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On weakly σ-permutable subgroups of finite groups

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Abstract. Let G be a finite group and $\sigma = \{\sigma_i | i \in I\}$ be a partition of the set of all primes $\mathbb{P}$. A set $\mathcal{H}$ of subgroups of G with $1 \in \mathcal{H}$ is said to be a complete Hall σ-set of G if every non-identity member of $\mathcal{H}$ is a Hall σ_i-subgroup of G and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every $\sigma_i \in \sigma(G)$. A subgroup H of G is said to be σ-permutable if G possesses a complete Hall σ-set $\mathcal{H}$ such that $HA^x = A^xH$ for all $A \in \mathcal{H}$ and all $x \in G$. We say that a subgroup H of G is weakly σ-permutable in G if there exists a σ-subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{\sigma G}$, where $H_{\sigma G}$ is the subgroup of H generated by all those subgroups of H which are σ-permutable in G. By using this new notion, we establish some new criteria for a group G to be a σ-soluble and supersoluble, and also we give the conditions under which a normal subgroup of G is hypercyclically embedded.

1. Introduction

Throughout this paper, all groups are finite and G always denotes a group. Moreover, n is an integer and $\mathbb{P}$ is the set of all primes. The symbol $\pi(n)$ denotes the set of all primes dividing n and $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G. $|G|_p$ denotes the order of the Sylow p-subgroup of G.

In what follows, $\sigma = \{\sigma_i | i \in I\}$ is some partition of $\mathbb{P}$, that is, $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. Π is always supposed to be a non-empty subset of σ and $\Pi' = \sigma \backslash \Pi$. We write $\sigma(n) = \{\sigma_i | \sigma_i \cap \pi(n) \neq \emptyset\}$ and $\sigma(G) = \sigma(|G|)$.

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Following [1], [2], we say that G is *σ-primary* if $G = 1$ or $|\sigma(G)| = 1$; we say that n is a *Π-number* if $\pi(n) \subseteq \bigcup_{\sigma_i \in \Pi} \sigma_i$; a subgroup H of G is called a *Π-subgroup* of G if $|H|$ is a Π-number; a subgroup H of G is called a *Hall Π-subgroup* of G if H is a Π-subgroup of G and $|G : H|$ is a $\Pi^{'}$-number. A set $\mathcal{H}$ of subgroups of G with $1 \in \mathcal{H}$ is said to be a *complete Hall Π-set* of G if every non-identity member of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $\sigma_i \in \Pi$, and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every $\sigma_i \in \Pi \cap \sigma(G)$. In particular, when $\Pi = \sigma$, we call $\mathcal{H}$ a complete Hall σ-set of G. G is said to be Π-*full* if G possesses a complete Hall Π-set; *a Π-full group of Sylow type* if every subgroup of G is a D_{σ_i}-group for all $\sigma_i \in \Pi \cap \sigma(G)$. In particular, G is said to be *σ-full* (or *σ-group*) if G possesses a complete Hall σ-set; *a σ-full group of Sylow type* if every subgroup of G is a D_{σ_i}-group for all $\sigma_i \in \sigma(G)$. A subgroup H of G is called [1] σ-subnormal in G if there is a subgroup chain $H = H_0 \leq H_1 \leq \cdots \leq H_t = G$ such that either H_{i-1} is normal in H_i or $H_i/(H_{i-1})_{H_i}$ is σ-primary for all $i = 1, \ldots, t$.

In the past 20 years, a large number of researches have involved finding and applying some generalized complemented subgroups. For example, a subgroup H of G is said to be c-normal [3] in G if G has a normal subgroup T of G such that $G = HT$ and $H \cap T \leq H_G$, where H_G is the normal core of H. A subgroup H of G is said to be weakly s-permutable [4] in G if G has a subnormal subgroup T such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the largest s-permutable subgroup of G contained in H (note that a subgroup H of G is said to be s-permutable in G if $HP = PH$ for any Sylow subgroup P of G). A subgroup H of G is said to be σ-permutable [1] in G if G possesses a complete Hall σ-set $\mathcal{H}$ such that $HA^x = A^xH$ for all $A \in \mathcal{H}$ and all $x \in G$. By using the above special supplemented subgroups and other generalized complemented subgroups, the researchers have obtained a series of interesting results (see [1], [3], [4], [5], [6], [7], [8], [9], [10], [11], and so on). Now, we consider the following new generalized supplemented subgroup:

Definition 1.1. A subgroup H of G is said to be weakly σ-permutable in G if there exists a σ-subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{\sigma G}$, where $H_{\sigma G}$ is the subgroup of H generated by all those subgroups of H which are σ-permutable in G.

Following [12], we call $H_{\sigma G}$ the σ-core of H.

It is clear that every c-normal subgroup, every s-permutable subgroup, every weakly s-permutable subgroup and every σ-permutable subgroup of G are all weakly σ-permutable in G. However, the following example shows that the converse is not true.

Example 1.2. Let $G = (C_7 \rtimes C_3) \times A_5$, where $C_7 \rtimes C_3$ is a non-abelian group of order 21, and A_5 is the alternating group of degree 5. Let B be a subgroup of A_5 of order 12, and A be a Sylow 5-subgroup of A_5. Let $\sigma = \{\sigma_1, \sigma_2\}$, where $\sigma_1 = \{2, 3, 5\}$ and $\sigma_2 = \{2, 3, 5\}^{'}$. Then B is weakly σ-permutable in G. In fact, let $T = (C_7 \rtimes C_3) \times A$. Then $C_7 \rtimes C_3 \le T_G$ and $|G : C_7 \rtimes C_3| = 5 \cdot 2^2 \cdot 3$ is a σ_1-number. Hence G/T_G is a σ_1-group, and so T is σ-subnormal in G. Since $T \cap B = 1$ and $G = BT$, which means that B is weakly σ-permutable in G. But B is neither weakly s-permutable in G nor c-normal in G. In fact, if there exists a subnormal subgroup K of G such that $G = BK$ and $B \cap K \le B_{sG}$, then B_{sG} is subnormal in G by [4, Lemma 2.8], and so is subnormal in A_5 by [13, A, Lemma 14.1]. It follows that $B_{sG} = 1$ for simple group A_5. Hence $|G : K| = |B| = 2^2 \cdot 3$. But since $1 < A_5 < A_5C_7 < G$ is a chief series of G and also a composition series of G, G has no subnormal subgroup K whose index is $2^2 \cdot 3$ by the Jordan–Hölder theorem. Therefore, B is not weakly s-permutable in G. Consequently, B is neither s-permutable nor c-normal in G.

Now let $H = BC_3$. Then H is weakly σ-permutable in G but not σ-permutable in G. Indeed, let $T = C_7A_5$. Then $G = HT$, T is normal in G and $H \cap T = B$. It is easy to see that $\mathcal{H} = \{A_5C_3, C_7\}$ is a complete Hall σ-set of G. Since $H_{\sigma G}$ is σ-subnormal in G by Lemma 2.3 (4) below and [1, Theorem B], $H_{\sigma G} \le O_{\sigma_1}(G)$ by Lemma 2.2 (8) below. Clearly, $O_{\sigma_1}(G) \le C_G(O_{\sigma_2}(G)) = C_G(C_7) = C_7A_5$. Hence $H_{\sigma G} \le C_7A_5$. But since $B(A_5C_3)^x = BA_5C_3^x = A_5C_3^x = C_3^xA_5 = (A_5C_3)^xB$ for all $x \in G$, B is σ-permutable in G for $C_7 \trianglelefteq G$. Hence $B \le H_{\sigma G} \le C_7A_5$, which implies that $B = H_{\sigma G}$. Thus H is weakly σ-permutable in G. But H is not σ-permutable in G for $H_{\sigma G} = B < H$.

Following [1], G is called:

(i) σ-soluble if every chief factor of G is σ-primary;

(ii) σ-nilpotent if $H/K \rtimes (G/C_G(H/K))$ is σ-primary for every chief factor H/K of G.

The results in [1], [4], [14], [15], [3] are the motivation for the following:

Question 1.3. *Let G be a σ-full group of Sylow type. What is the structure of G provided that some subgroups are weakly σ-permutable in G?*

In this paper, we obtain the following results.

Theorem 1.4. *Let G be a σ-full group of Sylow type, and suppose that every Hall σ_i-subgroup of G is weakly σ-permutable in G for every $\sigma_i \in \sigma(G)$. Then G is σ-soluble.*

Theorem 1.5. *Let G be a σ-full group of Sylow type, and $\mathcal{H} = \{W_1, \dots, W_t\}$ be a complete Hall σ-set of G such that W_i is a nilpotent σ_i-subgroup for all $i = 1, \dots, t$. Suppose that the maximal subgroups of any non-cyclic W_i are weakly σ-permutable in G. Then G is supersoluble.*

The following results immediately appear from Theorems 1.4 and 1.5.

Corollary 1.6. *If every Sylow subgroup of G is weakly s-permutable in G, then G is soluble.*

Corollary 1.7 (see Huppert [16, VI, Theorem 10.3]). *If every Sylow subgroup of G is cyclic, then G is supersoluble.*

Corollary 1.8 (see Miao [17, Corollary 3.4]). *If all maximal subgroups of every Sylow subgroup of G are weakly s-permutable in G, then G is supersoluble.*

Corollary 1.9 (see Skiba [4, Theorem 1.4]). *If all maximal subgroups of every non-cyclic Sylow subgroup of G are weakly s-permutable in G, then G is supersoluble.*

Corollary 1.10 (see Srinivasan [15, Theorem 1]). *If all maximal subgroups of every Sylow subgroup of G are normal in G, then G is supersoluble.*

Corollary 1.11 (see Srinivasan [15, Theorem 2]). *If all maximal subgroups of every Sylow subgroup of G are s-permutable in G, then G is supersoluble.*

Corollary 1.12 (see Wang [3, Theorem 4.1]). *If all maximal subgroups of every Sylow subgroup of G are c-normal in G, then G is supersoluble.*

Recall that a normal subgroup E of G is called hypercyclically embedded in G and is denoted by $E \leq Z_{\mathfrak{U}}(G)$ (see [18, p. 217]) if either $E = 1$ or $E \neq 1$ and every chief factor of G below E is cyclic, where the symbol $Z_{\mathfrak{U}}(G)$ denotes the $\mathfrak{U}$-hypercentre of G, that is, the product of all normal hypercyclically embedded subgroups of G. Hypercyclically embedded subgroups play an important role in the theory of groups (see [7], [8], [18], [19]) and the conditions under which a normal subgroup is hypercyclically embedded in G were found by many authors (see the books [7], [8], [18], [19], and the recent papers [10], [14], [20], [21], [22], [23]).

On the base of Theorem 1.5, we will prove the following result.

Theorem 1.13. *Let G be a σ-full group of Sylow type, and $\mathcal{H} = \{W_1, \dots, W_t\}$ be a complete Hall σ-set of G such that W_i is nilpotent for all $i = 1, \dots, t$. Let E be a normal subgroup of G. If every maximal subgroup of $W_i \cap E$ is weakly σ-permutable in G for all $i = 1, \dots, t$, then $E \leq Z_{\mathfrak{U}}(G)$.*

The following results directly follow from Theorem 1.13.

Corollary 1.14. *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups, and let E be a normal subgroup of G with $G/E \in \mathfrak{F}$. Suppose that G is a σ-full group of Sylow type, and $\mathcal{H} = \{W_1, \dots, W_t\}$ is a complete Hall σ-set of G such that W_i is nilpotent for all $i = 1, \dots, t$. If every maximal subgroup of $W_i \cap E$ is weakly σ-permutable in G for all $i = 1, \dots, t$, then $G \in \mathfrak{F}$.*

Corollary 1.15 (see Asaad [24, Theorem 4.1]). *Let G be a group with a normal subgroup E such that G/E is supersoluble. If every maximal subgroup of every Sylow subgroup of E is s-permutable in G, then G is supersoluble.*

Corollary 1.16 (see Asaad [25, Theorem 1.3]). *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups, and let E be a normal subgroup of G with $G/E \in \mathfrak{F}$. If the maximal subgroups of every Sylow subgroup of E are s-permutable in G, then $G \in \mathfrak{F}$.*

Corollary 1.17 (see Wei [11, Corollary 1]). *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups, and let E be a normal subgroup of G with $G/E \in \mathfrak{F}$. If the maximal subgroups of every Sylow subgroup of E are c-normal in G, then $G \in \mathfrak{F}$.*

All unexplained terminologies and notations are standard, as in [8] and [13].

2. Preliminaries

We use $\mathfrak{S}_\sigma$ and $\mathfrak{N}_\sigma$ to denote the classes of all σ-soluble groups and σ-nilpotent groups, respectively.

Lemma 2.1 (see [1, Lemma 2.1]). *The class $\mathfrak{S}_\sigma$ is closed under taking direct products, homomorphic images and subgroups. Moreover, any extension of the σ-soluble group by a σ-soluble group is a σ-soluble group as well.*

Following [1] and [2], $O^{\Pi}(G)$ denotes the subgroup of G generated by all its Π'-subgroups. Instead of $O^{\{\sigma_i\}}(G)$, we write $O^{\sigma_i}(G)$. $O_{\Pi}(G)$ denotes the subgroup of G generated by all its normal Π-subgroups.

Lemma 2.2 (see [1, Lemma 2.6] and [2, Lemma 2.1]). *Let A, K and N be subgroups of G. Suppose that A is σ-subnormal in G and N is normal in G.*

(1) *$A \cap K$ is σ-subnormal in K.*

(2) *If K is a σ-subnormal subgroup of A, then K is σ-subnormal in G.*

(3) *If K is σ-subnormal in G, then $A \cap K$ and $\langle A, K\rangle$ are σ-subnormal in G.*

(4) *AN/N is σ-subnormal in G/N.*

(5) *If $N \leq K$ and K/N is σ-subnormal in G/N, then K is σ-subnormal in G.*

(6) *If $K \leq A$ and A is σ-nilpotent, then K is σ-subnormal in G.*

(7) *If $|G : A|$ is a Π-number, then $O^{\Pi}(A) = O^{\Pi}(G)$.*

(8) *If G is Π-full and A is a Π-group, then $A \leq O_{\Pi}(G)$.*

Let $\mathcal{L}$ be some non-empty set of subgroups of G, and E a subgroup of G. Then a subgroup A of G is called $\mathcal{L}$-permutable if $AH = HA$ for all $H \in \mathcal{L}$; $\mathcal{L}^E$-permutable if $AH^x = H^xA$ for all $H \in \mathcal{L}$ and all $x \in E$. In particular, a subgroup H of G is σ-permutable if G possesses a complete Hall σ-set $\mathcal{H}$ such that H is $\mathcal{H}^G$-permutable.

Lemma 2.3 (see [1, Lemma 2.8] and [2, Lemma 2.2]). *Let H, K and N be subgroups of G. Let $\mathcal{H} = \{H_1, \ldots, H_t\}$ be a complete Hall σ-set of G and $\mathcal{L} = \mathcal{H}^K$. Suppose that H is $\mathcal{L}$-permutable and N is normal in G.*

(1) *If $H \leq E \leq G$, then H is $\mathcal{L}^*$-permutable, where $\mathcal{L}^* = \{H_1 \cap E, \ldots, H_t \cap E\}^{K\cap E}$. In particular, if G is a σ-full group of Sylow type and H is σ-permutable in G, then H is σ-permutable in E.*

(2) *The subgroup HN/N is $\mathcal{L}^{**}$-permutable, where $\mathcal{L}^{**} = \{H_1N/N, \ldots, H_tN/N\}^{KN/N}$.*

(3) *If G is a σ-full group of Sylow type and E/N is a σ-permutable subgroup of G/N, then E is σ-permutable in G.*

(4) *If K is $\mathcal{L}$-permutable, then $\langle H, K\rangle$ is $\mathcal{L}$-permutable* [13, A, Lemma 1.6(a)]. *In particular, $H_{\sigma G}$ is σ-permutable in G. Moreover, if G is a σ-full group of Sylow type, then $H_{\sigma G}$ is a σ-subnormal subgroup of G (see* [1, Theorems B and C]*).*

Lemma 2.4 (see [1, Lemma 3.1]). *Let H be a σ_1-subgroup of a σ-full group G. Then H is σ-permutable in G if and only if $O^{\sigma_1}(G) \leq N_G(H)$.*

Lemma 2.5. *Let G be a σ-full group of Sylow type and $H \leq K \leq G$.*

(1) *If H is weakly σ-permutable in G, then H is weakly σ-permutable in K.*

(2) *Suppose that N is a normal subgroup of G and $N \leq H$. Then H/N is weakly σ-permutable in G/N if and only if H is weakly σ-permutable in G.*

(3) *If N is a normal subgroup of G, then for every weakly σ-permutable subgroup H of G with $(|H|, |N|) = 1$, HN/N is weakly σ-permutable in G/N.*

Proof. (1) Suppose that there exists a σ-subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{\sigma G}$. Since $H \leq K$, $K = H(K \cap T)$. By Lemma 2.2 (1), $K \cap T$ is σ-subnormal in K. Moreover, $H \cap (K \cap T) = H \cap T \leq H_{\sigma G} \leq H_{\sigma K}$ by Lemma 2.3 (1)(4). Hence, H is weakly σ-permutable in K.

(2) First assume that there exists a σ-subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{\sigma G}$. Then $G/N = (H/N)(TN/N)$, TN/N is σ-subnormal in G/N by Lemma 2.2 (4), and $H/N \cap TN/N = (H \cap T)N/N \leq H_{\sigma G}N/N \leq (H/N)_{\sigma(G/N)}$ by Lemma 2.3 (2). This shows that H/N is weakly σ-permutable in G/N.

Conversely, assume that H/N is weakly σ-permutable in G/N. Then $G/N = (H/N)(T/N)$ and $H/N \cap T/N \leq (H/N)_{\sigma(G/N)}$, where T/N is σ-subnormal in G/N. So $G = HT$ and T is σ-subnormal in G by Lemma 2.2 (5). Let $(H/N)_{\sigma(G/N)} = E/N$. Then E is σ-permutable in G by Lemma 2.3 (3)(4). Hence $H \cap T \leq E \leq H_{\sigma G}$. This shows that H is weakly σ-permutable in G.

(3) Assume that there exists a σ-subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{\sigma G}$. Then $G/N = (HN/N)(TN/N)$. Since $(|H|, |N|) = 1$, $(|HT \cap N : H \cap N|, |HT \cap N : T \cap N|) = (|(HT \cap N)H : H|, |(HT \cap N)T : T|) = 1$. Hence $N = N \cap HT = (N \cap H)(N \cap T) = N \cap T$ by [13, A, Lemma 1.6]. It follows that $N \leq T$. Hence $(HN/N) \cap (TN/N) = (H \cap T)N/N \leq H_{\sigma G}N/N \leq (HN/N)_{\sigma(G/N)}$ by Lemma 2.3 (2)(4). Besides, by Lemma 2.2 (4), T/N is σ-subnormal in G/N. Thus HN/N is weakly σ-permutable in G/N. □

Lemma 2.6 (see [26, Lemma 2.12]). *Let P be a normal p-subgroup of G. If $P/\Phi(P) \leq Z_{\mathfrak{U}}(G/\Phi(P))$, then $P \leq Z_{\mathfrak{U}}(G)$.*

3. Proof of Theorem 1.4

Proof of Theorem 1.4. Assume that this is false, and let G be a counterexample of minimal order. Then $|\sigma(G)| > 1$.

(1) G/N is σ-soluble for every non-identity normal subgroup N of G.

Let N be a non-identity normal subgroup of G and H/N be any Hall σ_i-subgroup of G/N, where $\sigma_i \cap \pi(G/N) \neq \emptyset$. Then $H/N = H_iN/N$ for some Hall σ_i-subgroup H_i of G. By the hypothesis, there exists a σ-subnormal subgroup T of G such that $G = H_iT$ and $H_i \cap T \leq (H_i)_{\sigma G}$. Then $G/N = (H_iN/N)(TN/N) = (H/N)(TN/N)$. Since $|H_iN \cap T : H_i \cap T| = |(H_iN \cap T)H_i : H_i|$ is a σ_i'-number, $(|H_iN \cap T : H_i \cap T|, |H_iN \cap T : N \cap T|) = 1$. Hence $H_iN \cap T = (H_i \cap T)(N \cap T)$ by [13, A, Lemma 1.6]. Consequently, $(H_iN/N) \cap (TN/N) = (H_iN \cap T)N/N = (H_i \cap T)N/N \leq (H_i)_{\sigma G}N/N \leq (H_iN/N)_{\sigma(G/N)}$

by Lemma 2.3 (2)(4). By Lemma 2.2 (4), TN/N is σ-subnormal in G/N. Therefore, H/N is weakly σ-permutable in G/N. This shows that G/N satisfies the hypothesis. The minimal choice of G implies that G/N is σ-soluble.

(2) G is not a simple group.

Suppose that G is a non-abelian simple group. Then 1 is the only proper σ-subnormal subgroup of G. Let H_i be a non-identity Hall σ_i-subgroup of G, where $\sigma_i \in \sigma(G)$. By the hypothesis and $|\sigma(G)| > 1$, we have $G = H_iG$ and $H_i = H_i \cap G \leq (H_i)_{\sigma G}$. By Lemma 2.3 (4), $(H_i)_{\sigma G}$ is σ-subnormal in G, so $H_i = (H_i)_{\sigma G} = 1$, a contradiction. Hence we have (2).

(3) Let R be a minimal normal subgroup of G, then R is σ-soluble.

Let H be any Hall σ_i-subgroup of R, where $\sigma_i \cap \pi(R) \neq \emptyset$. Then there exists a Hall σ_i-subgroup H_i of G such that $H = H_i \cap R$. By the hypothesis, there exists a σ-subnormal subgroup T of G such that $G = H_iT$ and $H_i \cap T \leq (H_i)_{\sigma G}$. Since $|H_iT \cap R : H_i \cap R| = |(H_iT \cap R)H_i : H_i|$ is a σ_i'-number, $(|H_iT \cap R : H_i \cap R|, |H_iT \cap R : T \cap R|) = 1$. Hence $R = R \cap H_iT = (H_i \cap R)(R \cap T) = H(R \cap T)$ by [13, A, Lemma 1.6(c)]. Since $R \cap T$ is σ-subnormal in R by Lemma 2.2 (1), and $H \cap R \cap T = (R \cap H_i) \cap (R \cap T) \leq (H_i)_{\sigma G} \cap R \leq H_{\sigma R}$ by Lemma 2.3 (1)(4), R satisfies the hypothesis. The minimal choice of G implies that R is σ-soluble.

(4) Final contradiction.

In view of (1), (2) and (3), we have that G is σ-soluble by Lemma 2.1. The final contradiction completes the proof of the theorem. □

4. Proof of Theorem 1.5

First, we prove the following proposition, which is a main step of the proof of Theorem 1.5.

Proposition 4.1. *Let G be a σ-full group of Sylow type, and $\mathcal{H} = \{W_1, \ldots, W_t\}$ be a complete Hall σ-set of G such that W_i is a nilpotent σ_i-subgroup for all $i = 1, ..., t$, and let the smallest prime p of $\pi(G)$ belong to σ_1. If every maximal subgroup of W_1 is weakly σ-permutable in G, then G is soluble.*

Proof. First note that if G is σ-soluble, then every chief factor H/K of G is σ-primary, that is, H/K is a σ_i-group for some i. But since W_i is nilpotent, H/K is an elementary abelian group. It follows that G is soluble. Hence we only need to prove that G is σ-soluble. Suppose that the assertion is false, and let G be a counterexample of minimal order. Then clearly, $t > 1$, and $p = 2 \in \pi(W_1)$ by the Feit–Thompson theorem. Without loss of generality, we can assume that W_i is a σ_i-group for all $i = 1, \ldots, t$.

(1) $O_{\sigma_1}(G) = 1$.

Assume that $N = O_{\sigma_1}(G) \neq 1$. Note that if $W_1 = N$, then G/N is a σ_1'-group, so G/N is soluble by the Feit–Thompson theorem, and so G is σ-soluble. We may, therefore, assume that $W_1 \neq N$. Then W_1/N is a non-identity Hall σ_1-subgroup of G/N. Let M/N be a maximal subgroup of W_1/N. Then M is a maximal subgroup of W_1. By the hypothesis and Lemma 2.5 (2), M/N is weakly σ-permutable in G/N. The minimal choice of G implies that G/N is σ-soluble. Consequently, G is σ-soluble. This contradiction shows that (1) holds.

(2) $O_{\sigma_1'}(G) = 1$.

Assume that $R = O_{\sigma_1'}(G) \neq 1$. Then W_1R/R is a Hall σ_1-subgroup of G/R. Let M/R be a maximal subgroup of W_1R/R. Then $M = (M \cap W_1)R$. Since W_1 is nilpotent and $|W_1R/R : M/R| = |W_1R/R : (M \cap W_1)R/R| = |W_1 : M \cap W_1|$, $M \cap W_1$ is a maximal subgroup of W_1. By the hypothesis and Lemma 2.5 (3), $M/R = (M \cap W_1)R/R$ is weakly σ-permutable in G/R. This shows that G/R satisfies the hypothesis. The choice of G implies that G/R is σ-soluble. By the Feit–Thompson theorem, we know that R is soluble. It follows that G is σ-soluble, a contradiction.

(3) If $R \neq 1$ is a minimal normal subgroup of G. Then R is not σ-soluble and $G = RW_1$.

If R is σ-soluble, then R is a σ_i-group for some $\sigma_i \in \sigma(G)$. So $R \leq O_{\sigma_1}(G)$ or $R \leq O_{\sigma_1'}(G)$, a contradiction. Therefore, R is not σ-soluble. Assume that $RW_1 < G$. Then by the hypothesis and Lemma 2.5 (1), RW_1 satisfies the hypothesis. Hence RW_1 is σ-soluble by the choice of G. It follows from Lemma 2.1 that R is σ-soluble. This contradiction shows that $G = RW_1$.

(4) G has a unique minimal normal subgroup R.

By (3), $G = RW_1$ for every non-identity minimal normal subgroup R of G. Then clearly, G/R is σ-soluble. Hence by Lemma 2.1, G has a unique minimal normal subgroup, which is denoted by R.

(5) W_1 is a 2-group.

Let $q \in \pi(W_1)\backslash\{2\}$. Since W_1 is nilpotent, there exist two maximal subgroups M_1 and M_2 of W_1 such that $|W_1 : M_1| = q$ and $|W_1 : M_2| = 2$. By the hypothesis, there exist σ-subnormal subgroups T_i of G, such that $G = M_iT_i$ and $M_i \cap T_i \leq (M_i)_{\sigma G}$, $i = 1, 2$. By Lemma 2.3 (4), $(M_i)_{\sigma G}$ is σ-subnormal in G. Then by Lemma 2.2 (8), $(M_i)_{\sigma G} \leq O_{\sigma_1}(G) = 1$, $i = 1, 2$. Hence $M_i \cap T_i = 1, i = 1, 2$. Consequently, $|G : T_i| = |M_i : M_i \cap T_i| = |M_i|$, $i = 1, 2$, which implies that $|G : T_i|$ is a σ_i-number for $i = 1, 2$. Hence $O^{\sigma_1}(T_i) = O^{\sigma_1}(G)$ for $i = 1, 2$ by Lemma 2.2 (7). Since $t > 1$, $O^{\sigma_1}(G) > 1$. It follows that $1 \neq O^{\sigma_1}(G) \leq (T_i)_G$ for $i = 1, 2$. Then by (4), $R \leq (T_1)_G \cap (T_2)_G \leq T_1 \cap T_2$. It is clear that $W_1 \cap R$ is a Hall

σ_1-subgroup of R, and $W_1 \cap R \neq 1$ by (2). Hence $1 \neq W_1 \cap R \leq T_1 \cap T_2 \cap W_1$. Since $G = M_1T_1 = W_1T_1 = M_2T_2 = W_1T_2$, where $M_1 \cap T_1 = 1$ and $M_2 \cap T_2 = 1$, we have that $|W_1 \cap T_1| = |W_1 : M_1| = q$ and $|W_1 \cap T_2| = |W_1 : M_2| = 2$. Therefore, $(W_1 \cap T_1) \cap (W_1 \cap T_2) = 1$, which implies that $1 \neq W_1 \cap R \leq T_1 \cap T_2 \cap W_1 = (T_1 \cap W_1) \cap (T_2 \cap W_1) = 1$. This contradiction shows that W_1 is a 2-group.

(6) Final contradiction.

Let P_1 be a maximal subgroup of W_1. Then $|W_1 : P_1| = 2$. By the hypothesis, there exists a σ-subnormal subgroup K of G such that $G = P_1K$ and $P_1 \cap K \leq (P_1)_{\sigma G}$. By (1) and Lemma 2.2 (8), $(P_1)_{\sigma G} = 1$, Hence $|K|_2 = 2$, and so K is 2-nilpotent by [16, IV, Theorem 2.8]. Let $K_{2'}$ be the normal Hall $2^{'}$-subgroup of K. Then $1 \neq K_{2'}$ is σ-subnormal in G, and so $K_{2'} \leq O_{\sigma_1'}(G) = 1$ by Lemma 2.2(8). The final contradiction completes the proof. □

Proof of Theorem 1.5. Assume that the assertion is false, and let G be a counterexample of minimal order.

(1) G is soluble.

Let q be the smallest prime dividing $|G|$. Without loss of generality, we may assume that $q \in \pi(W_1)$. If W_1 is cyclic, then the Sylow q-subgroup of G is cyclic. Hence G is q-nilpotent by [16, IV, Theorem 2.8], and so G is soluble. If W_1 is non-cyclic, then by Proposition 4.1, G is soluble. Hence we always have that G is soluble.

(2) The hypothesis holds on G/R for every non-identity minimal normal subgroup R of G. Consequently, G/R is supersoluble.

It is clear that $\overline{\mathcal{H}} = \{W_1R/R, \ldots, W_tR/R\}$ is a complete Hall σ-set of G/R and $W_iR/R \simeq W_i/W_i \cap R$ is nilpotent. By (1), R is an elementary abelian p-group for some prime p. Without loss of generality, we can assume that $R \leq W_1$. If W_1/R is non-cyclic, then W_1 is non-cyclic. For every maximal subgroup M/R of W_1/R, we have that M is a maximal subgroup of W_1. Then by the hypothesis and Lemma 2.5 (2), M/R is weakly σ-permutable in G/R. Now assume that W_iR/R is non-cyclic for $i \neq 1$, and that M/R is a maximal subgroup of W_iR/R. Then $M = (M \cap W_i)R$. Since W_i is nilpotent, $|W_iR/R : M/R| = |W_iR/R : (M \cap W_i)R/R| = |W_i : M \cap W_i|$ is a prime. Hence $M \cap W_i$ is a maximal subgroup of W_i. By the hypothesis and Lemma 2.5 (3), $M/R = (M \cap W_i)R/R$ is weakly σ-permutable in G/R. This shows that the hypothesis holds for G/R. Hence G/R is supersoluble by the choice of G.

(3) R is the unique minimal normal subgroup of G, $\Phi(G) = 1$, $C_G(R) = R = F(G) = O_p(G)$ and $|R| > p$ for some prime p (it follows from (2)).

(4) For some $i \in \{1, \ldots, t\}$, W_i is a p-group. Without loss of generality, we may assume that W_1 is a p-group.

Since R is a p-group, $R \leq W_i$ for some $i \in \{1, \dots, t\}$. Moreover, since $C_G(R) = R$ and W_i is a nilpotent group, we have that W_i is a p-group.

(5) Final contradiction.

Since $\Phi(G) = 1$, $R \nleq \Phi(W_1)$ [16, III, Lemma 3.3]. Hence there exists a maximal subgroup V of W_1 such that $W_1 = RV$. Let $E = R \cap V$. Then $|R : E| = |RV : V| = |W_1 : V| = p$. Hence E is a maximal subgroup of R and $1 \neq E \trianglelefteq W_1$. Since $|R| > p$ and $R \leq W_1$, W_1 is non-cyclic. Hence by the hypothesis, there exists a σ-subnormal subgroup T of G such that $G = VT$ and $V \cap T \leq V_{\sigma G}$. Since $|G : T|$ is a p-number, $O^p(T) = O^{\sigma_1}(T) = O^{\sigma_1}(G)$ by Lemma 2.2 (7). So $|G : T_G|$ is a p-number. It follows that $T_G \neq 1$ and $R \leq T_G \leq T$ by (2). Since $V_{\sigma G}$ is σ-subnormal in G by Lemma 2.3 (4), we have that $V_{\sigma G} \leq O_{\sigma_1}(G) = O_p(G) = R$ by Lemma 2.2 (8). Hence $E = R \cap V \leq T \cap V \leq V_{\sigma G} \leq R$. But since E is a maximal subgroup of R, it follows that $V_{\sigma G} = R$ or $V_{\sigma G} = E$. In the former case, we have that $R \leq V$, a contradiction. In the latter case, $E = V_{\sigma G}$ is σ-permutable in G by Lemma 2.3 (4) and E is a σ_1-group. It follows from Lemma 2.4 that $O^{\sigma_1}(G) \leq N_G(E)$. Hence $E \trianglelefteq G$, which contradicts the minimality of R. The final contradiction completes the proof of the theorem. □

5. Proof of Theorem 1.13

Proof of Theorem 1.13. Assume that the assertion is false, and let (G, E) be a counterexample with minimal $|G| + |E|$. Without loss of generality, we can assume that W_i is a σ_i-group for all $i = 1, \dots, t$. We now proceed with the proof via the following steps.

(1) E is supersoluble.

In fact, $\{W_1 \cap E, \dots, W_t \cap E\}$ is a complete Hall σ-set of E and $W_i \cap E$ is nilpotent. Consequently, E is a σ-full group of Sylow type. Hence E is supersoluble by Lemma 2.5 (1) and Theorem 1.5.

(2) If R is a minimal normal subgroup of G contained in E, then R is a p-group for some prime p, and the hypothesis holds for $(G/R, E/R)$. Therefore, $E/R \leq Z_{\mathfrak{U}}(G/R)$.

By (1), R is a p-group for some p. Without loss of generality, we can assume that $R \leq W_1 \cap E$. It is clear that $\overline{\mathcal{H}} = \{W_1/R, \dots, W_tR/R\}$ is a complete Hall σ-set of G/R, and $W_iR/R \simeq W_i/W_i \cap R$ is nilpotent. Let M/R be a maximal subgroup of $(W_1 \cap E)/R$. Then by the hypothesis and Lemma 2.5 (2), M/R is weakly σ-permutable in G/R. Now let V/R be a maximal subgroup of $(W_iR/R) \cap (E/R) = (W_i \cap E)R/R$, $i = 2, \dots, t$. Then $V = (V \cap W_i)R$. Since $(W_iR/R) \cap$

(E/R) is nilpotent, $|W_i \cap E : V \cap W_i| = |W_iR \cap E : (V \cap W_i)R| = |(W_iR/R) \cap (E/R) : V/R|$ is a prime, so $V \cap W_i$ is a maximal subgroup of $W_i \cap E$. Then by the hypothesis and Lemma 2.5 (3), $V/R = (V \cap W_i)R/R$ is weakly σ-permutable in G/R, $i = 2, \ldots, t$. This shows that $(G/R, E/R)$ satisfies the hypothesis. Thus $E/R \leq Z_{\mathfrak{U}}(G/R)$ by the choice of (G, E).

(3) R is the unique minimal normal subgroup of G contained in E, $|R| > p$ and $O_{p'}(E) = 1$.

Let L be a minimal normal subgroup of G contained in E such that $R \neq L$. Then $E/R \leq Z_{\mathfrak{U}}(G/R)$ and $E/L \leq Z_{\mathfrak{U}}(G/L)$ by (2), and clearly, $|R| > p$. It follows that $LR/L \leq Z_{\mathfrak{U}}(G/L)$, so $|R| = p$ by the G-isomorphism $RL/L \simeq R$, a contradiction. Hence R is the unique minimal normal subgroup of G contained in E. Consequently, $O_{p'}(E) = 1$. Hence (3) holds.

Without loss of generality, we may assume $p \in \pi(W_1)$.

(4) E is a p-group, and so $E \cap W_1 = E$ and $E \cap W_i = 1$ for $i = 2, \ldots, t$.

Let q be the largest prime dividing $|E|$, and let Q be a Sylow q-subgroup of E. Since E is supersoluble by (1) (see [16, VI, Theorem 9.1]), Q is characteristic in E. Then Q is normal in G. Hence by (3), we have that $q = p$ and $F(E) = Q = O_p(E) = P$ is a Sylow p-subgroup of E. Thus $C_E(P) \leq P$ (see [27, Theorem 1.8.18]). But since $P \leq W_1 \cap E$ and $W_1 \cap E$ is nilpotent, we have that $P = W_1 \cap E$. Since $P \cap W_1 = P = W_1 \cap E$ and $P \cap W_i = 1$ for all $i = 2, \ldots, t$, the hypothesis holds for (G, P). If $P < E$, then $R \leq P \leq Z_{\mathfrak{U}}(G)$ by the choice of (G, E). It follows that $|R| = p$, a contradiction. Hence $E = P$ is a p-group, and so $E \leq W_1$.

(5) $\Phi(E) = 1$, so E is an elementary abelian p-group.

Assume that $\Phi(E) \neq 1$. Then clearly, $(G/\Phi(E), E/\Phi(E))$ satisfies the hypothesis. Hence $E/\Phi(E) \leq Z_{\mathfrak{U}}(G/\Phi(E))$. It follows from (4) and Lemma 2.6 that $E \leq Z_{\mathfrak{U}}(G)$, a contradiction. Thus we have (5).

(6) Final contradiction.

Let R_1 be a maximal subgroup of R such that $R_1 \trianglelefteq W_1$. Then $|R_1| > 1$ by (3). Claim (5) implies that R has a complement S in E. Let $V = R_1S$. Then $R \cap V = R_1$, and V is a maximal subgroup of E. Hence by (4) and the hypothesis, there exists a σ-subnormal subgroup T of G such that $G = VT$ and $V \cap T \leq V_{\sigma G}$. Then $G = VT = ET$ and $E = V(E \cap T)$. By (5), it is easy to see that $1 \neq E \cap T \trianglelefteq G$. Hence $R \leq E \cap T$ by (3), and so $R_1 = R \cap V \leq E \cap T \cap V = V \cap T \leq V_{\sigma G}$. Consequently, $R_1 \leq V_{\sigma G} \cap R \leq R$. It follows that $R = V_{\sigma G} \cap R$ or $R_1 = V_{\sigma G} \cap R$. In the former case, $R \leq V$, which contradicts the fact that $R_1 = R \cap V$. Thus $R_1 = V_{\sigma G} \cap R$. By Lemma 2.3(4), we have that $V_{\sigma G}$ is σ-permutable in G, so $O^{\sigma_1}(G) \leq N_G(V_{\sigma G})$ by Lemma 2.4. Hence $O^{\sigma_1}(G) \leq N_G(V_{\sigma G} \cap R) = N_G(R_1)$.

Moreover, since $R_1 \trianglelefteq W_1$, we obtain that $R_1 \trianglelefteq G$. This implies that $R_1 = 1$. The final contradiction completes the proof. □

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