B(\lambda)^{-1} + \lambda B \subset R_B(\mu)^{-1} + \mu B$. The reverse inclusion follows by symmetry. Hence,

$$R_B(\lambda)^{-1} + \lambda B = R_B(\mu)^{-1} + \mu B.$$

Define the linear relation A by $A = R_B(\lambda)^{-1} + \lambda B$, which is equivalent to $R_B(\lambda) = (A - \lambda B)^{-1}$. Clearly, the relation A is uniquely defined. Since $R_B(\lambda) \in [\mathfrak{Y}, \mathfrak{X}]$ for $\lambda \in \Omega$, this implies that $\lambda \in \rho_B(A)$. Hence $\Omega \subset \rho_B(A)$. □

Theorem 4.2. *Let A be a closed linear relation. Then*

$$\gamma_B(A) \cap \operatorname{clos} \rho_B(A) \subset \rho_B(A).$$

PROOF. Let $\mu \in \gamma_B(A) \cap \operatorname{clos} \rho_B(A)$, and let $(\mu_n) \subset \rho_B(A) \subset \gamma_B(A)$ such that $\mu_n \to \mu$. It follows from inequality (3.1) that the sequence $\left(\|(A - \mu_n B)^{-1}\|\right)$ is bounded. Hence, the resolvent identity implies that

$$\|(A - \mu_n B)^{-1} - (A - \mu_m B)^{-1}\| \to 0, \quad n, m \to \infty.$$

Hence, the B-resolvent $R_B(\lambda) = (A - \lambda B)^{-1}$ has an extension to μ, and the resolvent identity shows that $R_B(\lambda)$ extended to μ is a B-pseudo-resolvent, which implies that $\mu \in \rho_B(A)$. □

5. The B-spectrum

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two complex Banach spaces, $A \in L[\mathfrak{X}, \mathfrak{Y}]$ and $B \in [\mathfrak{X}, \mathfrak{Y}]$. It follows from $A - \lambda B = \{\{x, y - \lambda Bx\} : \{x, y\} \in A\}$ that

$$\ker(A - \lambda B) = \{x : \{x, \lambda Bx\} \in A\}.$$

A complex number $\lambda \in \mathbb{C}$ is said to be a B-eigenvalue of A when there is a non-zero element $x \in \ker(A - \lambda B)$. Furthermore, ∞ is said to be a B-eigenvalue of A when there is a non-zero element $m \in \operatorname{mul} A$. The B-point spectrum $\sigma_{pB}(A)$ of A is the set of all B-eigenvalues $\lambda \in \mathbb{C} \cup \{\infty\}$ of A. It may happen that $\sigma_{pB} = \mathbb{C} \cup \infty$. Indeed, if there is a non-zero element $z \in \mathfrak{X}$ such that $\{z, 0\} \in A$

and $\{0, Bz\} \in A$, then $\{z, \lambda Bz\} \in A$ for any $\lambda \in \mathbb{C}$. When $\lambda \in \mathbb{C}$, the identity $\mathrm{mul}(A - \lambda B)^{-1} = \ker(A - \lambda B)$ implies that

$$\lambda \in \sigma_{pB}(A) \ \Leftrightarrow \ (A - \lambda B)^{-1} \text{ is not an operator.}$$

The B-spectrum $\sigma_B(A)$ of A is defined by $\sigma_B(A) = \mathbb{C} \setminus \rho_B(A)$, and the B-approximative point spectrum (or B-spectral kernel) of A is defined by $\Pi_B(A) = \mathbb{C} \setminus \gamma_B(A)$.

Theorem 5.1. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two complex Banach spaces, $A \in L[\mathfrak{X}, \mathfrak{Y}]$ and $B \in [\mathfrak{X}, \mathfrak{Y}]$. The B-approximative point spectrum $\Pi_B(A)$ of A is contained in the B-spectrum $\sigma_B(A)$ of A, and both sets are closed. Moreover, $\lambda \in \Pi_B(A)$ if and only if there exists a sequence $(\{x_n, y_n\}) \subset A$ such that*

$$\|x_n\| = 1, \quad y_n - \lambda B x_n \to 0, \quad n \to \infty.$$

Proof. It follows from $\rho_B(A) \subset \gamma_B(A)$ that $\Pi_B(A) \subset \sigma_B(A)$. It has been already shown that the sets $\rho_B(A)$ and $\gamma_B(A)$ are open, so that their complements $\sigma_B(A)$ and $\Pi_B(A)$ are closed.

Assume that $\lambda \in \Pi_B(A)$. Then for each $\varepsilon > 0$ there exists an element $\{x_\varepsilon, y_\varepsilon\} \in A$ with $\|x_\varepsilon\| = 1$ and $\|y_\varepsilon - \lambda B x_\varepsilon\| \leq \varepsilon$. This implies the existence of the requested sequence. Conversely, assume that such a sequence exists. Then does not exist a number $\varepsilon_0 > 0$ such that $\|y - \lambda x\| \geq \varepsilon_0 \|x\|$ for all $\{x, y\} \in A$. This shows that $\lambda \in \Pi_B(A)$. □

Theorem 5.1 shows that $\sigma_{pB}(A) \setminus \{\infty\}$ is contained in the B-approximative point spectrum $\Pi_B(A)$. It is possible to separate various points of the spectrum. Observe that for any $\lambda \in \mathbb{C}$ there are three different situations with respect to $\ker(A - \lambda B)$:

K_1. $\ker(A - \lambda B) = \{0\}$, $(A - \lambda B)^{-1}$ is bounded;

K_2. $\ker(A - \lambda B) = \{0\}$, $(A - \lambda B)^{-1}$ is not bounded;

K_3. $\ker(A - \lambda B) \neq \{0\}$.

Similarly, there are three different situations with respect to $\mathrm{ran}(A - \lambda B)$:

R_1. $\mathrm{ran}(A - \lambda B) = \mathfrak{Y}$;

R_2. $\overline{\mathrm{ran}}\,(A - \lambda B) = \mathfrak{Y}$, $\mathrm{ran}(A - \lambda B) \neq \mathfrak{Y}$;

R_3. $\overline{\mathrm{ran}}\,(A - \lambda B) \neq \mathfrak{Y}$.

According to these possibilities, the complex plane $\mathbb{C}$ can be divided into nine mutually disjoint subsets. Furthermore, the following equivalences hold true:

(1) $\lambda \in \gamma_B(A)$ (points of B-regular type) $\Leftrightarrow \lambda \in K_1 \cap (R_1 \cup R_2 \cup R_3)$;

(2) $\lambda \in \Pi_B(A)$ (B-approximative point spectrum) $\Leftrightarrow$
$$\Leftrightarrow \lambda \in (K_2 \cup K_3) \cap (R_1 \cup R_2 \cup R_3);$$
(3) $\lambda \in \rho_B(A)$ (B-resolvent set) $\Leftrightarrow \lambda \in K_1 \cap (R_1 \cup R_2)$;

(4) $\lambda \in \sigma_B(A) \Leftrightarrow \lambda \in ((K_2 \cup K_3) \cap (R_1 \cup R_2 \cup R_3)) \cup (K_1 \cap R_3)$;

(5) $\lambda \in \sigma_{pB}(A)$ (B-point spectrum) $\Leftrightarrow \lambda \in K_3 \cap (R_1 \cup R_2 \cup R_3)$.

Theorem 5.2. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two complex Banach spaces, $A \in LC[\mathfrak{X}, \mathfrak{Y}]$ and $B \in [\mathfrak{X}, \mathfrak{Y}]$. Then the subsets $K_1 \cap R_2$ and $K_2 \cap R_1$ are empty.*

Proof. Assume that $\lambda \in K_1 \cap R_2$, so that $(A - \lambda B)^{-1}$ is a bounded closed operator with a closed domain of definition $\operatorname{dom}(A - \lambda B)^{-1} = \operatorname{ran}(A - \lambda B)$, a contradiction.

Assume now that $\lambda \in K_2 \cap R_1$, so that $(A - \lambda B)^{-1}$ is an unbounded closed operator with the domain of definition $\operatorname{dom}(A - \lambda B)^{-1} = \operatorname{ran}(A - \lambda B) = \mathfrak{Y}$, which leads to a contradiction by the closed graph theorem. $\square$

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