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# Almost simple groups with the socle $PSL(2, p^n)$ are determined by their complex group algebras

By SOMAYEH HEYDARI (Shahre-kord) and NEDA AHANJIDEH (Shahre-kord)

**Abstract.** Let $n$ be a natural number, $p$ be a prime number and $L$ be an almost simple group with the socle $PSL(2, p^n)$. In this paper, we prove that $L$ is uniquely determined by the first column of its character table. As a consequence, we show that $L$ is uniquely determined by its complex group algebra.

## 1. Introduction and notation

Throughout this paper, all groups are finite. Let $G$ be a group, $p$ be a prime number and $n$ be a natural number. $G$ is said to be almost simple if there is a simple group $S$ such that $S \leq G \leq \mathrm{Aut}(S)$. $G$ is called quasisimple if $G = G'$ and $\frac{G}{Z(G)}$ is a non-abelian simple group. The set of prime divisors of $|G|$ forgetting multiplicities is shown by $\pi(G)$. We say that $G$ is a $K_n$-group if $|\pi(G)| = n$. Also, if $G$ is simple, then $G$ is called a simple $K_n$-group. The set of irreducible characters of $G$ is shown by $\mathrm{Irr}(G)$, $X_1(G)$ is the set of irreducible character degrees of $G$ counting multiplicities, and the set of irreducible character degrees of $G$ forgetting multiplicities is denoted by $\mathrm{cd}(G)$. In fact, $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$. In 2000, B. Huppert in [14] conjectured that if $G$ is a group, and $S$ is a non-abelian simple group such that $\mathrm{cd}(G) = \mathrm{cd}(S)$, then for some abelian group $A$, $G \cong S \times A$. Huppert in [14] and [15] proved his conjecture for the Suzuki groups and $PSL(2, q)$, where $q$ is a prime power. In [11], it was proved that some simple $K_4$-groups can be uniquely determined by their orders and two of their

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irreducible character degrees. The set of prime divisors of the elements of $\mathrm{cd}(G)$ is denoted by $\rho(G)$. Let $(a, b)$ be the greatest common divisor of natural numbers $a$ and $b$. The character degree graph of $G$, which is shown by $\Delta(G)$, is a graph with the vertex set $\rho(G)$, and there is an edge between two vertices $a$ and $b$ if there is some $f \in \mathrm{cd}(G)$ such that $ab \mid f$. In [13], the authors proved that some simple groups are determined by their character degree graphs and orders. In [1], a question (No. 126) is posed as:

*Question 1.1.* Let $X_1(G) = X_1(S_n)$. Is it true that $G \cong S_n$? Moreover, if $X_1(G) = X_1(H)$, where $H$ is a simple group, then do we conclude that $G \cong H$?

If $H$ is a group which is not simple, then the second part of Question 1.1 is not necessarily true. For example, $X_1(D_8) = X_1(Q_8)$, while $Q_8 \not\cong D_8$. But this problem, in the case that $H$ is simple, is completely different.

H. P. Tong-Viet in [23], [24], [25] and [26] proved that the answer to Question 1.1 is positive. After that, some authors tried to verify Question 1.1 when $H$ is not simple. For example, in [10] and [12], it was shown that some almost simple groups with the socle $PSL(2, p^n)$ are uniquely determined by their $X_1$.

Let $\mathbb{C}$ be the complex number field. Then the group algebra of $G$ over $\mathbb{C}$ is denoted by $\mathbb{C}G$. By Molien's theorem ([1], Theorem 2.13), for the groups $G$ and $L$, $X_1(G) = X_1(L)$ if and only if $\mathbb{C}G \cong \mathbb{C}L$. Thus it was proved that the non-abelian simple groups and some almost simple groups with the socle $PSL(2, p^n)$ are determined by their complex group algebras.

Also, in [2], [18] and [19], the authors showed that the quasisimple groups, which are not simple, are determined by their complex group algebras. In this paper, we prove that:

**Theorem 1.2.** *Let $L$ be an almost simple group with the socle $PSL(2, p^n)$, and let $G$ be a group. Then $L \cong G$ if and only if $X_1(G) = X_1(L)$.*

As a consequence of Theorem 1.2 and Molien's theorem, it is proved that:

**Corollary 1.3.** *Let $L$ be an almost simple group with the socle $PSL(2, p^n)$. Then for a group $G$, $L \cong G$ if and only if $\mathbb{C}G \cong \mathbb{C}L$.*

Throughout this paper, we use the following notations:

If $\chi = \sum_{i=1}^{t} n_i \chi_i$, where for every $1 \leq i \leq t$, $\chi_i \in \mathrm{Irr}(G)$, then those $\chi_i$ with $n_i > 0$ are called irreducible constituents of $\chi$. Also, if $N$ is a normal subgroup of $G$, then a character $\theta \in \mathrm{Irr}(N)$ is called extendible to $G$ if there is a character $\chi \in \mathrm{Irr}(G)$ such that $\chi_N = \theta$, where $\chi_N$ is the restriction of $\chi$ to $N$. The common divisor graph of $G$, which is denoted by $\Gamma(G)$, is a graph with the vertex set

$\mathrm{cd}(G)-\{1\}$, and two vertices $a$ and $b$ are adjacent to each other if $(a,b)>1$. Let $a$ be a prime, and let $b$ and $m$ be natural. If $a^m \parallel b$, i.e., $a^m \mid b$, but $a^{m+1} \nmid b$, then we write $|b|_a = a^m$. If $G$ is a group of Lie type in characteristic $p$, then $G$ has a character of degree $|G|_p$, which is called the Steinberg character and is denoted by 'St'. Finally, $\Phi_i(q)$ is the value of the $i$-th cyclotomic polynomial evaluated at $q$.

## 2. Preliminaries

In this section, we bring some lemmas which will be used in the proof of Theorem 1.2.

In the following lemma, let $\delta$ be a diagonal automorphism of order 2, and let $\phi$ be a field automorphism of order $n$ of $PSL(2,p^n)$. It is known that

$$\mathrm{Aut}(PSL(2,p^n)) = \begin{cases} PSL(2,p^n)(\langle\delta\rangle\times\langle\phi\rangle), & \text{if } p\neq 2,\\ PSL(2,p^n)\langle\phi\rangle, & \text{if } p=2.\end{cases}$$

Lemma 2.1 determines irreducible character degrees of $PSL(2,q)$ and almost simple groups with the socle $PSL(2,q)$ that we will use this lemma in the proof of Theorem 1.2.

**Lemma 2.1** ([28])**.** *Let $q$ be a prime power.*

(a) *If $q$ is odd, then $\mathrm{cd}(PSL(2,q)) = \{1, q-1, q, q+1, \frac{q+\varepsilon}{2}\}$, where $\varepsilon = (-1)^{\frac{(q-1)}{2}}$.*

(b) *If $q$ is even, then $\mathrm{cd}(PSL(2,q)) = \{1, q-1, q, q+1\}$.*

(c) *Let $S = PSL(2,q)$, where $q = p^n > 3$ for a prime $p$, $A = \mathrm{Aut}(S)$, and let $S \leq G \leq A$. Set $H = PGL(2,q)$ if $\delta \in G$, and $H = S$ if $\delta \notin G$, and let $[G:H] = d = 2^a m$, $m$ odd. Also, if $p$ is odd, then set $\epsilon = (-1)^{\frac{(q-1)}{2}}$. Then*

$$\mathrm{cd}(G) = \left\{1, q, \frac{q+\epsilon}{2}\right\} \cup \{(q-1)2^a l : l \mid m\} \cup \{(q+1)j : j \mid d\},$$

*with the following exceptions:*

(i) *If either $p$ is odd with $G \not\leq S\langle\phi\rangle$ or $p=2$, then $\frac{q+\epsilon}{2}$ is not a degree of $G$.*

(ii) *If $n$ is odd, $p=3$ and $G = S\langle\phi\rangle$, then $l \neq 1$.*

(iii) *If $n$ is odd, $p=3$ and $G = A$, then $j \neq 1$.*

(iv) *If $n$ is odd, $p = 2, 3$ or $5$ and $G = S\langle\phi\rangle$, then $j \neq 1$.*

(v) *If $n \equiv 2 \pmod 4$, $p = 2$ or $3$, and $G = S\langle\phi\rangle$ or $G = S\langle\delta\phi\rangle$, then $j \neq 2$.*

Lemmas 2.2 and 2.3 help us to obtain information about the normal subgroups of a group $G$ according to its irreducible character degrees.

**Lemma 2.2** (Ito's theorem, [16, Theorem 6.15])**.** *Let $A$ be an abelian normal subgroup of $G$. Then $\chi(1) \mid [G : A]$, for all $\chi \in \mathrm{Irr}(G)$.*

**Lemma 2.3** ([16, Lemma 6.8 and Corollary 11.29])**.** *Let $N$ be a normal subgroup of $G$ and $\chi \in \mathrm{Irr}(G)$. Let $\theta$ be an irreducible constituent of $\chi_N$. Then $\theta(1) \mid \chi(1)$ and $\frac{\chi(1)}{\theta(1)} \mid [G : N]$.*

Lemma 2.4 illustrates the structure of a solvable group $G$, when $G'$ is the unique normal minimal subgroup of $G$. This lemma will be used in the proof of Lemma 3.6.

**Lemma 2.4** ([16, Lemma 12.3])**.** *Let $G$ be solvable, and assume that $G'$ is the unique normal minimal subgroup of $G$. Then all non-linear irreducible characters of $G$ have equal degree $f$, and one of the following situations obtains:*

(a) *$G$ is a $p$-group, $Z(G)$ is cyclic and $\frac{G}{Z(G)}$ is elementary abelian of order $f^2$.*

(b) *$G$ is a Frobenius group with an abelian Frobenius complement of order $f$. Also, $G'$ is the Frobenius kernel and is an elementary abelian $p$-group.*

**Lemma 2.5** ([3])**.** *If $\Gamma(G)$ is a complete graph, then $G$ is a solvable group.*

The following lemma shows that alternating groups and simple groups of Lie type have some irreducible character degrees which extend to their corresponding almost simple groups.

**Lemma 2.6.** (a) ([3]) *If $n \geq 6$, then $A_n$ has irreducible characters of degrees $\frac{n(n-3)}{2}$ and $\frac{(n-1)(n-2)}{2}$ that extend to $\mathrm{Aut}(A_n)$.*
(b) (Schmid, [20], [21]) *Let $N$ be a normal subgroup of a group $G$, and let $N$ be isomorphic to a simple group of Lie type. If $\theta$ is the Steinberg character for $N$, then $\theta$ extends to $G$.*

Lemma 2.7 gives us information about extendible irreducible character degrees of a normal minimal subgroup of a group $G$.

**Lemma 2.7** ([3])**.** *Let $N$ be a normal minimal subgroup of $G$ such that for a non-abelian simple group $S$, $N \cong \underbrace{S \times \cdots \times S}_{t}$. Let $A$ be the automorphism group of $S$. If $\sigma \in \mathrm{Irr}(S)$ extends to $A$, then $\underbrace{\sigma \times \cdots \times \sigma}_{t} \in \mathrm{Irr}(N)$ extends to $G$.*

**Lemma 2.8** ([27])**.** *Let $S$ be a simple group of Lie type in characteristic $p$. Then no proper multiple of $\mathrm{St}(1)$ is a character degree of $S$.*

**Lemma 2.9** ([12]). *Let $L$ be an almost simple group with the socle $PSL(2, p^n)$ such that $p \nmid [L : PSL(2, p^n)]$. Then for a group $G$, $L \cong G$ if and only if $X_1(G) = X_1(L)$.*

Lemma 2.10 classifies the simple groups according to their character degree graphs, which will be used in the proof of Theorem 1.2, Steps c–e.

**Lemma 2.10** ([29, Corollary 1.2]). *Let $G$ be a simple group. The graph $\Delta(G)$ is disconnected if and only if $G \cong PSL(2, q)$ for some prime power $q$. If $\Delta(G)$ is connected, then the diameter of $\Delta(G)$ is at most 3 and $\Delta(G)$ is a complete graph except in the following cases:*

(1) *The diameter of $\Delta(G)$ is 3 if and only if $G \cong J_1$.*
(2) *The diameter of $\Delta(G)$ is 2 if and only if $G$ is isomorphic to one of the following groups:*
   (a) *the sporadic Mathieu group $M_{11}$ or $M_{23}$,*
   (b) *the alternating group $A_8$,*
   (c) *the Suzuki group ${}^2B_2(q^2)$, where $q^2 = 2^{2m+1}$ and $m \geq 1$,*
   (d) *the linear group $PSL(3, q)$, where $q > 2$ is even or $q$ is odd, and*
   (e) *the unitary group $PSU(3, q)$, where $q > 2$ and $q + 1$ is divisible by a prime other than 2 or 3.*

Note that among the alternating groups, the only groups whose character degree graphs are disconnected are $A_5$ and $A_6$. Since $A_5 \cong PSL(2, 4)$ and $A_6 \cong PSL(2, 9)$, in the above lemma, they are considered as linear groups.

The following lemma leads us to classify the simple $K_3$, $K_4$ and $K_5$-groups that we will use it in the proof of Lemma 3.9.

**Lemma 2.11.**

(i) ([9]) *If $G$ is a simple $K_3$-group, then $G$ is isomorphic to one of the following groups:*
   *$A_5, A_6, PSL(2, 7), PSL(2, 8), PSL(2, 17), PSL(3, 3), PSU(3, 3), PSU(4, 2)$.*
(ii) ([4], [22]) *If $G$ is a simple $K_4$-group, then $G$ is isomorphic to one of the following groups:*
   (1) *$A_7$, $A_8$, $A_9$, $A_{10}$, $M_{11}$, $M_{12}$, $J_2$, ${}^2B_2(8)$, ${}^2B_2(32)$, ${}^3D_4(2)$, ${}^2F_4(2)'$, $PSL(3, 4)$, $PSL(3, 5)$, $PSL(3, 7)$, $PSL(3, 8)$, $PSL(3, 17)$, $PSL(4, 3)$, $PSp(4, 4)$, $PSp(4, 5)$, $PSp(4, 7)$, $PSp(4, 9)$, $PSp(6, 2)$, $P\Omega^+(8, 2)$, $PSU(3, 4)$, $PSU(3, 5)$, $PSU(3, 7)$, $PSU(3, 8)$, $PSU(3, 9)$, $PSU(4, 3)$, $PSU(5, 2)$, $G_2(3)$;*

(2) *$PSL(2,q)$, where $q$ is a prime power such that $q(q^2-1) = (2,q-1)2^{\alpha_1}3^{\alpha_2}v^{\alpha_3}r^{\alpha_4}$, with $v, r > 3$ distinct prime numbers and for $1 \leq i \leq 4$, $\alpha_i \in \mathbb{N}$.*

(iii) ([17]) *If $G$ is a simple $K_5$-group, then $G$ is isomorphic to one of the following groups: $PSL(2,q)$, where $|\pi(q^2-1)| = 4$, $PSL(3,q)$, where $|\pi((q^2-1)(q^3-1))| = 4$, $PSU(3,q)$, where $|\pi((q^2-1)(q^3+1))| = 4$, $\Omega(5,q)$, where $|\pi(q^4-1)| = 4$, ${}^2B_2(q^2)$, where $q^2 = 2^{2k+1}$ and $|\pi((q^2-1)(q^4+1))| = 4$, ${}^2G_2(q^2)$, where $q^2$ is an odd power of 3 and $|\pi((q^4-1)(q^4-q^2+1))| = 4$ or one of the following simple groups:*
*$PSL(4,4)$, $PSL(4,5)$, $PSL(4,7)$, $PSL(5,2)$, $PSL(5,3)$, $PSL(6,2)$, $\Omega(7,3)$, $\Omega(9,2)$, $PSp(6,3)$, $PSp(8,2)$, $PSU(4,4)$, $PSU(4,5)$, $PSU(4,7)$, $PSU(4,9)$, $PSU(5,3)$, $PSU(6,2)$, $P\Omega^+(8,3)$, $P\Omega^-(8,2)$, $A_{11}$, $A_{12}$, $M_{22}$, $J_3$, $HS$, $He$, $McL$, ${}^3D_4(3)$, $G_2(4)$, $G_2(5)$, $G_2(7)$, $G_2(9)$.*

Let $a > 1$ be an integer, and let $m$ be a natural number. If there is a prime $l$ such that $l \mid a^m - 1$ and $l \nmid a^i - 1$ for $i < m$, then $l$ is called a primitive prime divisor of $a^m - 1$, which is shown by $r_m(a)$. Also, if $m$ is even, then the definition of primitive prime divisor of $a^m - 1$ forces $r_m(a) \nmid a^{\frac{m}{2}} - 1$, and so, $r_m(a) \mid a^{\frac{m}{2}} + 1$. Hence, $r_m(a)$ is called a primitive prime divisor of $a^{\frac{m}{2}} + 1$.

From [8], we can see when $a^m - 1$ and $a^m + 1$ have primitive prime divisors.

**Lemma 2.12** (Zsigmondy's theorem)**.** *Let $a > 1$ be an integer and $m \in \mathbb{N}$. Then $a^m - 1$ has a primitive prime divisor except when $a = 2$ and $m \in \{1, 6\}$ or $m = 2$ and $a = 2^t - 1$, for some natural number $t$. Also, $a^m + 1$ has a primitive prime divisor except when $a = 2$ and $m = 3$ or $m = 1$ and $a = 2^t - 1$.*

*Remark 2.13.* Let $\epsilon' = \pm$, $a$ be a prime, $m$ be a natural number and $r$ be a primitive prime divisor of $a^m + \epsilon' 1$. Then Fermat's little theorem shows that if $\epsilon' = 1$, then $2m \mid r - 1$, therefore $r \nmid 2m$, and if $\epsilon' = -1$, then $m \mid r - 1$, hence $r \nmid m$, and so, $r \nmid 2m$, because $r$ is odd.

**Lemma 2.14** ([7])**.**

(a) *Except the relations $(239)^2 - 2(13)^4 = -1$ and $(3)^5 - 2(11)^2 = 1$, every solution of the equation*

$$a^r - 2b^s = \pm 1 \quad a, b \text{ prime}; \quad r, s > 1,$$

*has exponents $r = s = 2$; i.e., it comes from a unit $a - b.2^{\frac{1}{2}}$ of the quadratic field $Q(2^{1/2})$ for which the coefficients $a, b$ are primes.*

(b) *The equation* $a^r - b^s = 1$, *where* $a$ *and* $b$ *are primes and* $r, s > 1$, *has only one solution, namely,* $3^2 - 2^3 = 1$.

**Lemma 2.15** ([30]).

$$(a^s-1, a^t+1) = \begin{cases} a^{(s,t)}+1, & \text{if } \frac{s}{(s,t)} \text{ is even and } \frac{t}{(s,t)} \text{ is odd,} \\ (a+1,2), & \text{otherwise.} \end{cases}$$

## 3. Main results

Throughout this section, we suppose that $PSL(2,p^n) \lesssim L \lesssim \mathrm{Aut}(PSL(2,p^n))$, where $p$ is prime. If $\delta \in L$, then set $H = PGL(2,p^n)$, and if $\delta \notin L$, then set $H = PSL(2,p^n)$. Suppose that $[L : H] = 2^a m$, where $m$ is an odd number, and if $p$ is odd, then set $\epsilon = (-1)^{\frac{(p^n-1)}{2}}$. Also, in Table 1, we bring some irreducible character degrees of the simple groups of Lie type. Note that the irreducible character degrees of the simple exceptional groups of Lie type available in Table 1 can be found in ([5, §13.9]), and the irreducible character degrees of the simple classical groups of Lie type mentioned in Table 1 can be calculated by formulas from ([5, §13.8]). Also, for the irreducible character degrees of the simple groups of small orders, we refer the reader to [6].

In the following lemmas, let $p \mid [L : PSL(2,p^n)]$, and let $G$ be a group with $X_1(G) = X_1(L)$. Thus, since $\mathrm{cd}(G) = \mathrm{cd}(L)$, we can use Lemma 2.1(c) to obtain $\mathrm{cd}(G)$.

*Remark 3.1.* Since $X_1(G) = X_1(L)$,

$$|G| = \sum_{\chi \in \mathrm{Irr}(G)} (\chi(1))^2 = \sum_{\chi \in \mathrm{Irr}(L)} (\chi(1))^2 = |L| = \frac{2^{a+i} m p^n (p^n+1)(p^n-1)}{2},$$

where $i = 1$ or $i = 0$.

*Remark 3.2.* If $p=2$ and $n=3$, then, since $2 \nmid [\mathrm{Aut}(PSL(2,2^3)) : PSL(2,2^3)] = 3$, Lemma 2.9 shows that $G \cong L$. So, in the following, we assume that either $p \neq 2$ or $n \neq 3$.

**Lemma 3.3.** $|\frac{G}{G'}| = 2^{a+i}m$, *and also,* $|G'| = |PSL(2,p^n)|$.

Proof. Since $\frac{L}{PSL(2,p^n)} \lesssim \frac{\mathrm{Aut}(PSL(2,p^n))}{PSL(2,p^n)}$ and $\frac{\mathrm{Aut}(PSL(2,p^n))}{PSL(2,p^n)}$ is abelian, we conclude that $\frac{L}{PSL(2,p^n)}$ is abelian, and hence, $L' \lesssim PSL(2,p^n)$. Now, since $PSL(2,p^n)$ is simple and $L$ is non-abelian, $L' \cong PSL(2,p^n)$. On the other hand, we know that for every finite group $K$, $|\frac{K}{K'}|$ is the number of linear characters of $K$. Thus, since $X_1(G) = X_1(L)$, $|\frac{G}{G'}| = |\frac{L}{L'}| = 2^{a+i}m$. Moreover, by Remark 3.1, $|L| = |G|$. Now, since we have $|L'| = |PSL(2,p^n)|$, we conclude that $|G'| = |PSL(2,p^n)|$. □

**Lemma 3.4.** *Let $p^n+1$ and $p^n-1$ have primitive prime divisors. Then the only character degrees of $G$ of prime powers are $p^n$, $\frac{p^n-1}{2} = \frac{3^3-1}{2}$, a Mersenne prime $p^n-1 = 2^n-1$ or a Fermat prime $p^n+1 = 2^n+1$.*

Proof. Considering Lemma 2.14 (a,b) and cd of $G$ completes the proof. □

Hereafter, let $K$ be maximal such that $K$ is normal in $G$ and $\frac{G}{K}$ is non-abelian.

*Remark 3.5.* Our assumption on $K$ shows that $(\frac{G}{K})'$ is the unique normal minimal subgroup of $\frac{G}{K}$, so,

$$\left(\frac{G}{K}\right)' \lesssim \frac{G}{K} \lesssim \mathrm{Aut}\left(\left(\frac{G}{K}\right)'\right). \tag{3.1}$$

**Lemma 3.6.** *$\frac{G}{K}$ is not solvable.*

Proof. On the contrary, let $\frac{G}{K}$ be solvable. We are going to get a contradiction. Since $(\frac{G}{K})'$ is the unique normal minimal subgroup of $\frac{G}{K}$, and $\frac{G}{K}$ is solvable, Lemma 2.4 shows that for some $f$, $\mathrm{cd}(\frac{G}{K}) = \{1, f\}$, and one of the following cases holds:

(a) $\frac{G}{K}$ is a $s$-group ($s \in \pi(G)$), and $\frac{\frac{G}{K}}{Z(\frac{G}{K})}$ is of order $f^2$. Thus $f^2 \mid |\frac{G}{K}|$, and so, $f^2 \mid |G|$. Now, since $\mathrm{cd}(\frac{G}{K}) \subseteq \mathrm{cd}(G)$ and $f \in \mathrm{cd}(\frac{G}{K})$, we conclude that $f \in \mathrm{cd}(G)$. But considering $\mathrm{cd}(G)$ shows that there does not exist any $f \in \mathrm{cd}(G) - \{1\}$ such that $f^2 \mid |G|$, which is a contradiction.

(b) $\frac{G}{K}$ is a Frobenius group with the Frobenius kernel $(\frac{G}{K})'$ and $[\frac{G}{K} : (\frac{G}{K})'] = f$. Now, by Lemma 3.3, $[G : G'] = 2^{a+i}m$. So, $f \mid 2^{a+i}m$. But since $f \in \mathrm{cd}(G)$, considering $\mathrm{cd}(G)$ leads us to get a contradiction.

These contradictions show that $\frac{G}{K}$ is not solvable. □

**Lemma 3.7.** *If $(\frac{G}{K})' \cong PSL(2,p^n)$, then $PSL(2,p^n) \lesssim G \lesssim \mathrm{Aut}(PSL(2,p^n))$.*

Proof. By Lemma 3.3, $|G'| = |PSL(2,p^n)|$. Thus, since $(\frac{G}{K})' = \frac{G'K}{K} \cong \frac{G'}{G' \cap K}$ and $(\frac{G}{K})' \cong PSL(2,p^n)$, we deduce that $|G' \cap K| = 1$. It follows that $G' \cong PSL(2,p^n)$ and $K \leq Z(G)$.

Now, we claim that $K = 1$. On the contrary, suppose that $K \neq 1$. By Lemma 2.1(c), $G$ has irreducible character degrees $p^n, 2^a m(p^n-1)$ and $\alpha(p^n+1)$, where $\alpha$ is a natural number with $\alpha \mid 2^a m$. Let $\chi, \beta, \eta \in \mathrm{Irr}(G)$ such that $\chi(1) = p^n$, $\beta(1) = 2^a m(p^n - 1)$ and $\eta(1) = \alpha(p^n + 1)$. Then, applying Ito's theorem to $K, \chi, \beta$ and $\eta$ shows that for some $y \geq 1$, $|K| = 2^j p^y$, where $j = 0$ or 1. Set $M = G' \times K$, and let $\kappa \in \mathrm{Irr}(M)$ such that $[\beta_M, \kappa] \neq 0$. Then Lemma 2.3 implies that $\frac{\beta(1)}{\kappa(1)} \mid \frac{|G|}{|M|}$ and so, $|K|(p^n - 1) \mid \kappa(1)$. But this is a contradiction, because $\mathrm{cd}(M) = \mathrm{cd}(G') = \mathrm{cd}(PSL(2,p^n))$, and by Lemma 2.1(a,b),

$$\{1, p^n, p^n - 1, p^n + 1\} \subseteq \mathrm{cd}(PSL(2,p^n)) \subseteq \left\{1, p^n, p^n - 1, p^n + 1, \frac{p^n + \epsilon}{2}\right\}.$$

Thus $K = 1$, and hence 3.1 shows that $G' \lesssim G \lesssim \mathrm{Aut}(G')$, and since $G' \cong PSL(2,p^n)$, the proof is complete. □

**Lemma 3.8.** *If* $PSL(2,p^n) \lesssim G \lesssim \mathrm{Aut}(PSL(2,p^n))$, *then* $G \cong L$.

PROOF. The proof goes back to [12], Steps 5–11 in the proof of the main theorem. □

**Lemma 3.9.** *If* $p = 2$ *and* $n = 2$ *or* $n = 6$, *then* $G \cong L$.

PROOF. Let $p = 2$ and $n = 6$. As was mentioned before, assume that $K$ is maximal such that $K$ is normal in $G$ and $\frac{G}{K}$ is non-abelian. Then Lemma 3.6 shows that $\frac{G}{K}$ is non-solvable. Hence, $(\frac{G}{K})'$ is isomorphic to a direct product of some non-abelian simple group $S$. Now, by an easy calculation, we can see that $2^6.3^2.5.7.13 \mid |L|$ and $|L| \mid 2^7.3^3.5.7.13$ and so, $\pi(L) = \{2,3,5,7,13\}$. Moreover, Remark 3.1 shows that $|G| = |L|$, and so, $G$ is a $K_5$-group. Now, since $\pi(S) \subseteq \pi(G)$, we conclude that $S$ is a simple $K_3$, $K_4$ or $K_5$-group. Hence, $5, 7$ or 13 belongs to $\pi(S)$.

On the other hand, $|G|_{13} = 13$, $|G|_7 = 7$ and $|G|_5 = 5$. Thus $|\frac{G}{K}|_{13} \mid 13$, $|\frac{G}{K}|_7 \mid 7$ and $|\frac{G}{K}|_5 \mid 5$, and so, $|(\frac{G}{K})'|_{13} \mid 13$, $|(\frac{G}{K})'|_7 \mid 7$ and $|(\frac{G}{K})'|_5 \mid 5$. It follows that $(\frac{G}{K})'$ is simple. Hence, considering Lemma 2.11 and $|G|$ shows that $(\frac{G}{K})'$ is isomorphic to one of the following groups:

$$\begin{aligned}
&A_5, A_6, A_7, A_8, PSL(3,3), PSU(3,3), PSL(3,4), PSL(2,7), {}^2B_2(8), PSL(2,8),\\
&PSL(2,q), \text{ where } |\pi(q^2-1)| = 3, PSL(2,q), \text{where } |\pi(q^2-1)| = 4, PSL(3,q),\\
&\text{where } |\pi((q^2-1)(q^3-1))| = 4, PSU(3,q), \text{where } |\pi((q^2-1)(q^3+1))| = 4, \Omega(5,q),\\
&\text{where } |\pi(q^4-1)| = 4, {}^2G_2(q^2), \text{where } q^2 = 3^{2m+1}, m \geq 1\\
&\text{and } |\pi((q^4-1)(q^4-q^2+1))| = 4, {}^2B_2(q^2),\\
&\text{where } q^2 = 2^{2k+1} \text{and } |\pi((q^2-1)(q^4+1))| = 4. \qquad (3.2)
\end{aligned}$$

By our assumption, $p = 2 \mid [L : PSL(2, p^n)]$, and so, $L \ncong PSL(2, p^n)$, $PSL(2, p^n).Z_3$. Thus, considering Lemma 2.1(c) shows that

$$\mathrm{cd}(L) \subseteq \{1, 2^6, 2^6+1, 2(2^6+1), 3(2^6+1), 6(2^6+1), 2(2^6-1), 6(2^6-1)\}. \quad (3.3)$$

Now, since $\mathrm{cd}(G) = \mathrm{cd}(L)$ and $\mathrm{cd}(\frac{G}{K}) \subseteq \mathrm{cd}(G)$, we conclude that $\mathrm{cd}(\frac{G}{K})$ is a subset of 3.3.

If $(\frac{G}{K})'$ is a simple group of Lie type, then Lemma 2.6(b) shows that 'St' extends to $\frac{G}{K}$, where 'St' is the Steinberg character of $(\frac{G}{K})'$. Now, since St(1) is a prime power, considering $\mathrm{cd}(\frac{G}{K})$ (mentioned in 3.3) shows that $\mathrm{St}(1) = 2^6$.

Now, let $(\frac{G}{K})'$ be a simple $K_3$ or $K_4$-group. Then 3.2 shows that $(\frac{G}{K})' \cong PSL(3,4)$, $A_5$, $A_6$, $A_7$, $A_8$, $PSL(3,3)$, $PSU(3,3)$, $PSL(2,7)$, $PSL(2,8)$, ${}^2B_2(8)$ or $PSL(2,q)$, where $|\pi(q^2-1)| = 3$. First, suppose that $(\frac{G}{K})' \cong PSL(3,4)$. Then $20 \in \mathrm{cd}(PSL(3,4)) = \mathrm{cd}((\frac{G}{K})')$. Now, let $\theta \in \mathrm{Irr}((\frac{G}{K})')$ and $\Upsilon \in \mathrm{Irr}(\frac{G}{K})$ such that $\theta(1) = 20$ and $[\Upsilon_{(\frac{G}{K})'}, \theta] \neq 0$. Then 3.1 shows that $|\frac{\frac{G}{K}}{(\frac{G}{K})'}| \mid |\frac{\mathrm{Aut}((\frac{G}{K})')}{(\frac{G}{K})'}| = 12$. Hence, Lemma 2.3 implies that $\Upsilon(1) = 20, 40, 60, 120$ or $240$, which is a contradiction by considering $\mathrm{cd}(\frac{G}{K})$ (mentioned in 3.3). Also, the same reasoning as above rules out $(\frac{G}{K})' \cong A_5$, $A_6$, $A_7$, $A_8$, $PSL(3,3)$, $PSU(3,3)$, $PSL(2,7)$, $PSL(2,8)$ or ${}^2B_2(8)$. Thus $(\frac{G}{K})' \cong PSL(2,q)$, where $|\pi(q^2-1)| = 3$. Hence, by Table 1, $\mathrm{St}(1) = q$. Also, as was mentioned above, the degree of the Steinberg character of $(\frac{G}{K})'$ is $2^6$, and so, $q = 2^6$. Thus $q^2 - 1 = 3^2.5.7.13$, which is a contradiction to $|\pi(q^2-1)| = 3$.

Therefore, $(\frac{G}{K})'$ is a simple $K_5$-group. So, 3.2 shows that $(\frac{G}{K})'$ is a simple group of Lie type, and we use Table 1 to obtain the degree of the Steinberg character of $(\frac{G}{K})'$. Now, as was mentioned, $\mathrm{St}(1) = 2^6$. Thus $(\frac{G}{K})'$ is not isomorphic to ${}^2G_2(q^2)$, where $q^2 = 3^{2m+1}, m \geq 1$, because the degree of the Steinberg character of ${}^2G_2(q^2)$ is $q^6 = 3^{3(2m+1)}$. If $(\frac{G}{K})' \cong \Omega(5,q)$, then $\mathrm{St}(1) = q^4 = 2^6$, which is impossible. Let $(\frac{G}{K})' \cong PSL(3,q)$, where $|\pi((q^2-1)(q^3-1))| = 4$. Then $\mathrm{St}(1) = q^3 = 2^6$, and so, $q = 4$. Thus $(q^2-1)(q^3-1) = 3^3.5.7$, which is a contradiction, because $|\pi((q^2-1)(q^3-1))| = 4$. Also, the same argument leads us to see that $(\frac{G}{K})'$ is not isomorphic to $PSU(3,q)$ or ${}^2B_2(q^2)$.

These contradictions show that $(\frac{G}{K})' \cong PSL(2,q)$, where $|\pi(q^2-1)| = 4$. Thus $q = \mathrm{St}(1) = 2^6$, and so, $(\frac{G}{K})' \cong PSL(2, 2^6)$. Therefore, Lemma 3.7 forces $PSL(2,2^6) \lesssim G \lesssim \mathrm{Aut}(PSL(2,2^6))$, and hence, by Lemma 3.8, $G \cong L$, as wanted.

Also, when $p = n = 2$, the same argument completes the proof. □

*Remark 3.10.* Since $\mathrm{cd}(\frac{G}{K}) \subseteq \mathrm{cd}(G)$, we conclude that

$$\mathrm{cd}\left(\frac{G}{K}\right) \subseteq \{1, p^n\} \cup \{j(p^n+1) : \ j \mid 2^a m\} \cup \{2^a l(p^n-1) : \ l \mid m\} \cup \left\{\frac{p^n+\epsilon}{2}\right\}.$$

If $\mathrm{cd}(\frac{G}{K}) \subseteq \{1, p^n\}$, then $\Gamma(\frac{G}{K})$ is complete and hence, Lemma 2.5 shows that $\frac{G}{K}$ is solvable, which is a contradiction to Lemma 3.6. Thus for some natural numbers $r, l$ with $r \mid 2^a m$ and $l \mid m$, $\frac{G}{K}$ has some irreducible character degree $j(p^n+1)$, $\frac{p^n+\epsilon}{2}$ (if $p$ is odd) or $2^a l(p^n-1)$.

*Remark 3.11.* If $p^n+1$ and $p^n-1$ have primitive prime divisors and $\Omega, \Psi \in \mathrm{Irr}(\frac{G}{K})$ such that $r_{2n}(p) \mid \Omega(1)$ and $r_n(p) \mid \Psi(1)$, then considering $\mathrm{cd}(\frac{G}{K})$ (mentioned in Remark 3.10) and the facts that $(p^n+1, p^n-1) \mid 2$ and $(p^n+1, p) = (p^n-1, p) = 1$ shows that $\Omega(1) = f(p^n+1)$ or $\Omega(1) = \frac{p^n+1}{2}$ (if $p$ is odd and $\epsilon = 1$) and $\Psi(1) = 2^a k(p^n-1)$ or $\Psi(1) = \frac{p^n-1}{2}$ (if $p$ is odd and $\epsilon = -1$), where $f, k$ are natural numbers with $f \mid 2^a m$ and $k \mid m$.

Now, we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** If $G \cong L$, then it is obvious that $X_1(G) = X_1(L)$. Thus in the following, we assume that $X_1(G) = X_1(L)$. If $p \nmid [L : PSL(2,p^n)]$, then Lemma 2.9 completes the proof of Theorem 1.2. So, we assume that $p \mid [L : PSL(2,p^n)]$. By considering Remark 3.2 and Lemma 3.9, we can assume that either $p \neq 2$ or $n \neq 2, 3, 6$. Since either $p \neq 2$ or $n \neq 3, 6$, Lemma 2.12 guarantees that $p^n+1$ and $p^n-1$ have primitive prime divisors $r_{2n}(p)$ and $r_n(p)$, respectively. Let $b$ and $c$ be natural numbers such that $(r_{2n}(p))^c \parallel p^n+1$ and $(r_n(p))^b \parallel p^n-1$. Also, since $PSL(2,p^n) \lesssim L \lesssim \mathrm{Aut}(PSL(2,p^n))$ and $\mathrm{cd}(G) = \mathrm{cd}(L)$, Lemma 2.1(c) helps us to obtain $\mathrm{cd}(G)$.

Let $K$ be maximal such that $K$ is normal in $G$ and $\frac{G}{K}$ is non-abelian. Then $(\frac{G}{K})'$ is the unique normal minimal subgroup of $\frac{G}{K}$. Now, Lemma 3.6 shows that $\frac{G}{K}$ is non-solvable. Hence, there exists a simple group $S$ and a natural number $t$ such that $(\frac{G}{K})' \cong \underbrace{S \times \cdots \times S}_{t}$. In the following steps, we are going to prove that $t = 1$ and $(\frac{G}{K})' \cong PSL(2,p^n)$. Next, Lemmas 3.7 and 3.8 prove that $G \cong L$.

*Step (a).* (i) Let $\omega \in \mathrm{Irr}(\frac{G}{K})$ such that $\omega(1) \in \{2^a l(p^n-1), j(p^n+1), \frac{p^n+\epsilon}{2}\}$, where $l, j$ are natural numbers with $l \mid m$ and $j \mid 2^a m$, and let $\varpi \in \mathrm{Irr}((\frac{G}{K})')$ such that $[\omega_{(\frac{G}{K})'}, \varpi] \neq 0$. If $\omega(1) = j(p^n+1)$ or $\frac{p^n+1}{2}$, then $(r_{2n}(p))^c \parallel \varpi(1)$ and if $\omega(1) = 2^a l(p^n-1)$ or $\frac{p^n-1}{2}$, then $(r_n(p))^b \parallel \varpi(1)$.

(ii) If $\varsigma \in \mathrm{Irr}((\frac{G}{K})')$ such that $r_{2n}(p) \mid \varsigma(1)$ or $r_n(p) \mid \varsigma(1)$, then $(r_{2n}(p))^c \parallel \varsigma(1)$ or $(r_n(p))^b \parallel \varsigma(1)$, respectively.

(iii) If $\theta \in \mathrm{Irr}((\frac{G}{K})')$ such that $\theta(1)$ is not a prime power, then only one of $r_n(p)$ and $r_{2n}(p)$ divides $\theta(1)$.

Proof. (i) Let $\omega \in \mathrm{Irr}(\frac{G}{K})$, and let $\varpi \in \mathrm{Irr}((\frac{G}{K})')$ such that $[\omega_{(\frac{G}{K})'}, \varpi] \neq 0$, and $\omega(1) = 2^a l(p^n-1)$ or $\frac{p^n-1}{2}$ ($\omega(1) = j(p^n+1)$ or $\frac{p^n+1}{2}$). Then Lemma 2.3

implies that $\frac{\omega(1)}{\varpi(1)} \mid [\frac{G}{K} : (\frac{G}{K})']$. Now, since $[\frac{G}{K} : (\frac{G}{K})'] \mid [G : G']$, considering Lemma 3.3 shows that $[\frac{G}{K} : (\frac{G}{K})'] \mid 2^{a+i}m$. It follows that $\frac{\omega(1)}{(\omega(1),2^{a+i}m)} \mid \varpi(1)$. On the other hand, $2^{a+i}m \mid 2n$, and so, Remark 2.13 implies that $r_{2n}(p), r_n(p) \nmid 2^{a+i}m$. Thus $(r_n(p))^b \parallel \frac{\omega(1)}{(\omega(1),2^{a+i}m)}$ $((r_{2n}(p))^c \parallel \frac{\omega(1)}{(\omega(1),2^{a+i}m)})$, and hence, $(r_n(p))^b \parallel \varpi(1)$ $((r_{2n}(p))^c \parallel \varpi(1))$.

(ii) Now, let $\varsigma \in \mathrm{Irr}((\frac{G}{K})')$ such that $r_{2n}(p) \mid \varsigma(1)$, and let $\Theta \in \mathrm{Irr}(\frac{G}{K})$ such that $[\Theta_{(\frac{G}{K})'}, \varsigma] \neq 0$. Then Lemma 2.3 shows that $\varsigma(1) \mid \Theta(1)$, and so, $r_{2n}(p) \mid \Theta(1)$. Hence, Remark 3.11 shows that for some natural number $e$ with $e \mid 2^{a+i}m$, $\Theta(1) = e(p^n+1)$ or $p$ is odd, $\epsilon = 1$ and $\Theta(1) = \frac{p^n+1}{2}$. Thus (i) implies that $(r_{2n}(p))^c \parallel \varsigma(1)$. Also, when $r_n(p) \mid \varsigma(1)$, the same argument completes the proof.

(iii) Let $\theta \in \mathrm{Irr}((\frac{G}{K})')$ such that $\theta(1)$ is not a prime power, and let $\varrho \in \mathrm{Irr}(\frac{G}{K})$ such that $[\varrho_{(\frac{G}{K})'}, \theta] \neq 0$. Then Lemma 2.3 shows that $\theta(1) \mid \varrho(1)$. Thus since $\theta(1)$ is not a prime power, considering $\mathrm{cd}(\frac{G}{K})$ shows that $\varrho(1) \in \{2^a l(p^n-1), j(p^n+1), \frac{p^n+\epsilon}{2}\}$, where $l, j$ are natural numbers with $l \mid m$ and $j \mid 2^a m$. Hence, part (i) implies that one of $r_n(p)$ and $r_{2n}(p)$ divides $\theta(1)$. Now, suppose that $r_n(p)r_{2n}(p) \mid \theta(1)$. Then since $\theta(1) \mid \varrho(1)$, we obtain $r_n(p)r_{2n}(p) \mid \varrho(1)$, which is a contradiction by considering $\mathrm{cd}(\frac{G}{K})$. □

*Step (b).* $(\frac{G}{K})'$ is simple.

Proof. On the contrary, suppose that $t > 1$. According to Remark 3.10, $\frac{G}{K}$ has some irreducible character degree $r(p^n+1)$, $\frac{p^n+\epsilon}{2}$ (if $p$ is odd) or $2^a l(p^n-1)$, where $r, l$ are natural numbers with $r \mid 2^a m$ and $l \mid m$.

In the following, without loss of generality, we suppose that $\varphi \in \mathrm{Irr}(\frac{G}{K})$ such that $\varphi(1) = r(p^n+1)$. Also, let $\gamma \in \mathrm{Irr}((\frac{G}{K})')$ such that $[\varphi_{(\frac{G}{K})'}, \gamma] \neq 0$. Then since $(\frac{G}{K})' \cong \underbrace{S \times \cdots \times S}_{t}$, for every $1 \leq j \leq t$, there exists some $\gamma_j \in \mathrm{Irr}(S)$ such that $\gamma(1) = \gamma_1(1) \times \gamma_2(1) \times \cdots \times \gamma_t(1)$. Moreover, Step (a)(i) shows that $(r_{2n}(p))^c \parallel \gamma(1)$. Without loss of generality, we suppose that $r_{2n}(p) \mid \gamma_1(1)$. Since $(\gamma_1(1))^t, \gamma_1(1) \times \underbrace{1 \times \cdots \times 1}_{t-1} \in \mathrm{cd}((\frac{G}{K})')$, Step (a)(ii) shows that $(r_{2n}(p))^c \parallel (\gamma_1(1))^t, \gamma_1(1)$, which is a contradiction. This contradiction proves that $t = 1$, and hence $(\frac{G}{K})'$ is simple. □

*Step (c).* Let $(\frac{G}{K})'$ be a simple group of Lie type. Then the degree of the Steinberg character of $(\frac{G}{K})'$ is $p^n$.

Proof. On the contrary, let $\mathrm{St}(1) \neq p^n$. By Lemma 2.6(b), the Steinberg character of $(\frac{G}{K})'$ extends to $\frac{G}{K}$, and so, $\mathrm{St}(1) \in \mathrm{cd}(\frac{G}{K})$. Since $\mathrm{St}(1)$ is a prime power and $\mathrm{cd}(\frac{G}{K}) \subseteq \mathrm{cd}(G)$, considering Lemma 3.4 shows that St(1) is a Fermat prime $p^n + 1 = 2^n + 1 = r_{2n}(2)$, a Mersenne prime $p^n - 1 = 2^n - 1 = r_n(2)$ or $\frac{p^n-1}{2} = \frac{3^3-1}{2} = r_3(3)$. Thus St(1) is an odd prime.

First, we are going to prove that $\Delta((\frac{G}{K})')$ is disconnected. For this reason, we prove that for every $l \in \mathrm{cd}((\frac{G}{K})') - \{\mathrm{St}(1)\}$, $(l, \mathrm{St}(1)) = 1$. On the contrary, let $\mathrm{St}(1) \neq l \in \mathrm{cd}((\frac{G}{K})')$ and $(l, \mathrm{St}(1)) \neq 1$. Then, since St(1) is prime, we conclude that $\mathrm{St}(1) \mid l$, which is a contradiction to Lemma 2.8. This contradiction shows that $\Delta((\frac{G}{K})')$ is disconnected.

So, Lemma 2.10 implies that for some prime power $q$, $(\frac{G}{K})' \cong PSL(2, q)$. Hence, $q = \mathrm{St}(1)$ is an odd prime, and by considering Lemma 2.1(a), we can see that

$$\mathrm{cd}((\frac{G}{K})') = \mathrm{cd}(PSL(2,q)) = \left\{1, q, q-1, q+1, \frac{q+(-1)^{\frac{q-1}{2}}}{2}\right\}.$$

Now, we consider the following cases:

(i) Let $p = 2$, and let $q$ be a Fermat prime $2^n + 1$. Then, since $m$ is odd and $m \mid n$, $m = 1$. Now, let $\varsigma \in \mathrm{Irr}((\frac{G}{K})')$ and $\gamma \in \mathrm{Irr}(\frac{G}{K})$ such that $[\gamma_{(\frac{G}{K})'}, \varsigma] \neq 0$ and $\varsigma(1) = q + 1 = 2^n + 2$. Then Lemma 2.3 implies that $2^n + 2 \mid \gamma(1)$. But considering $\mathrm{cd}(\frac{G}{K})$ (mentioned in Remark 3.10) and the facts that $2^n + 2 \nmid 2^n$, $(2^n + 2, 2^n + 1) = 1$, $2^n + 2 \nmid n$ and $m = 1$ shows that $\gamma(1) = 2^a(2^n - 1)$. Hence, $2^n + 2 \mid 2^a(2^n - 1)$, and so, $2^{n-1} + 1 \mid 2^n - 1$. Thus since, $(2^{n-1} + 1, 2^n - 1) \mid 3$, we conclude that $2^{n-1} + 1 = 1$ or $2^{n-1} + 1 = 3$. The first case is impossible. Also, the latter case implies that $2^{n-1} = 2$ and so, $n = 2$. But this is a contradiction, because by our assumption either $p \neq 2$ or $n \neq 2$.

(ii) Let $p = 2$, and let $q$ be a Mersenne prime $2^n - 1$. Since $2 = p \mid [L : PSL(2, p^n)]$, $[L : PSL(2, p^n)] \mid n$ and $n$ is prime, we deduce that $n = 2$. But this is a contradiction, because by our assumption either $p \neq 2$ or $n \neq 2$.

(iii) Let $\mathrm{St}(1) = q = \frac{p^n-1}{2} = \frac{3^3-1}{2} = 13$. Also, let $\iota \in \mathrm{Irr}((\frac{G}{K})')$ such that $\iota(1) = q - 1 = 12$. Then, since $\iota(1)$ is not a prime power, Step (a)(iii) shows that $r_6(3) = 7$ or $r_3(3) = 13$ divides $\iota(1)$, which is impossible.

These contradictions show that $\mathrm{St}(1) = p^n$. □

*Step (d).* $\Delta((\frac{G}{K})')$ is not complete.

Proof. On the contrary, let $\Delta((\frac{G}{K})')$ be complete. By Remark 3.10, for some natural numbers $r$ and $k$ with $r \mid 2^a m$ and $k \mid m$, $\frac{G}{K}$ has irreducible character degree $r(p^n + 1)$, $2^a k(p^n - 1)$ or $\frac{p^n+\epsilon}{2}$. Thus $r_{2n}(p)$ or $r_n(p)$ belongs to $\rho(\frac{G}{K})$.

Now, we claim that only one of $r_{2n}(p)$ and $r_n(p)$ belongs to $\rho(\frac{G}{K})$. On the contrary, suppose that $\zeta, \nu \in \mathrm{Irr}(\frac{G}{K})$ such that $r_{2n}(p) \mid \zeta(1)$ and $r_n(p) \mid \nu(1)$.

Then, by considering Remark 3.11 and without loss of generality, we can assume that $\zeta(1) = r(p^n+1)$ and $\nu(1) = 2^a k(p^n-1)$. Thus Step (a)(i) guarantees that $r_{2n}(p), r_n(p) \in \rho((\frac{G}{K})')$. Now, since $\Delta((\frac{G}{K})')$ is complete, we conclude that there exists some $\theta \in \mathrm{Irr}((\frac{G}{K})')$ such that $r_{2n}(p)r_n(p) \mid \theta(1)$. But this is a contradiction to Step (a)(iii).

In the following, without loss of generality, we suppose that $r_{2n}(p) \in \rho(\frac{G}{K})$. If $\psi \in \mathrm{Irr}(\frac{G}{K})$ such that $r_{2n}(p) \mid \psi(1)$, then Remark 3.11 shows that $\psi(1) = r(p^n+1)$ or $\psi(1) = \frac{p^n+1}{2}$ (if $p$ is odd and $\epsilon = 1$), where $r$ is a natural number with $r \mid 2^a m$. Now, if $p^n \notin \mathrm{cd}(\frac{G}{K})$, then, since for every $\gamma \in \mathrm{Irr}(\frac{G}{K}) - \{1\}$, $r_{2n}(p) \mid \gamma(1)$, we conclude that $\Gamma(\frac{G}{K})$ is complete, which is a contradiction by considering Lemma 2.5 and the fact that $\frac{G}{K}$ is non-solvable. This contradiction shows that $p^n \in \mathrm{cd}(\frac{G}{K})$.

Now, the classification theorem of the simple groups shows that $(\frac{G}{K})'$ is an alternating group of degree at least 5, a sporadic simple group or a simple group of Lie type. Thus, by considering Lemma 2.10, we have the following cases:

*Case 1.* Let for some $8 \neq l \geq 7$, $(\frac{G}{K})' \cong A_l$. Then Lemma 2.6(a) shows that $(\frac{G}{K})'$ has two irreducible characters as $\delta$ and $\vartheta$ such that $\delta(1) = \frac{l(l-3)}{2}$ and $\vartheta(1) = \frac{(l-1)(l-2)}{2}$, and these characters extend to $\mathrm{Aut}(\frac{G}{K})$. Hence, Lemma 2.7 guarantees that $\delta(1), \vartheta(1) \in \mathrm{cd}(\frac{G}{K})$. Also, we can see that $\delta(1)$ and $\vartheta(1)$ are consecutive integers.

First, note that since $l \geq 7$, $l-1, l-2 \neq 1$. Hence, $\vartheta(1)$ cannot be a prime power, and so, considering $\mathrm{cd}(\frac{G}{K})$ shows that $\vartheta(1) = k(p^n+1)$ or $\frac{p^n+1}{2}$ (if $p$ is odd and $\epsilon = 1$), where $k \mid 2^a m$. Thus, since $(\delta(1), \vartheta(1)) = 1$, considering $\mathrm{cd}(\frac{G}{K})$ leads us to see that $\delta(1) = \frac{l(l-3)}{2} = p^n$.

On the other hand, $(l, l-3) \mid 3$. It follows that $p = 3$, and so, for some natural numbers $t, z$, $l = 3^t$ and $\frac{l-3}{2} = 3^z$. Now, an easy calculation shows that $z = 1$ and $t = 2$, hence $l = 9$ and $n = 3$, and so, $(\frac{G}{K})' \cong A_9$. Thus 3.1 implies that $A_9 \lesssim \frac{G}{K} \lesssim \mathrm{Aut}(A_9) = S_9$, and hence, $\frac{G}{K} \cong A_9$ or $S_9$. Now, since $8 \in \mathrm{cd}(A_9), \mathrm{cd}(S_9)$, we conclude that $8 \in \mathrm{cd}(\frac{G}{K})$. It follows that $8 \in \mathrm{cd}(G)$, which is a contradiction, because

$$\mathrm{cd}(G) \subseteq \{1, 3^3, 3^3+1, 2(3^3+1), 3(3^3+1), 6(3^3+1), 3^3-1, 2(3^3-1), 3(3^3-1), 6(3^3-1)\}.$$

*Case 2.* Let $(\frac{G}{K})'$ be isomorphic to one of 27 sporadic simple groups. Let $(\frac{G}{K})' \cong B$. Then

$$\{4371, 96255, 1139374, 9458750\} \subseteq \mathrm{cd}(B).$$

Now, since the mentioned degrees are not prime power, Step a(iii) guarantees that $(r_n(p))^b$ or $(r_{2n}(p))^c$ divides these degrees. Now, since $r_n(p) \notin \rho((\frac{G}{K})')$, we get $(r_{2n}(p))^c \mid 4371, 96255, 1139374, 9458750$. Hence, $r_{2n}(p) \mid (4371, 96255) = 3.31$ and $r_{2n}(p) \mid (1139374, 9458750) = 2.23.47$, which is impossibe. Now, let $(\frac{G}{K})' \cong J_2$. Then $\{21, 160\} \subseteq \mathrm{cd}(J_2)$. Since 160 and 21 are not prime powers, Step (a)(iii) and the fact that $r_n(p) \notin \rho((\frac{G}{K})')$ show that $(r_{2n}(p))^c \mid 21$ and $(r_{2n}(p))^c \mid 160$, and so, $r_{2n}(p) \mid (21, 160) = 1$, which is impossible.

Also, the same argument as used in the above cases rules out the possibility that $(\frac{G}{K})'$ is isomorphic to the other sporadic simple groups.

| Groups | Labels | Degrees |
|---|---|---|
| ${}^2G_2(q^2)$, | St | $q^6$ |
| $q^2 = 3^{2k+1},\ k \geq 1$ | $\mu$ | $\frac{1}{\sqrt{3}} q\Phi_1(q)\Phi_2(q)\phi_4(q)$ |
| $G_2(q)$ | St | $q^6$ |
| | $\mu$ | $\frac{1}{3} q\Phi_1^2(q)\Phi_2^2(q)$ |
| $E_6(q)$ | St | $q^{36}$ |
| | $\mu$ | $q\Phi_8(q)\Phi_9(q)$ |
| ${}^2F_4(q^2)$, | St | $q^{24}$ |
| $q^2 = 2^{2k+1},\ k \geq 1$ | $\mu$ | $\frac{1}{3} q^4\Phi_1^2(q)\Phi_2^2(q)\Phi_4^2(q)\Phi_8^2(q)$ |
| $E_8(q)$ | St | $q^{120}$ |
| | $\mu$ | $q\Phi_4^2(q)\Phi_8(q)\Phi_{12}(q)\Phi_{20}(q)\Phi_{24}(q)$ |
| ${}^2B_2(q^2)$, | St | $q^4$ |
| $q^2 = 2^{2k+1},\ k \geq 1$ | $\mu$ | $\frac{1}{\sqrt{2}} q\Phi_1(q)\phi_2(q)$ |
| $\Omega(2k+1, q)$ or | St | $q^{k^2}$ |
| $PSp(2k, q),\ k \geq 3$ | $\mu$ | $\frac{q(q^k+1)(q^{k-1}-1)}{2(q-1)}$ |
| $P\Omega^+(2k, q)$, | St | $q^{k(k-1)}$ |
| $k \geq 4$ | $\mu$ | $\frac{q(q^{k-2}+1)(q^k-1)}{q^2-1}$ |

| Groups | Labels | Degrees |
|---|---|---|
| $PSU(4,q)$ | St | $q^6$ |
| | $\mu$ | $q^2\Phi_4(q)$ |
| $PSL(2,q)$ | St | $q$ |
| ${}^3D_4(q^3)$ | St | $q^{12}$ |
| | $\mu$ | $\frac{1}{2}q^3\Phi_1^2(q)\Phi_3^2(q)$ |
| $F_4(q)$ | St | $q^{24}$ |
| | $\mu$ | $\frac{1}{2}q^{13}\Phi_2^2(q)\Phi_6^2(q)\Phi_8(q)$ |
| $E_7(q)$ | St | $q^{63}$ |
| | $\mu$ | $q\Phi_7(q)\Phi_{12}(q)\Phi_{14}(q)$ |
| ${}^2E_6(q^2)$ | St | $q^{36}$ |
| | $\mu$ | $q\Phi_8(q)\Phi_{18}(q)$ |
| $PSL(k+1,q)$, | St | $q^{\frac{k(k+1)}{2}}$ |
| $k \geq 2$ | $\mu$ | $\frac{q(q^k-1)}{q-1}$ |
| $P\Omega^-(2k,q)$, | St | $q^{k(k-1)}$ |
| $k \geq 4$ | $\mu$ | $\frac{q(q^{k-2}-1)(q^k+1)}{q^2-1}$ |
| $PSU(k+1,q)$, | St | $q^{\frac{k(k+1)}{2}}$ |
| $k \geq 2$, $k \neq 3$ | $\mu$ | $\frac{q(q^k-(-1)^k)}{q+1}$ |
| $\Omega(5,q)$ | St | $q^4$ |
| | $\mu$ | $\frac{q(q+1)^2}{2}$ |

Table 1: Irreducible character degrees of the simple groups of Lie type

*Case 3.* Let $(\frac{G}{K})'$ be isomorphic to a simple group of Lie type over a finite field with $q$-elements ($q^2$-elements), where for a prime number $l$ and a natural number $f$, $q = l^f (q^2 = l^f)$. Also, let $\mu, \mathrm{St} \in \mathrm{Irr}((\frac{G}{K})')$ be as shown in Table 1. Then, according to Step (c), $\mathrm{St}(1) = p^n$. On the other hand, Table 1 shows that $\mathrm{St}(1)$ is a power of $q$, and so, $q$ is a power of $p$. Hence, an easy calculation implies that $\mu(1)$ is as in Table 2, and so, $\mu(1)$ is not a prime power. Thus, considering Step (a)(iii) and the fact that $r_n(p) \notin \rho((\frac{G}{K})')$ leads us to see that $r_{2n}(p) \mid \mu(1)$.

Now, for example, let for $k \geq 2$, $(\frac{G}{K})' \cong \Omega(2k+1,q)$. First, suppose that

$k=2$. Then, since $r_{2n}(p) \mid \mu(1) = \frac{p^{\frac{n}{4}}(p^{\frac{n}{4}}+1)^2}{2}$, $r_{2n}(p) \mid p^{\frac{n}{4}}+1$. But this contradicts the definition of the primitive prime divisor, because $\frac{n}{4} < n$. Now, assume that $k \geq 3$. Then again, $r_{2n}(p) \mid \mu(1) = \frac{p^{\frac{n}{k^2}}(p^{\frac{n}{k}}+1)(p^{\frac{n(k-1)}{k^2}}-1)}{2(p^{\frac{n}{k^2}}-1)}$, and so, $r_{2n}(p) \mid p^{\frac{n}{k}}+1$ or $p^{\frac{n(k-1)}{k^2}}-1$. Now, since $k \geq 3$, the first case contradicts the definition of the primitive prime divisor. Thus $r_{2n}(p) \mid p^{\frac{n(k-1)}{k^2}}-1$, and hence, $r_{2n}(p) \mid (p^{\frac{n(k-1)}{k^2}}-1, p^n+1)$. Therefore, Lemma 2.15 shows that $r_{2n}(p) \mid \left(p^{\frac{n(k-1)}{k^2}}-1, p^n+1\right) = p^{(\frac{n(k-1)}{k^2},n)}+1$ or $r_{2n}(p) \mid (2,p+1)$. Since $r_{2n}(p)$ is odd, the latter case is impossible. Thus $r_{2n}(p) \mid p^{(\frac{n(k-1)}{k^2},n)}+1$. Now, since $k \geq 3$, we can see that $\frac{n(k-1)}{k^2} < n$, and hence, $(\frac{n(k-1)}{k^2}, n) < n$, which is a contradiction by considering the definition of the primitive prime divisor.

Also, the same argument rules out the possibility that $(\frac{G}{K})'$ is isomorphic to the other simple groups whose character degree graphs are complete.

| Groups | $q(q^2)$ | $\mu(1)$ |
|---|---|---|
| ${}^2G_2(q^2)$, $q^2 = 3^{2k+1}, k \geq 1$ | $p^{\frac{n}{3}}$ | $3^k(p^{\frac{2n}{3}}-1)$ |
| $G_2(q)$ | $p^{\frac{n}{6}}$ | $\frac{1}{3}p^{\frac{n}{6}}(p^{\frac{n}{3}}-1)^2$ |
| ${}^2F_4(q^2)$, $q^2 = 2^{2k+1}, k \geq 1$ | $p^{\frac{n}{12}}$ | $\frac{1}{3}p^{\frac{n}{6}}(p^{\frac{n}{3}}-1)^2$ |
| ${}^2E_6(q^2)$ | $p^{\frac{n}{36}}$ | $p^{\frac{n}{36}}(p^{\frac{n}{9}}+1)(p^{\frac{n}{6}}-p^{\frac{n}{12}}+1)$ |
| ${}^2B_2(q^2)$, $q^2 = 2^{2k+1},\ k \geq 1$ | $p^{\frac{n}{2}}$ | $2^k(p^{\frac{n}{2}}-1)$ |
| $PSL(k+1,q)$, $k \geq 2$ | $p^{\frac{2n}{k(k+1)}}$ | $\frac{p^{\frac{2n}{k(k+1)}}(p^{\frac{2n}{k+1}}-1)}{p^{\frac{2n}{k(k+1)}}-1}$ |
| $\Omega(2k+1,q)$ or $PSp(2k,q),\ k \geq 3$ | $p^{\frac{n}{k^2}}$ | $\frac{p^{\frac{n}{k^2}}(p^{\frac{n}{k}}+1)(p^{\frac{n(k-1)}{k^2}}-1)}{2(p^{\frac{n}{k^2}}-1)}$ |
| $P\Omega^-(2k,q)$, $k \geq 4$ | $p^{\frac{n}{k(k-1)}}$ | $\frac{p^{\frac{n}{k(k-1)}}(p^{\frac{n(k-2)}{k(k-1)}}-1)(p^{\frac{n}{k-1}}+1)}{p^{\frac{2n}{k(k-1)}}-1}$ |

| Groups | $q(q^2)$ | $\mu(1)$ |
|---|---|---|
| $P\Omega^+(2k,q)$, $k \geq 4$ | $p^{\frac{n}{k(k-1)}}$ | $\frac{p^{\frac{n}{k(k-1)}}(p^{\frac{n(k-2)}{k(k-1)}}+1)(p^{\frac{n}{k-1}}-1)}{p^{\frac{2n}{k(k-1)}}-1}$ |
| $PSU(k+1,q)$, $k \geq 2$, $k \neq 3$ | $p^{\frac{2n}{k(k+1)}}$ | $\frac{p^{\frac{2n}{k(k+1)}}(p^{\frac{2n}{(k+1)}}-(-1)^k)}{p^{\frac{2n}{k(k+1)}}+1}$ |
| $PSU(4,q)$ | $p^{\frac{n}{6}}$ | $p^{\frac{n}{3}}(p^{\frac{n}{3}}+1)$ |
| $\Omega(5,q)$ | $p^{\frac{n}{4}}$ | $\frac{p^{\frac{n}{4}}(p^{\frac{n}{4}}+1)^2}{2}$ |
| $E_7(q)$ | $p^{\frac{n}{63}}$ | $p^{\frac{n}{63}}(p^{\frac{4n}{63}}-p^{\frac{2n}{63}}+1)$ $\times(p^{\frac{2n}{21}}-p^{\frac{5n}{63}}+p^{\frac{4n}{63}}-p^{\frac{n}{21}}+p^{\frac{2n}{63}}-p^{\frac{n}{63}}+1)$ $\times(p^{\frac{2n}{21}}+p^{\frac{5n}{63}}+p^{\frac{4n}{63}}+p^{\frac{n}{21}}+p^{\frac{2n}{63}}+p^{\frac{n}{63}}+1)$ |
| $F_4(q)$ | $p^{\frac{n}{24}}$ | $\frac{1}{2}p^{\frac{13n}{24}}(p^{\frac{n}{24}}+1)^2(p^{\frac{n}{12}}-p^{\frac{n}{24}}+1)^2(p^{\frac{n}{6}}+1)$ |
| $E_6(q)$ | $p^{\frac{n}{36}}$ | $p^{\frac{n}{36}}(p^{\frac{n}{9}}+1)(p^{\frac{n}{6}}+p^{\frac{n}{12}}+1)$ |
| ${}^3D_4(q^3)$ | $p^{\frac{n}{12}}$ | $\frac{1}{2}p^{\frac{n}{4}}(p^{\frac{n}{12}}-1)^2(p^{\frac{n}{6}}+p^{\frac{n}{12}}+1)^2$ |
| $E_8(q)$ | $p^{\frac{n}{120}}$ | $p^{\frac{n}{120}}(p^{\frac{n}{60}}+1)^2(p^{\frac{n}{30}}+1)(p^{\frac{n}{30}}-p^{\frac{n}{60}}+1)(p^{\frac{n}{15}}-p^{\frac{n}{30}}+1)$ $\times(p^{\frac{n}{15}}-p^{\frac{n}{20}}+p^{\frac{n}{30}}-p^{\frac{n}{60}}+1)$ |

Table 2: Irreducible character degrees of the simple groups of Lie type

Thus we proved that $\Delta((\frac{G}{K})')$ is not complete. □

*Step (e).* $(\frac{G}{K})' \cong PSL(2,p^n)$.

Proof. By Step (d), $\Delta((\frac{G}{K})')$ is not complete. Now, we claim that $\Delta((\frac{G}{K})')$ is disconnected. On the contrary, suppose that $\Delta((\frac{G}{K})')$ is connected. Before beginning the proof, we note that if $(\frac{G}{K})'$ is a simple group of Lie type, then we suppose that $\mu, \mathrm{St} \in \mathrm{Irr}((\frac{G}{K})')$ be as shown in Table 1. Then, by the same argument as used in Case (3) of Step d, we can see that $\mu(1)$ is not a prime power and it is as in Table 2.

Now, by considering Lemma 2.10, we have the following cases:

(1) Let $(\frac{G}{K})' \cong {}^2B_2(q^2)$, where $q^2 = 2^{2k+1}$, $k \geq 1$. Since $\mu(1)$ is not a prime power, by Step (a)(iii), we can see that either $r_{2n}(p) \mid \mu(1)$ or $r_n(p) \mid \mu(1)$. Hence,

$r_n(p) \mid q-1 = p^{\frac{n}{2}} - 1$ or $r_{2n}(p) \mid q-1 = p^{\frac{n}{2}} - 1$. But the first case contradicts the definition of the primitive prime divisor, because $\frac{n}{2} < n$. Therefore, $r_{2n}(p) \mid p^{\frac{n}{2}} - 1$, and so, $r_{2n}(p) \mid (p^{\frac{n}{2}} - 1, p^n + 1)$. Now, Lemma 2.15 implies that $r_{2n}(p) \mid (p^{\frac{n}{2}} - 1, p^n + 1) = (p-1, 2)$, which is impossible.

(2) Let $(\frac{G}{K})' \cong PSL(3, q)$ or $PSU(3, q)$, where $q$ is a prime power. Since $\mu(1)$ is not a prime power, Step (a)(iii) leads us to see that if $(\frac{G}{K})' \cong PSL(3, q)$, then one of $r_{2n}(p)$ and $r_n(p)$ divides $p^{\frac{n}{3}} + 1$, and if $(\frac{G}{K})' \cong PSU(3, q)$, then either $r_{2n}(p)$ or $r_n(p)$ divides $p^{\frac{n}{3}} - 1$. If $r_{2n}(p) \mid p^{\frac{n}{3}} + 1$ or $r_n(p) \mid p^{\frac{n}{3}} - 1$, then, since $\frac{n}{3} < n$, considering the definition of the primitive prime divisor leads us to get a contradiction. If $r_n(p) \mid p^{\frac{n}{3}} + 1$, then $r_n(p) \mid p^{\frac{2n}{3}} - 1$, which is a contradiction. Also, if $r_{2n}(p) \mid p^{\frac{n}{3}} - 1$, then $r_{2n}(p) \mid (p^{\frac{n}{3}} - 1, p^n + 1)$, and so, Lemma 2.15 shows that $r_{2n}(p) \mid (p+1, 2)$, which is a contradiction.

(3) Let $(\frac{G}{K})' \cong M_{11}, A_8$, $M_{23}$ or $J_1$. Let $(\frac{G}{K})' \cong M_{11}$. Then $\{10, 44, 55\} \subseteq \mathrm{cd}(M_{11})$. Since 10 and 44 are not prime powers and $(10, 44) = 2$, by considering Step (a)(iii), we conclude that one of $r_{2n}(p)$ and $r_n(p)$ divides 10, and the other one divides 44. Hence, $\{r_n(p), r_{2n}(p)\} = \{5, 11\}$. But since $55 \in \mathrm{cd}((\frac{G}{K})')$, considering Step(a)(iii) leads us to get a contradiction. The same argument rules out $(\frac{G}{K})' \cong A_8$, $M_{23}$ or $J_1$.

These contradictions show that $\Delta((\frac{G}{K})')$ is not connected. Thus Lemma 2.10 implies that $(\frac{G}{K})' \cong PSL(2, q)$, where $q$ is a prime power. Hence, $\mathrm{St}(1) = q$. Moreover, by Step (c), $\mathrm{St}(1) = p^n$. Therefore, $q = p^n$, and so, we get that $(\frac{G}{K})' \cong PSL(2, p^n)$. □

*Step (f).* $L \cong G$.

Proof. We get from Step (e) and Lemma 3.7 that $PSL(2, p^n) \lesssim G \lesssim \mathrm{Aut}(PSL(2, p^n))$, and hence, Lemma 3.8 shows that $G \cong L$, as desired. □

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SOMAYEH HEYDARI
DEPARTMENT OF PURE MATHEMATICS
FACULTY OF MATHEMATICAL SCIENCES
SHAHRE-KORD UNIVERSITY
P. O. BOX 115
SHAHRE-KORD
IRAN

*E-mail:* heydari.somayeh@stu.sku.ac.ir

NEDA AHANJIDEH
DEPARTMENT OF PURE MATHEMATICS
FACULTY OF MATHEMATICAL SCIENCES
SHAHRE-KORD UNIVERSITY
P. O. BOX 115
SHAHRE-KORD
IRAN

*E-mail:* ahanjideh.neda@sci.sku.ac.ir

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